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EXISTENCE RESULTS FOR COUPLED NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH COUPLED STRIP AND INFINITE POINT BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate a class of nonlinear fractional differential system supplemented with coupled strip and infinite point boundary conditions. Existence results for the given problem are obtained by using the Banach's fixed point theorem and the $\|\cdot\|_e$ norm. The Lipschitz type conditions on nonlinearities are needed and it seems that the continuity assumptions used previously are not sufficient. The proposed problem is of quite a general nature as it covers several special cases. Finally, we present an example to illustrate our main results.
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1. Introduction

Fractional differential equations have been of increasing importance for their deep backgrounds in science and engineering, such as, the memory of a variety of materials, signal identification and image processing, optical systems, thermal system materials and mechanical systems, control system, etc., see [7, 11]. In recent years, much attention has been focused on the study of the existence and multiplicity of solutions for boundary value problems of fractional differential equations, see [10, 14, 13, 16, 17], and the references therein.

Coupled systems of differential equations have also been investigated by many authors. Such systems appear in the modeling of many real world problems, for example, see [3, 12]. For some recent results on the topic, we refer the reader to a series of papers [1, 8, 9, 15, 6, 2], and the references therein.

Cui [4] studied the following differential system with coupled integral boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), v(t)), \quad -v''(t) = g(t, u(t), v(t)), \\ u(0) = v(0) = 0, \quad u(1) = \int_0^1 v(t)dA(t), \quad v(1) = \int_0^1 u(t)dB(t), \end{cases}$$

where $\int_0^1 v(t)dA(t)$, $\int_0^1 u(t)dB(t)$ are Riemann-Stieltjes integrals. By means of μ_0-positive operator, they obtained the uniqueness of solution for the above differential system under the assumption that f, g are Lipschitz continuous functions. It should be mentioned that the Lipschitz constant is dependant of the spectral radius corresponding to the related linear operators. Therefore, their result is new and meaningful. However, it is rather difficult to determine the value of the spectral radius.

Here, we consider the existence of solutions for the boundary value problem of fractional differential systems with mixed fractional derivatives

$$\begin{cases} {}^C D_{0+}^{\beta-1}(D+k)D_{0+}^{\alpha} u(t) = f(t, u(t), v(t)), \quad \text{for } a.e.\ t \in [0,1], \\ {}^C D_{0+}^{\beta-1}(D+k)D_{0+}^{\alpha} v(t) = g(t, u(t), v(t)), \quad \text{for } a.e.\ t \in [0,1], \end{cases} \tag{1.1}$$

subject to the following coupled boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = D_{0+}^{\alpha} u(0) = 0, \\ \quad D_{0+}^{\alpha} u(1) = a_1 \int_0^{\sigma} D_{0+}^{\alpha} v(s)ds, \quad D_{0+}^{\gamma} u(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j) + \eta_0 \int_0^1 D_{0+}^{\gamma} u(\tau) d\mathcal{A}(\tau), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = D_{0+}^{\alpha} v(0) = 0, \\ \quad D_{0+}^{\alpha} v(1) = a_2 \int_0^{\sigma} D_{0+}^{\alpha} u(s)ds, \quad D_{0+}^{\gamma} v(1) = \sum_{j=1}^{\infty} \varepsilon_j v(\xi_j) + \varepsilon_0 \int_0^1 D_{0+}^{\gamma} v(\tau) d\mathcal{B}(\tau), \end{cases} \tag{1.2}$$

where D_{0+}^{ς} denote the Riemann-Liouville fractional derivative of order ς with $\varsigma = \alpha, \gamma$, ${}^C D_{0+}^{\beta-1}$ is the Caputo fractional derivative, D is the ordinary derivative, $\alpha > 2$, $n-1 < \alpha \le n$, $1 < \beta \le 2$, γ is a fixed number and $\gamma \in [1, n-2]$ $(n \ge 3)$, $\eta_j, \varepsilon_j \ge 0$ $(j = 1, 2, \cdots)$, $\eta_0, \varepsilon_0 \in [0,1)$, $\xi_j \in (0,1)$, $k, a_1, a_2 \ge 0$, $\sigma \in [0,1]$, $\int_0^1 u(s)d\mathcal{A}(s)$, $\int_0^1 u(s)d\mathcal{B}(s)$ are Riemann-Stieltjes integrals, $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

To the best of our knowledge, there are fewer works concerned the existence results for fractional differential systems with coupled strip and infinite point boundary conditions. In order to enrich the theoretical knowledge of the above, we investigate the existence of solutions for the problem (1.1)–(1.2) by using the Banach's fixed point theorem and the $\|\cdot\|_e$ norm. The relations between the linear Caputo fractional differential systems and the corresponding linear integral systems are studied. It should be mentioned that the Lipschitz type conditions on nonlinearities are needed in the study of the nonlinear Caputo fractional differential systems. The proposed problem is of quite a general nature as it covers several special cases. Thus the obtained results will be a useful and novel contribution to the existing literature on fractional differential systems.

The paper is organized as follows. In Section 2, we present some notations, definitions of fractional calculus and give some useful lemmas. In Section 3, we discuss the existence of solutions of the boundary value problem (1.1)–(1.2). In Section 4, we give an example to illustrate our results.

2. Preliminaries

In this section, we introduce notations, definitions of fractional calculus and present some lemmas before stating our main results.

Definition 2.1. [7] For a continuous function $y : (0,\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral of order α is defined as

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \quad \alpha > 0,$$

where Γ is the Gamma function.

Definition 2.2. [7] The Riemann-Liouville fractional derivative of order α for a continuous function $y(t)$ is defined by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)ds, \quad \alpha > 0,$$

where Γ is the Gamma function, n is the smallest integer greater than or equal to α.

Let $\gamma \in [1, n-2]$. Then we have

$$\begin{aligned} D_{0+}^{\gamma} y(t) &= D^m I_{0+}^{m-\gamma} y(t) \\ &= D^m D^{n-m} I_{0+}^{n-m} I_{0+}^{m-\gamma} y(t) \\ &= D^n I_{0+}^{n-\gamma} y(t) \\ &= D^n I_{0+}^{n-\alpha} I_{0+}^{\alpha-\gamma} y(t) \\ &= D_{0+}^{\alpha} I_{0+}^{\alpha-\gamma} y(t), \end{aligned}$$

where m is the smallest integer greater than or equal to γ. This implies that $D_{0+}^{\gamma} y(t)$ exists if $D_{0+}^{\alpha} y(t)$ exists.

Let $AC[0,1]$ be the space of functions f which are absolutely continuous on $[0,1]$. For $n \in \mathbb{N} := 1, 2, 3, \cdots$, we denote by $AC^n[0,1]$ the space of functions f which have continuous derivatives up to order $n-1$ on $[0,1]$ such that $f^{(n-1)} \in AC[0,1]$.

Definition 2.3. [7] If $y(t) \in AC^n[0,1]$, then the Caputo fractional derivative of order α for $y(t)$ is defined by

$$ {}^C D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)ds, \;\; \alpha > 0, $$

where Γ is the Gamma function, n is the smallest integer greater than or equal to α.

Throughout, we assume that the following conditions are satisfied.

(H1) $a_1 a_2 < \left(\frac{1-e^{-k}}{\sigma - \frac{1}{k}(1-e^{-k\sigma})} \right)^2$.

(H2) $\mathcal{A}, \mathcal{B} : [0,1] \to \mathbb{R}$ are nondecreasing functions,

$$ 0 \le \int_0^1 d\mathcal{A}(t) < 1, \quad 0 \le \int_0^1 d\mathcal{B}(t) < 1, $$

$$ \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1} < \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \left(1 - \eta_0 \int_0^1 t^{\alpha-\gamma-1} d\mathcal{A}(t) \right), $$

$$ \sum_{j=1}^{\infty} \varepsilon_j \xi_j^{\alpha-1} < \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \left(1 - \varepsilon_0 \int_0^1 t^{\alpha-\gamma-1} d\mathcal{B}(t) \right). $$

Lemma 2.4. [8] *Let $\alpha \in (0,1)$. Then the following assertions hold:*
(i) I_{0+}^{α} maps $AC[0,1]$ to $AC[0,1]$;
(ii) For each $f \in AC[0,1]$, $I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t)$ for a.e. $t \in [0,1]$.

Lemma 2.5. *For $1 < \beta \le 2$ and $h_1, h_2 \in AC[0,1]$, the solution of the linear system of fractional differential equations*

$$ \begin{cases} {}^C D_{0+}^{\beta-1}(D+k)x(t) = h_1(t), & \text{for a.e. } t \in [0,1], \\ \\ {}^C D_{0+}^{\beta-1}(D+k)y(t) = h_2(t), & \text{for a.e. } t \in [0,1], \end{cases} \tag{2.1} $$

supplemented with the boundary conditions

$$ \begin{cases} x(0) = 0, \;\; x(1) = a_1 \int_0^{\sigma} y(s)ds, \\ \\ y(0) = 0, \;\; y(1) = a_2 \int_0^{\sigma} x(s)ds, \end{cases} \tag{2.2} $$

is equivalent to functions x, y satisfy

$$ \begin{aligned} x(t) = & \frac{e^{-kt}}{\Gamma(\beta-1)} \int_0^t \int_0^s e^{ks}(s-\tau)^{\beta-2} h_1(\tau) d\tau ds \\ & + \frac{1-e^{-kt}}{k\Gamma(\beta-1)} \Big(\frac{a_1 A}{A^2 - A_1 A_2} \int_0^{\sigma} \int_0^t \int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2} h_2(\tau) d\tau ds dt \\ & + \frac{a_2 A_1}{A^2 - A_1 A_2} \int_0^{\sigma} \int_0^t \int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2} h_1(\tau) d\tau ds dt \\ & - \frac{A}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)}(s-\tau)^{\beta-2} h_1(\tau) d\tau ds \\ & - \frac{A_1}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)}(s-\tau)^{\beta-2} h_2(\tau) d\tau ds \Big), \end{aligned} \tag{2.3} $$

$$
\begin{aligned}
y(t) = & \frac{e^{-kt}}{\Gamma(\beta-1)} \int_0^t \int_0^s e^{ks}(s-\tau)^{\beta-2} h_2(\tau) d\tau ds \\
& + \frac{1-e^{-kt}}{k\Gamma(\beta-1)} \Big(\frac{a_1 A_2}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} h_2(\tau) d\tau ds dt \\
& + \frac{a_2 A}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} h_1(\tau) d\tau ds dt \\
& - \frac{A_2}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} h_1(\tau) d\tau ds \\
& - \frac{A}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} h_2(\tau) d\tau ds \Big),
\end{aligned} \tag{2.4}
$$

where

$$
A = \frac{1}{k}(1-e^{-k}), \quad A_1 = \frac{a_1}{k}(\sigma - \frac{1}{k} + \frac{1}{k}e^{-k\sigma}), \quad A_2 = \frac{a_2}{k}(\sigma - \frac{1}{k} + \frac{1}{k}e^{-k\sigma}).
$$

Proof. Solving the fractional differential equations (2.1) in a standard manner, we get

$$
x(t) = \frac{e^{-kt}}{\Gamma(\beta-1)} \int_0^t \int_0^s e^{ks}(s-\tau)^{\beta-2} h_1(\tau) d\tau ds + c_1 \frac{1-e^{-kt}}{k},
$$

$$
y(t) = \frac{e^{-kt}}{\Gamma(\beta-1)} \int_0^t \int_0^s e^{ks}(s-\tau)^{\beta-2} h_2(\tau) d\tau ds + c_2 \frac{1-e^{-kt}}{k},
$$

for some $c_1, c_2 \in \mathbb{R}$. Using the boundary conditions (2.2), together with (H1), we obtain

$$
\begin{aligned}
c_1 = & \frac{1}{\Gamma(\beta-1)} \Big(\frac{a_1 A}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} h_2(\tau) d\tau ds dt \\
& + \frac{a_2 A_1}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} h_1(\tau) d\tau ds dt \\
& - \frac{A}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} h_1(\tau) d\tau ds \\
& - \frac{A_1}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} h_2(\tau) d\tau ds \Big),
\end{aligned}
$$

$$
\begin{aligned}
c_2 = & \frac{1}{\Gamma(\beta-1)} \Big(\frac{a_1 A_2}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} h_2(\tau) d\tau ds dt \\
& + \frac{a_2 A}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} h_1(\tau) d\tau ds dt \\
& - \frac{A_2}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} h_1(\tau) d\tau ds \\
& - \frac{A}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} h_2(\tau) d\tau ds \Big).
\end{aligned}
$$

Consequently, $x(t),\ y(t)$ satisfy (2.3) and (2.4).

Conversely,

$$x(t) = \frac{e^{-kt}}{\Gamma(\beta-1)} \int_0^t \int_0^s e^{ks}(s-\tau)^{\beta-2} h_1(\tau) d\tau ds + c_1 \frac{1-e^{-kt}}{k},$$

$$y(t) = \frac{e^{-kt}}{\Gamma(\beta-1)} \int_0^t \int_0^s e^{ks}(s-\tau)^{\beta-2} h_2(\tau) d\tau ds + c_2 \frac{1-e^{-kt}}{k},$$

where $c_1,\ c_2$ are given by above equalities. Hence,

$$(D+k)x(t) = I_{0+}^{\beta-1} h_1(t) + c_1,\ t \in [0,1],$$

$$(D+k)y(t) = I_{0+}^{\beta-1} h_2(t) + c_2,\ t \in [0,1].$$

First, we consider $1 < \beta < 2$. Since $h_1, h_2 \in AC[0,1]$, by Lemma 2.4 (i), we have $I_{0+}^{\beta-1} h_j(t) \in AC[0,1],\ j = 1,2$. Thus, ${}^C D_{0+}^{\beta-1}(D+k)x(t)$ and ${}^C D_{0+}^{\beta-1}(D+k)y(t)$ exist. Moreover, by Lemma 2.4 (ii), we can get

$${}^C D_{0+}^{\beta-1} I_{0+}^{\beta-1} h_j(t) = I_{0+}^{1-(\beta-1)} D I_{0+}^{\beta-1} h_j(t) = I_{0+}^{2-\beta} D_{0+}^{2-\beta} h_j(t) = h_j(t),$$

for a.e. $t \in [0,1]$, $j = 1,2$.

In what follows we consider $\beta = 2$. Then $I_{0+}^{\beta-1} h_j(t) = I_{0+}^{1} h_j(t) \in AC[0,1],\ j = 1,2$ and ${}^C D_{0+}^{\beta-1} I_{0+}^{\beta-1} h_j(t) = {}^C D_{0+}^{1} I_{0+}^{1} h_j(t) = h_j(t),\ t \in [0,1]$. Consequently, (2.1) holds for $1 < \beta \leq 2$. By simple computation, it is easy to see that $x(t),\ y(t)$ satisfy (2.2). Thus, $x(t),\ y(t)$ is a solution of the problem (2.1)–(2.2). The proof is completed.

Remark 2.6. As mentioned in [8], $h_1, h_2 \in C[0,1]$ are not sufficient in the study of the equivalence between (2.1)–(2.2) and (2.3)–(2.4). We can prove the solutions of (2.1)–(2.2) satisfy (2.3)–(2.4) under the condition $h_1, h_2 \in C[0,1]$. However, we can not prove that if $h_1, h_2 \in C[0,1]$, then functions $x(t),\ y(t)$ satisfy (2.3)–(2.4) is a solution of the problem (2.1)–(2.2).

Lemma 2.7. *Let $n-1 < \alpha \leq n$ and $h \in C[0,1]$. The boundary value problem*

$$D_{0+}^{\alpha} u(t) = h(t), \quad \textit{for a.e. } t \in [0,1], \tag{2.5}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \tag{2.6}$$

$$D^{\gamma} u(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j) + \eta_0 \int_0^1 D^{\gamma} u(\tau) d\mathcal{A}(\tau), \tag{2.7}$$

has a unique solution

$$u(t) = -\int_0^1 G(t,s) h(s) ds,$$

where

$$G(t,s) = \frac{t^{\alpha-1}\eta_0}{\Lambda\Gamma(\alpha-\gamma)} \int_0^1 G_1(\tau,s) d\mathcal{A}(\tau) + G_2(t,s),$$

$$G_1(t,s) = \begin{cases} -(t-s)^{\alpha-\gamma-1} + (1-s)^{\alpha-\gamma-1} t^{\alpha-\gamma-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-\gamma-1} t^{\alpha-\gamma-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.8}$$

$$G_2(t,s)=\begin{cases}\frac{\rho(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-\rho(0)(t-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)}, & 0\le s\le t\le 1,\\ \\ \frac{\rho(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)}, & 0\le t\le s\le 1,\end{cases} \tag{2.9}$$

$$\rho(s)=\Gamma(\alpha)\Big(1-\eta_0\int_0^1 t^{\alpha-\gamma-1}d\mathcal{A}(t)\Big)-\Gamma(\alpha-\gamma)\sum_{s\le\xi_j}\eta_j(\frac{\xi_j-s}{1-s})^{\alpha-1}(1-s)^{\gamma},$$

$$\Lambda=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}\Big(1-\eta_0\int_0^1 t^{\alpha-\gamma-1}d\mathcal{A}(t)\Big)-\sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1}.$$

Proof. Applying the operator I_{0+}^{α} on both sides of (2.5), we have

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

for some $c_1,c_2,\cdots,c_n\in\mathbb{R}$. From (2.6), (2.7) it follows $c_2=\cdots=c_{n-1}=c_n=0$, and

$$c_1=\sum_{j=1}^{\infty}\frac{\eta_j}{\Lambda\Gamma(\alpha)}\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}h(s)ds+\frac{\eta_0}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1\int_s^1(t-s)^{\alpha-\gamma-1}d\mathcal{A}(t)h(s)ds$$
$$-\frac{1}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1(1-s)^{\alpha-\gamma-1}h(s)ds.$$

Consequently, $u(t)$ satisfies

$$u(t)=-\int_0^1\Big(\frac{t^{\alpha-1}\eta_0}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1 G_1(\tau,s)d\mathcal{A}(\tau)+G_2(t,s)\Big)h(s)ds=-\int_0^1 G(t,s)h(s)ds.$$

Conversely, if $u(t)=-\int_0^1 G(t,s)h(s)ds$, then

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_2=\cdots=c_{n-1}=c_n=0$, and

$$\begin{aligned}c_1=&\frac{1}{\Lambda}\sum_{j=1}^{\infty}\frac{\eta_j}{\Gamma(\alpha)}\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}h(s)ds\\&+\frac{\eta_0}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1\left(\int_s^1(t-s)^{\alpha-\gamma-1}d\mathcal{A}(t)\right)h(s)ds\\&-\frac{1}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1(1-s)^{\alpha-\gamma-1}h(s)ds.\end{aligned}$$

Hence,

$$D_{0+}^{\alpha}u(t)=h(t).$$

Moreover, $u(t)$ satisfies (2.6), (2.7). Thus, $u(t)$ is a solution of the problem (2.5)–(2.7). The proof is completed.

Lemma 2.8. *Let $n-1<\alpha\le n$ and $h\in C[0,1]$. The boundary value problem*

$$D_{0+}^{\alpha}v(t)=h(t),\quad \textit{for a.e. } t\in[0,1],$$

$$v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0,$$
$$D^\gamma v(1) = \sum_{j=1}^{\infty} \varepsilon_j v(\xi_j) + \varepsilon_0 \int_0^1 D^\gamma v(\tau) d\mathcal{B}(\tau),$$

has a unique solution

$$v(t) = -\int_0^1 H(t,s)h(s)ds,$$

where

$$H(t,s) = \frac{t^{\alpha-1}\varepsilon_0}{\Lambda_1 \Gamma(\alpha-\gamma)} \int_0^1 G_1(\tau,s) d\mathcal{B}(\tau) + G_3(t,s),$$

$G_1(t,s)$ *is given by* (2.8),

$$G_3(t,s) = \begin{cases} \frac{\rho_1(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - \rho_1(0)(t-s)^{\alpha-1}}{\Lambda_1\Gamma(\alpha)\Gamma(\alpha-\gamma)}, & 0 \le s \le t \le 1, \\ \frac{\rho_1(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Lambda_1\Gamma(\alpha)\Gamma(\alpha-\gamma)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.10}$$

$$\rho_1(s) = \Gamma(\alpha)\left(1 - \varepsilon_0 \int_0^1 t^{\alpha-\gamma-1} d\mathcal{B}(t)\right) - \Gamma(\alpha-\gamma) \sum_{s \le \xi_j} \varepsilon_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^\gamma,$$

$$\Lambda_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \left(1 - \varepsilon_0 \int_0^1 t^{\alpha-\gamma-1} d\mathcal{B}(t)\right) - \sum_{j=1}^{\infty} \varepsilon_j \xi_j^{\alpha-1}.$$

Proof. The proof is the same as that of Lemma 2.7, so it is omitted.

Lemma 2.9. *Suppose that (H2) holds, and* $n-1 < \alpha \le n$. *Function* $G_2(t,s)$, $G_3(t,s)$ *satisfy the following conditions*
(i) $G_i(t,s) \ge 0$, $\frac{\partial}{\partial t} G_i(t,s) \ge 0$, *for* $t,s \in (0,1)$, $i = 2,3$.
(ii) $\max_{0 \le t \le 1} G_i(t,s) = G_i(1,s)$, *for* $0 \le s \le 1$, $i = 2,3$.
(iii) $G_i(t,s) \ge t^{\alpha-1} G_i(1,s)$, $0 \le t,s \le 1$, $i = 2,3$.

Proof. First, we prove the properties of $G_2(t,s)$. From (H2), it is easy to see that

$$\rho'(s) = \Gamma(\alpha-\gamma) \sum_{s \le \xi_j} \eta_j (\xi_j - s)^{\alpha-2} (1-s)^{-\alpha+\gamma} ((1-\xi_j)(\alpha-1) + \gamma(\xi_j - s)) > 0.$$

Thus, ρ is increasing on $[0,1]$. Since

$$\rho(0) = \Gamma(\alpha)(1 - \eta_0 \int_0^1 t^{\alpha-\gamma-1} d\mathcal{A}(t)) - \Gamma(\alpha-\gamma) \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1} > 0,$$

we deduce $\rho(s) \ge \rho(0) > 0$, $s \in [0,1]$. By simple computation, we can get $G_2(t,s) \ge 0$, for $t,s \in (0,1)$. Note that

$$\frac{\partial}{\partial t} G_2(t,s) = \begin{cases} \frac{(\alpha-1)\rho(s)t^{\alpha-2}(1-s)^{\alpha-\gamma-1} - \rho(0)(\alpha-1)(t-s)^{\alpha-2}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)}, & 0 \le s \le t \le 1, \\ \frac{(\alpha-1)\rho(s)t^{\alpha-2}(1-s)^{\alpha-\gamma-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)}, & 0 \le t \le s \le 1, \end{cases}$$

we have $\frac{\partial}{\partial t}G_2(t,s) \geq 0$, for $t, s \in (0,1)$. Then $G_2(t,s)$ is nondecreasing with respect to t, $\max\limits_{0\leq t\leq 1} G_2(t,s) = G_2(1,s)$, for $0 \leq s \leq 1$. In the following, we will prove (iii). For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} G_2(t,s) &= \frac{\rho(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - \rho(0)(t-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)} \\ &= \frac{\rho(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - \rho(0)t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)} \\ &\geq \frac{\rho(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - \rho(0)t^{\alpha-1}(1-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)} \\ &= t^{\alpha-1}G_2(1,s). \end{aligned}$$

For $0 \leq t \leq s \leq 1$, it is clear that $G_2(t,s) \geq t^{\alpha-1}G_2(1,s)$. Now, we are in the position to prove the properties of $G_3(t,s)$. Repeating arguments similar to that above we can get (i)–(iii). The proof is completed.

Let $(X, \|\cdot\|)$ be a Banach space with a cone K in X, and let $\leq$ denote the partial order defined by K, that is $x \leq y$ if and only if $y - x \in K$. A cone is called normal if there exists $M > 0$ such that, for all $0 \leq x \leq y$, it follows that $\|x\| \leq M\|y\|$.

Let $C[0,1]$ with the norm $\|u\| = \max\limits_{0\leq t\leq 1} |u(t)|$. Then $C[0,1]$ is a Banach space. Denote

$$P = \{u \in C[0,1] : u(t) \geq 0,\ t \in [0,1]\}.$$

Obviously, P is a positive cone in $C[0,1]$. Thus $E = C[0,1] \times C[0,1]$ is a Banach space with the norm $\|(u,v)\|_1 = \max\{\|u\|, \|v\|\}$, and $P_1 = P \times P$ is a cone in E.

Definition 2.10. Let P be a cone in real Banach space E and $e \in P \setminus \{\theta\}$, set

$$E_e = \{x \in E : \exists \lambda > 0,\ -\lambda e \leq x \leq \lambda e, \}$$

and

$$\|x\|_e = \inf\{\lambda > 0 : -\lambda e \leq x \leq \lambda e, \},\ \forall\ x \in E_e.$$

It is easy to see that E_e becomes a normed linear space under the norm $\|\cdot\|_e$. $\|x\|_e$ is called the e-norm of the element $x \in E_e$.

Lemma 2.11. [5] *Let cone P be normal. Then E_e is a Banach space with e-norm, and there exists a constant $m > 0$ such that $\|x\| \leq m\|x\|_e$, $\forall\ x \in E_e$.*

3. Main results

Now we are in the position to establish the main results.

In the forthcoming analysis, we need the following assumption:

(H3) There exists $l_0, l_1, l_2, l_3, l_4, l_5 \in [0, +\infty)$ such that

$$|f(t_1,u_1,v_1) - f(t_2,u_2,v_2)| \leq l_0|t_1-t_2| + l_1|u_1-u_2| + l_2|v_1-v_2|,$$

$$|g(t_1,u_1,v_1) - g(t_2,u_2,v_2)| \leq l_3|t_1-t_2| + l_4|u_1-u_2| + l_5|v_1-v_2|,$$

for any $t_1, t_2 \in [0,1], u_1, u_2, v_1, v_2 \in \mathbb{R}$.

In view of Lemma 2.5, Lemma 2.7 and Lemma 2.8, we define an operator F as

$$F(u,v)(t) = (F_1(u,v)(t), F_2(u,v)(t)),\quad (u,v) \in E,$$

where $F_1, F_2 : E \to C[0,1]$ are defined by

$$F_1(u,v)(t) = -\int_0^1 G(t,s)\Big[\frac{e^{-ks}}{\Gamma(\beta-1)}\int_0^s\int_0^m e^{km}(m-\tau)^{\beta-2}f(\tau,u(\tau),v(\tau))d\tau dm$$
$$+\frac{1-e^{-ks}}{k\Gamma(\beta-1)}\Big(\frac{a_1A}{A^2-A_1A_2}\int_0^\sigma\int_0^t\int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2}g(\tau,u(\tau),v(\tau))d\tau dsdt$$
$$+\frac{a_2A_1}{A^2-A_1A_2}\int_0^\sigma\int_0^t\int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2}f(\tau,u(\tau),v(\tau))d\tau dsdt$$
$$-\frac{A}{A^2-A_1A_2}\int_0^1\int_0^s e^{-k(1-s)}(s-\tau)^{\beta-2}f(\tau,u(\tau),v(\tau))d\tau ds$$
$$-\frac{A_1}{A^2-A_1A_2}\int_0^1\int_0^s e^{-k(1-s)}(s-\tau)^{\beta-2}g(\tau,u(\tau),v(\tau))d\tau ds\Big)\Big]ds,$$

$$F_2(u,v)(t) = -\int_0^1 H(t,s)\Big[\frac{e^{-ks}}{\Gamma(\beta-1)}\int_0^s\int_0^m e^{km}(m-\tau)^{\beta-2}g(\tau,u(\tau),v(\tau))d\tau dm$$
$$+\frac{1-e^{-ks}}{k\Gamma(\beta-1)}\Big(\frac{a_1A_2}{A^2-A_1A_2}\int_0^\sigma\int_0^t\int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2}g(\tau,u(\tau),v(\tau))d\tau dsdt$$
$$+\frac{a_2A}{A^2-A_1A_2}\int_0^\sigma\int_0^t\int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2}f(\tau,u(\tau),v(\tau))d\tau dsdt$$
$$-\frac{A_2}{A^2-A_1A_2}\int_0^1\int_0^s e^{-k(1-s)}(s-\tau)^{\beta-2}f(\tau,u(\tau),v(\tau))d\tau ds$$
$$-\frac{A}{A^2-A_1A_2}\int_0^1\int_0^s e^{-k(1-s)}(s-\tau)^{\beta-2}g(\tau,u(\tau),v(\tau))d\tau ds\Big)\Big]ds.$$

Lemma 3.1. *Suppose that (H1)–(H3) hold, then the fixed points of the operator F are solutions of the problem* (1.1)–(1.2).

Proof. Assume that $(u,v) \in E$ is a fixed point of F. Then

$$u(t) = -\int_0^1 G(t,s)T_1(s)ds,$$

$$v(t) = -\int_0^1 H(t,s)T_2(s)ds,$$

where

$$T_1(t) = \frac{e^{-kt}}{\Gamma(\beta-1)}\int_0^t\int_0^m e^{km}(m-\tau)^{\beta-2}f(\tau,u(\tau),v(\tau))d\tau dm + \frac{1-e^{-kt}}{k\Gamma(\beta-1)}c_1,$$

$$T_2(t) = \frac{e^{-kt}}{\Gamma(\beta-1)}\int_0^t\int_0^m e^{km}(m-\tau)^{\beta-2}g(\tau,u(\tau),v(\tau))d\tau dm + \frac{1-e^{-kt}}{k\Gamma(\beta-1)}c_2,$$

and

$$
\begin{aligned}
c_1 = & \frac{a_1 A}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} g(\tau, u(\tau), v(\tau)) d\tau ds dt \\
& + \frac{a_2 A_1}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} f(\tau, u(\tau), v(\tau)) d\tau ds dt \\
& - \frac{A}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} f(\tau, u(\tau), v(\tau)) d\tau ds \\
& - \frac{A_1}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} g(\tau, u(\tau), v(\tau)) d\tau ds,
\end{aligned}
$$

$$
\begin{aligned}
c_2 = & \frac{a_1 A_2}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} g(\tau, u(\tau), v(\tau)) d\tau ds dt \\
& + \frac{a_2 A}{A^2 - A_1 A_2} \int_0^\sigma \int_0^t \int_0^s e^{-k(t-s)} (s-\tau)^{\beta-2} f(\tau, u(\tau), v(\tau)) d\tau ds dt \\
& - \frac{A_2}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} f(\tau, u(\tau), v(\tau)) d\tau ds \\
& - \frac{A}{A^2 - A_1 A_2} \int_0^1 \int_0^s e^{-k(1-s)} (s-\tau)^{\beta-2} g(\tau, u(\tau), v(\tau)) d\tau ds.
\end{aligned}
$$

Since f, g are continuous, we have $T_1(t), T_2(t) \in C[0,1]$. By Lemma 2.7 and 2.8, we can get $D_{0+}^\alpha u(t) = T_1(t)$, $D_{0+}^\alpha v(t) = T_2(t)$, and

$$
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,
$$

$$
v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0,
$$

$$
D^\gamma u(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j) + \eta_0 \int_0^1 D^\gamma u(\tau) d\mathcal{A}(\tau),
$$

$$
D^\gamma v(1) = \sum_{j=1}^{\infty} \varepsilon_j v(\xi_j) + \varepsilon_0 \int_0^1 D^\gamma v(\tau) d\mathcal{B}(\tau).
$$

Moreover, $u'(t), v'(t)$ exist for $t \in [0,1]$ and $\max\{|u'(t)|\}, \max\{|v'(t)|\} < \infty, t \in [0,1]$. This, together with the condition (H3), implies that for $(u, v) \in E$ and $t_1, t_2 \in [0,1]$,

$$
\begin{aligned}
& |f(t_1, u(t_1), v(t_1)) - f(t_2, u(t_2), v(t_2))| \\
& \le l_0 |t_1 - t_2| + l_1 |u(t_1) - u(t_2)| + l_2 |v(t_1) - v(t_2)| \\
& \le \Big(l_0 + l_1 \max\{|u'(t)|\} + l_2 \max\{|v'(t)|\} \Big) |t_1 - t_2|,
\end{aligned}
$$

and

$$
\begin{aligned}
& |g(t_1, u(t_1), v(t_1)) - g(t_2, u(t_2), v(t_2))| \\
& \le l_3 |t_1 - t_2| + l_4 |u(t_1) - u(t_2)| + l_5 |v(t_1) - v(t_2)| \\
& \le \Big(l_3 + l_4 \max\{|u'(t)|\} + l_5 \max\{|v'(t)|\} \Big) |t_1 - t_2|
\end{aligned}
$$

hold. It follows that $f(\cdot,u(\cdot),v(\cdot)), g(\cdot,u(\cdot),v(\cdot)) \in AC[0,1]$. By Lemma 2.5, we have

$$\begin{cases} {}^C D_{0+}^{\beta-1}(D+k)T_1(t) = f(t,u(t),v(t)), & \text{for } a.e.\ t\in[0,1],\\ {}^C D_{0+}^{\beta-1}(D+k)T_2(t) = g(t,u(t),v(t)), & \text{for } a.e.\ t\in[0,1], \end{cases}$$

and

$$\begin{cases} T_1(0)=0, \ \ T_1(1) = a_1\int_0^\sigma T_2(s)ds,\\ T_2(0)=0, \ \ T_2(1) = a_2\int_0^\sigma T_1(s)ds. \end{cases}$$

Hence, $(u,v)\in E$ is a solution of the problem (1.1)–(1.2).

Theorem 3.2. *Suppose that (H1)–(H3) hold. Then the problem* (1.1)–(1.2) *has a unique solution if*

$$\max\{M_3, M_4\}l < 1,$$

where

$$l = \max\{l_1,l_2,l_4,l_5\},\ M_1 = \max_{t\in[0,1]}|f(t,0,0)|,\ M_2 = \max_{t\in[0,1]}|g(t,0,0)|,$$

and

$$M_3 = \frac{2\Gamma(\alpha)}{\Lambda\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta+1)}\Big(e^k + \frac{a_1A\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)} + \frac{a_2A_1\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)} + \frac{A(\alpha+\beta)}{k(A^2-A_1A_2)} + \frac{A_1(\alpha+\beta)}{k(A^2-A_1A_2)}\Big),$$

$$M_4 = \frac{2\Gamma(\alpha)}{\Lambda_1\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta+1)}\Big(e^k + \frac{a_1A_2\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)} + \frac{a_2A\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)} + \frac{A_2(\alpha+\beta)}{k(A^2-A_1A_2)} + \frac{A(\alpha+\beta)}{k(A^2-A_1A_2)}\Big).$$

Proof. From Lemma 2.7 and Lemma 2.9, we can get

$$\begin{aligned} 0 &\le t^{\alpha-1}G_2(1,s) \le G(t,s) = \frac{t^{\alpha-1}\eta_0}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1 G_1(\tau,s)d\mathcal{A}(\tau) + G_2(t,s)\\ &\le \frac{t^{\alpha-1}\eta_0}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}t^{\alpha-\gamma-1}d\mathcal{A}(t) + \frac{\rho(s)t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Lambda\Gamma(\alpha)\Gamma(\alpha-\gamma)}\\ &\le \frac{t^{\alpha-1}\eta_0}{\Lambda\Gamma(\alpha-\gamma)}\int_0^1 t^{\alpha-\gamma-1}d\mathcal{A}(t) + \frac{\Big(1-\eta_0\int_0^1 t^{\alpha-\gamma-1}d\mathcal{A}(t)\Big)t^{\alpha-1}}{\Lambda\Gamma(\alpha-\gamma)} = \frac{t^{\alpha-1}}{\Lambda\Gamma(\alpha-\gamma)}. \end{aligned}$$

In the same way, from Lemma 2.8 and Lemma 2.9, we have

$$0\le t^{\alpha-1}G_3(1,s) \le H(t,s) \le \frac{t^{\alpha-1}}{\Lambda_1\Gamma(\alpha-\gamma)}.$$

For $(u,v)\in E$, by (H3) and the above two inequalities, we obtain

$$
\begin{aligned}
&|F_1(u,v)(t)|\\
&\leq \int_0^1 \frac{G(t,s)}{k\Gamma(\beta-1)}\Big(\int_0^s\int_0^m ke^{km}(m-\tau)^{\beta-2}(|f(\tau,u(\tau),v(\tau))-f(\tau,0,0)|+M_1)d\tau dm\\
&+\frac{a_1A}{A^2-A_1A_2}\int_0^\sigma\int_0^t\int_0^s (s-\tau)^{\beta-2}(|g(\tau,u(\tau),v(\tau))-g(\tau,0,0)|+M_2)d\tau dsdt\\
&+\frac{a_2A_1}{A^2-A_1A_2}\int_0^\sigma\int_0^t\int_0^s (s-\tau)^{\beta-2}(|f(\tau,u(\tau),v(\tau))-f(\tau,0,0)|+M_1)d\tau dsdt\\
&+\frac{A}{A^2-A_1A_2}\int_0^1\int_0^s (s-\tau)^{\beta-2}(|f(\tau,u(\tau),v(\tau))-f(\tau,0,0)|+M_1)d\tau ds\\
&+\frac{A_1}{A^2-A_1A_2}\int_0^1\int_0^s (s-\tau)^{\beta-2}(|g(\tau,u(\tau),v(\tau))-g(\tau,0,0)|+M_2)d\tau ds\Big)ds\\
&\leq \frac{t^{\alpha-1}}{\Lambda\Gamma(\alpha-\gamma)}\Big[\Big(\frac{e^k}{\Gamma(\beta+2)}+\frac{a_1A\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}+\frac{a_2A_1\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}\\
&+\frac{A}{k\Gamma(\beta+1)(A^2-A_1A_2)}+\frac{A_1}{k\Gamma(\beta+1)(A^2-A_1A_2)}\Big)l(\|u\|+\|v\|)\\
&+\Big(\frac{e^k}{\Gamma(\beta+2)}+\frac{a_2A_1\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}+\frac{A}{k\Gamma(\beta+1)(A^2-A_1A_2)}\Big)M_1\\
&+\Big(\frac{a_1A\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}+\frac{A_1}{k\Gamma(\beta+1)(A^2-A_1A_2)}\Big)M_2\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&|F_2(u,v)(t)|\\
&\leq \frac{t^{\alpha-1}}{\Lambda_1\Gamma(\alpha-\gamma)}\Big[\Big(\frac{e^k}{\Gamma(\beta+2)}+\frac{a_1A_2\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}+\frac{a_2A\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}\\
&+\frac{A_2}{k\Gamma(\beta+1)(A^2-A_1A_2)}+\frac{A}{k\Gamma(\beta+1)(A^2-A_1A_2)}\Big)l(\|u\|+\|v\|)\\
&+\Big(\frac{e^k}{\Gamma(\beta+2)}+\frac{a_1A_2\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}+\frac{A}{k\Gamma(\beta+1)(A^2-A_1A_2)}\Big)M_2\\
&+\Big(\frac{a_2A\sigma^{\beta+1}}{k\Gamma(\beta+2)(A^2-A_1A_2)}+\frac{A_2}{k\Gamma(\beta+1)(A^2-A_1A_2)}\Big)M_1\Big].
\end{aligned}
$$

Consequently, F maps all of E into the following vector subspace

$$E_e=\{(x,y)\in E:\exists\lambda>0,\ -\lambda e(t)\leq(x(t),y(t))\leq\lambda e(t),\ t\in[0,1]\},$$

where $e(t)=(t^{\alpha-1},t^{\alpha-1})$. In view of Definition 2.10 and Lemma 2.11, E_e is a subspace of E and E_e is a Banach space with the norm

$$\|(x,y)\|_e=\inf\{\lambda>0:-\lambda e(t)\leq(x(t),y(t))\leq\lambda e(t),\ t\in[0,1]\}.$$

Hence, it suffices to consider the fixed point of F in E_e. Let $(u_1, v_1), (u_2, v_2) \in E_e$. Then, we have

$$\begin{aligned}
&|F_1(u_1,v_1)(t) - F_1(u_2,v_2)(t)| \\
&\leq \int_0^1 \frac{G(t,s)}{\Gamma(\beta-1)}\Big(\int_0^s\int_0^m e^k(m-\tau)^{\beta-2}(l_1|u_1(\tau)-u_2(\tau)|+l_2|v_1(\tau)-v_2(\tau)|)d\tau dm \\
&+\frac{a_1A}{k(A^2-A_1A_2)}\int_0^\sigma\int_0^t\int_0^s(s-\tau)^{\beta-2}(l_4|u_1(\tau)-u_2(\tau)|+l_5|v_1(\tau)-v_2(\tau)|)d\tau dsdt \\
&+\frac{a_2A_1}{k(A^2-A_1A_2)}\int_0^\sigma\int_0^t\int_0^s(s-\tau)^{\beta-2}(l_1|u_1(\tau)-u_2(\tau)|+l_2|v_1(\tau)-v_2(\tau)|)d\tau dsdt \\
&+\frac{A}{k(A^2-A_1A_2)}\int_0^1\int_0^s(s-\tau)^{\beta-2}(l_1|u_1(\tau)-u_2(\tau)|+l_2|v_1(\tau)-v_2(\tau)|)d\tau ds \\
&+\frac{A_1}{k(A^2-A_1A_2)}\int_0^1\int_0^s(s-\tau)^{\beta-2}(l_4|u_1(\tau)-u_2(\tau)|+l_5|v_1(\tau)-v_2(\tau)|)d\tau ds\Big)ds \\
&\leq \int_0^1 \frac{G(t,s)}{\Gamma(\beta-1)}\Big(\int_0^s\int_0^m e^k(m-\tau)^{\beta-2}(l_1+l_2)\|(u_1,v_1)-(u_2,v_2)\|_e\tau^{\alpha-1}d\tau dm \\
&+\frac{a_1A}{k(A^2-A_1A_2)}\int_0^\sigma\int_0^t\int_0^s(s-\tau)^{\beta-2}(l_4+l_5)\|(u_1,v_1)-(u_2,v_2)\|_e\tau^{\alpha-1}d\tau dsdt \\
&+\frac{a_2A_1}{k(A^2-A_1A_2)}\int_0^\sigma\int_0^t\int_0^s(s-\tau)^{\beta-2}(l_1+l_2)\|(u_1,v_1)-(u_2,v_2)\|_e\tau^{\alpha-1}d\tau dsdt \\
&+\frac{A}{k(A^2-A_1A_2)}\int_0^1\int_0^s(s-\tau)^{\beta-2}(l_1+l_2)\|(u_1,v_1)-(u_2,v_2)\|_e\tau^{\alpha-1}d\tau ds \\
&+\frac{A_1}{k(A^2-A_1A_2)}\int_0^1\int_0^s(s-\tau)^{\beta-2}(l_4+l_5)\|(u_1,v_1)-(u_2,v_2)\|_e\tau^{\alpha-1}d\tau ds\Big)ds \\
&\leq \frac{t^{\alpha-1}\Gamma(\alpha)}{\Lambda\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta+1)}\Big(e^k+\frac{a_1A\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)}+\frac{a_2A_1\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)} \\
&+\frac{A(\alpha+\beta)}{k(A^2-A_1A_2)}+\frac{A_1(\alpha+\beta)}{k(A^2-A_1A_2)}\Big)2l\|(u_1,v_1)-(u_2,v_2)\|_e \\
&= M_3l\|(u_1,v_1)-(u_2,v_2)\|_e t^{\alpha-1},
\end{aligned}$$

and

$$\begin{aligned}
&|F_2(u_1,v_1)(t) - F_2(u_2,v_2)(t)| \\
&\leq \frac{t^{\alpha-1}\Gamma(\alpha)}{\Lambda_1\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta+1)}\Big(e^k+\frac{a_1A_2\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)}+\frac{a_2A\sigma^{\alpha+\beta}}{k(A^2-A_1A_2)} \\
&+\frac{A_2(\alpha+\beta)}{k(A^2-A_1A_2)}+\frac{A(\alpha+\beta)}{k(A^2-A_1A_2)}\Big)2l\|(u_1,v_1)-(u_2,v_2)\|_e \\
&= M_4l\|(u_1,v_1)-(u_2,v_2)\|_e t^{\alpha-1}.
\end{aligned}$$

The above two inequalities imply that

$$\|F(u_1,v_1)-F(u_2,v_2)\|_e \leq \max\{M_3,M_4\}l\|(u_1,v_1)-(u_2,v_2)\|_e.$$

Notice that $\max\{M_3, M_4\}l < 1$, the operator F is a contraction. Hence, F has a unique fixed point (u, v) in E_e which is a solution of the boundary value problem (1.1)–(1.2).

We now prove the uniqueness of solution of (1.1)–(1.2) in E_e. Suppose that $(\widetilde{u}, \widetilde{v}) \in E_e$ is another solution and let $D^{\alpha}_{0+}\widetilde{u}(t) = x(t)$, $D^{\alpha}_{0+}\widetilde{v}(t) = y(t)$. Then we have

$$\begin{cases} {}^C D^{\beta-1}_{0+}(D + k)x(t) = f(t, \widetilde{u}(t), \widetilde{v}(t)), & \text{for } a.e.\ t \in [0, 1], \\ {}^C D^{\beta-1}_{0+}(D + k)y(t) = g(t, \widetilde{u}(t), \widetilde{v}(t)), & \text{for } a.e.\ t \in [0, 1], \end{cases}$$

and

$$\begin{cases} x(0) = 0,\ x(1) = a_1 \int_0^{\sigma} y(s)ds, \\ y(0) = 0,\ y(1) = a_2 \int_0^{\sigma} x(s)ds. \end{cases}$$

Let

$$h_1(t) = f(t, \widetilde{u}(t), \widetilde{v}(t)), \quad h_2(t) = g(t, \widetilde{u}(t), \widetilde{v}(t)).$$

Then we obtain $h_1(t) \in C[0, 1]$, $h_2(t) \in C[0, 1]$. From Remark 2.6, we have x, y satisfy (2.3) and (2.4). It implies that $x(t), y(t) \in C[0, 1]$. Since $(\widetilde{u}, \widetilde{v})$ satisfy (1.1) and (1.2), we have

$$\widetilde{u}(0) = \widetilde{u}'(0) = \cdots = \widetilde{u}^{(n-2)}(0) = \widetilde{v}(0) = \widetilde{v}'(0) = \cdots = \widetilde{v}^{(n-2)}(0) = 0,$$

$$D^{\gamma}\widetilde{u}(1) = \sum_{j=1}^{\infty} \eta_j \widetilde{u}(\xi_j) + \eta_0 \int_0^1 D^{\gamma}\widetilde{u}(\tau)d\mathcal{A}(\tau),$$

$$D^{\gamma}\widetilde{v}(1) = \sum_{j=1}^{\infty} \varepsilon_j \widetilde{v}(\xi_j) + \varepsilon_0 \int_0^1 D^{\gamma}\widetilde{v}(\tau)d\mathcal{B}(\tau).$$

By Lemma 2.7 and 2.8, we can get

$$\widetilde{u}(t) = -\int_0^1 G(t, s)x(s)ds, \quad \widetilde{v}(t) = -\int_0^1 H(t, s)y(s)ds.$$

Hence, $(\widetilde{u}, \widetilde{v}) \in E_e$ is the fixed point of the operator F. Thus, (u, v) is the unique solution of the boundary value problem (1.1)–(1.2) if $\max\{M_3, M_4\}l < 1$. The proof is completed.

Remark 3.3. From the above argument, it may be favorable to consider the solution of the boundary value problem (1.1)–(1.2) in E_e. If we consider the problem in E, the result of Theorem 3.2 remains true except that the condition $\max\{M_3, M_4\}l < 1$ is replaced by $\max\{M_5, M_6\}l < 1$, where

$$\begin{aligned} M_5 = \frac{2}{\Lambda\Gamma(\alpha - \gamma)\Gamma(\beta + 2)} \Big(e^k + \frac{a_1 A \sigma^{\beta+1}}{k(A^2 - A_1A_2)} + \frac{a_2 A_1 \sigma^{\beta+1}}{k(A^2 - A_1A_2)} \\ + \frac{A(\beta + 1)}{k(A^2 - A_1A_2)} + \frac{A_1(\beta + 1)}{k(A^2 - A_1A_2)} \Big), \end{aligned}$$

$$M_6 = \frac{2}{\Lambda_1\Gamma(\alpha-\gamma)\Gamma(\beta+2)}\Big(e^k + \frac{a_1A_2\sigma^{\beta+1}}{k(A^2-A_1A_2)} + \frac{a_2A\sigma^{\beta+1}}{k(A^2-A_1A_2)} + \frac{A_2(\beta+1)}{k(A^2-A_1A_2)} + \frac{A(\beta+1)}{k(A^2-A_1A_2)}\Big).$$

In view of the conditions $\alpha > 2,\ 1 < \beta \le 2$ and $\sigma \in [0,1]$, we have $\sigma^{\alpha+\beta} \le \sigma^{\beta+1}$ and

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} < \frac{1}{(\alpha+\beta)(\alpha+\beta-1)} < \frac{1}{6} \le \frac{1}{\Gamma(\beta+2)},$$
$$\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} < \frac{1}{\alpha+\beta-1} < \frac{1}{2} \le \frac{1}{\Gamma(\beta+1)},$$

which yield $\max\{M_3, M_4\} < \max\{M_5, M_6\}$. Thus, Theorem 3.2 provides the same results with weaker condition.

4. Examples

In this section, we give an example to illustrate our main results.

Example 4.1. Consider

$$\begin{cases} {}^CD^{0.5}_{0+}(D+1)D^{3.5}_{0+}u(t) = \frac{1}{30\sqrt{900+t}}(\frac{|u(t)|}{1+|u(t)|} + \sin v(t)), & t\in[0,1],\\ {}^CD^{0.5}_{0+}(D+1)D^{3.5}_{0+}v(t) = \frac{1}{40\sqrt{1600+t^2}}(\frac{|v(t)|}{1+|v(t)|} + \sin u(t)), & t\in[0,1], \end{cases} \tag{4.1}$$

subject to the following coupled boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = D^{3.5}_{0+}u(0) = 0,\\ \quad D^{3.5}_{0+}u(1) = \int_0^{0.5} D^{3.5}_{0+}v(s)ds, \quad D^{1.5}_{0+}u(1) = \sum_{j=1}^{\infty}\frac{1}{2j^2}u(\frac{1}{j}) + \frac{1}{2}\int_0^1 D^{1.5}_{0+}u(\tau)d\mathcal{A}(\tau),\\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = D^{3.5}_{0+}v(0) = 0,\\ \quad D^{3.5}_{0+}v(1) = \int_0^{0.5} D^{3.5}_{0+}u(s)ds, \quad D^{1.5}_{0+}v(1) = \sum_{j=1}^{\infty}\frac{1}{2j^2}v(\frac{1}{j}) + \frac{1}{2}\int_0^1 D^{1.5}_{0+}v(\tau)d\mathcal{B}(\tau), \end{cases} \tag{4.2}$$

where $\alpha = 3.5,\ \beta = 1.5,\ \gamma = 1.5,\ \eta_j = \varepsilon_j = \frac{1}{2j^2},\ \xi_j = \frac{1}{j},\ j = 1, 2, \cdots,\ \eta_0 = \varepsilon_0 = \frac{1}{2}$, $\mathcal{A}(t) = \mathcal{B}(t) = \frac{t^3}{3},\ \sigma = 0.5,\ a_1 = a_2 = 1,\ k = 1$,

$$f(t,u(t),v(t)) = \frac{1}{30\sqrt{900+t}}(\frac{|u(t)|}{1+|u(t)|} + \sin v(t)),$$
$$g(t,u(t),v(t)) = \frac{1}{40\sqrt{1600+t^2}}(\frac{|v(t)|}{1+|v(t)|} + \sin u(t)).$$

Clearly,

$$\int_0^1 d\mathcal{A}(t) = \int_0^1 d\mathcal{B}(t) = \frac{1}{3}, \quad \sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1} = \sum_{j=1}^{\infty}\varepsilon_j\xi_j^{\alpha-1} = 0.5412,$$

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}(1-\eta_0\int_0^1 t^{\alpha-\gamma-1}d\mathcal{A}(t)) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}\Big(1-\varepsilon_0\int_0^1 t^{\alpha-\gamma-1}d\mathcal{B}(t)\Big) = 2.9079,$$

$$a_1a_2 = 1 < \Big(\frac{1-e^{-k}}{\sigma-\frac{1}{k}(1-e^{-k\sigma})}\Big)^2 = 35.212,$$

$$A = 0.63212,\quad A_1 = A_2 = 0.10653,\quad \Lambda = \Lambda_1 = 2.3667,$$

$$\sigma^{\alpha+\beta} = 0.0313 < \sigma^{\beta+1} = 0.1768,$$

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} = 0.0277 < \frac{1}{(\alpha+\beta)(\alpha+\beta-1)} = 0.0500 < \frac{1}{6} < \frac{1}{\Gamma(\beta+2)} = 0.3009,$$

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} = 0.1385 < \frac{1}{\alpha+\beta-1} = 0.2500 < \frac{1}{2} < \frac{1}{\Gamma(\beta+1)} = 0.7523,$$

$$M_3 = M_4 = 0.2876,\quad M_5 = M_6 = 1.9862,$$

$$\begin{aligned}
&|f(t_1,u_1,v_1)-f(t_2,u_2,v_2)|\\
=&\left|\frac{1}{30\sqrt{900+t_1}}\left(\frac{|u_1|}{1+|u_1|}+\sin v_1\right)-\frac{1}{30\sqrt{900+t_2}}\left(\frac{|u_2|}{1+|u_2|}+\sin v_2\right)\right|\\
=&\left|\frac{1}{30\sqrt{900+t_1}}\left(\frac{|u_1|}{1+|u_1|}+\sin v_1-\frac{|u_2|}{1+|u_2|}-\sin v_2\right)\right.\\
+&\left.\left(\frac{1}{30\sqrt{900+t_1}}-\frac{1}{30\sqrt{900+t_2}}\right)\left(\frac{|u_2|}{1+|u_2|}+\sin v_2\right)\right|\\
\le&\frac{1}{900}\left(\left|\frac{|u_1|}{1+|u_1|}-\frac{|u_2|}{1+|u_2|}\right|+|\sin v_1-\sin v_2|\right)+2\left|\frac{1}{30\sqrt{900+t_1}}-\frac{1}{30\sqrt{900+t_2}}\right|\\
\le&\frac{1}{900}(||u_1|-|u_2||+|v_1-v_2|)+\frac{2|t_2-t_1|}{30\sqrt{900+t_1}\sqrt{900+t_2}(\sqrt{900+t_2}+\sqrt{900+t_1})}\\
\le&\frac{1}{900}(|u_1-u_2|+|v_1-v_2|)+\frac{1}{900^2}|t_1-t_2|,
\end{aligned}$$

$$\begin{aligned}
&|g(t_1,u_1,v_1)-g(t_2,u_2,v_2)|\\
=&\left|\frac{1}{40\sqrt{1600+t_1^2}}\left(\frac{|v_1|}{1+|v_1|}+\sin u_1\right)-\frac{1}{40\sqrt{1600+t_2^2}}\left(\frac{|v_2|}{1+|v_2|}+\sin u_2\right)\right|\\
\le&\frac{1}{1600}(||v_1|-|v_2||+|u_1-u_2|)+2\left|\frac{(\sqrt{1600+t_2^2}-\sqrt{1600+t_1^2})}{40\sqrt{1600+t_1^2}\sqrt{1600+t_2^2}}\right|\\
\le&\frac{1}{1600}(|v_1-v_2|+|u_1-u_2|)+\frac{1}{1600\times 800}|t_1-t_2|.
\end{aligned}$$

Thus, we have

$$l_0=\frac{1}{900^2},\ l_1=l_2=\frac{1}{900},\ l_3=\frac{1}{1600\times 800},\ l_4=l_5=\frac{1}{1600},\ l=\frac{1}{900},$$

and

$$\max\{M_3,M_4\} = 0.2876 < \max\{M_5,M_6\} = 1.9862,$$

$$\max\{M_3,M_4\}l < 1,\quad \max\{M_5,M_6\}l < 1.$$

Hence, by Theorem 3.2 or Remark 3.3, the boundary value problem (4.1)–(4.2) has a unique solution.

5. Conclusion

We have studied the solution of the linear Caputo fractional differential system with coupled boundary conditions. The result shows that the absolutely continuous assumptions on h_1, h_2 in (2.1) are needed. We have also discussed the existence and uniqueness of solution for a class of nonlinear Caputo fractional differential system supplemented with coupled strip and infinite point boundary value conditions. The results specialize to several different cases for appropriate values of the parameters. For instance, our results correspond to the ones for nonlocal infinite-point boundary conditions if we fix $a_1 = a_2 = \eta_0 = \varepsilon_0 = 0$ in (1.2). We obtain the results associated with coupled strip and integral boundary conditions for $\eta_j = \varepsilon_j = 0,\ j = 1, 2, \cdots$ in (1.2). In case we take $a_1 = a_2 = \eta_j = \varepsilon_j = 0,\ j = 0, 1, \cdots$, we get the results for a fractional-order coupled system equipped with two-point boundary conditions. At the foundation of this paper, one can consider boundary value problems of fractional differential systems involving p-Laplacian operator, and also can make further research on eigenvalue problems of fractional differential systems.

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