Fixed Point Theory, 22(2021), No. 2, 887-898
DOI: 10.24193/fpt-ro.2021.2.58
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ITERATIVE APPROXIMATIONS OF FIXED POINTS FOR OPERATORS SATISFYING $(B_{\gamma,\mu})$ CONDITION

KIFAYAT ULLAH* AND JUNAID AHMAD**

*Department of Mathematics, University of Science and Technology,
Bannu-28100,Khyber Pakhtunkhwa, Pakistan;
Department of Mathematics, University of Lakki Marwat,
Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan
E-mail: Kifayatmath@yahoo.com

**Department of Mathematics, University of Science and Technology,
Bannu-28100,Khyber Pakhtunkhwa, Pakistan;
Department of Mathematics, International Islamic University,
H-10, Islamabad -44000, Pakistan
E-mail: ahmadjunaid436@gmail.com

Abstract. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to satisfy $(B_{\gamma,\mu})$ condition if there exists $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ satisfying $2\mu \leq \gamma$ such that for each $x, y \in C$,

$$\gamma||x - Tx|| \leq ||x - y|| + \mu||y - Ty||$$

$$\text{implies } ||Tx - Ty|| \leq (1 - \gamma)||x - y|| + \mu(||x - Ty|| + ||y - Tx||).$$

In this paper, we obtain some convergence theorems for such mappings using M iterative process in uniformly convex Banach space setting. Our results extend and improve many results in the literature.

Key Words and Phrases: Condition $(B_{\gamma,\mu})$, weak convergence, strong convergence, M iteration, Banach space.
2020 Mathematics Subject Classification: 47H05, 47H09, 47H10.

1. Introduction and preliminaries

Throughout the paper, $\mathbb{N}$ will represent the set of all positive integers. Let C be a self mapping on a subset C of a Banach space X. A point $q \in C$ is called a fixed point of T if $Tq = q$. We shall denote by $F(T)$ the fixed point set of T. The mapping $T : C \to C$ is called quasi-nonexpansive (resp. nonexpansive) whenever $||Tx-Tq|| \leq ||x-q||$ for all $x \in C$ and $q \in F(T)$ (resp. $||Tx-Ty|| \leq ||x-y||$, for each $x, y \in C$). The existence of fixed points for self nonexpansive mappings in the context of Banach spaces was studied independently by Browder [4], Gohde [7] and Kirk [12]. They differently proved that every self nonexpansive map acting on a bounded closed convex subset of a uniformly convex Banach space admits a fixed point. The first remarkable generalization of nonexpansive mappings was considered by Suzuki [21].

In fact, Suzuki [21] introduced a new class of mappings (which contains the class of nonexpansive mappings) known as Suzuki generalized nonexpansive mappings (a class of mappings which satisfy the (C) condition). A mapping $T : C \to C$ is said to satisfy (C) condition (or said to be Suzuki generalized nonexpansive mapping) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \;\Rightarrow\; ||Tx - Ty|| \leq ||x - y||,$$

for each $x, y \in C$.

Recently, Patir et al. [16] introduced the wider class of generalized nonexpansive mappings (which contains the class of Suzuki generalized nonexpansive mappings and hence the class of nonexpansive mappings) in the setting of Banach spaces. This class of mappings satisfies $(B_{\gamma,\mu})$ condition which is more general than the (C) condition. A mapping $T : C \to C$ is said to satisfy $(B_{\gamma,\mu})$ condition if there exists $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ satisfying $2\mu \leq \gamma$ such that for each $x, y \in C$,

$$\gamma||x - Tx|| \leq ||x - y|| + \mu||y - Ty||$$

$$\text{implies } ||Tx - Ty|| \leq (1 - \gamma)||x - y|| + \mu(||x - Ty|| + ||y - Tx||).$$

They also showed that, if a mapping satisfies the (C) condition, then it satisfies the $(B_{\gamma,\mu})$ condition but the converse does not hold in general.

Example 1.1. Define a mapping $T : [0, 2] \to \mathbb{R}$ by

$$Tx = \begin{cases} 0 & \text{for} \quad x \neq 2 \\ 1 & \text{for} \quad x = 2. \end{cases}$$

Here T satisfies $(B_{\gamma,\mu})$ condition, but does not satisfy (C) condition.

Finding of solutions for operator equations is an active and important research field on its own. From the celebrated Banach contraction mapping principle (BCMP), we know that, if T is a contraction map then the approximate solution of an operator equation $x = Tx$ can be obtained by the Picard iterative process, that is, $x_{n+1} = Tx_n$. Nevertheless, if T is nonexpansive then the Picard iterative process does not work. To approximate fixed points of nonexpansive mappings and to get better rate of convergence, many iterative processes can be found in the literature (e.g., Mann [13], Ishikawa [9], S [3], Noor [14], Abbas [1], SP [17], S* [10], CR [5], Normal-S [18], Picard-Mann hybrid [11], Picard-S [8], Thakur et al. [23] and so on).

The following iterative process is essentially due to Mann [13]:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\alpha_n \in (0, 1)$.

The following iterative process is essentially due to Ishikawa [9]:

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\alpha_n, \beta_n \in (0, 1)$.

In 2007, Agarwal et al. [3] slightly modified the Ishikawa iterative process and call it S iterative process:

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\alpha_n, \beta_n \in (0,1)$.

Agarwal et al. [3] proved that the S iterative is better than the Mann iterative process for Banach contraction mappings.

In 2014, Gursoy and Karakaya [8] proposed the Picard-S hybrid iterative process as follows:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n T z_n, \\ x_{n+1} = Ty_n, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\alpha_n, \beta_n \in (0,1)$.

Gursoy and Karakaya [8] proved that Picard-S hybrid iterative process is better than all of the Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas, and Normal-S iterative process for Banach contraction mappings.

In 2016, Thakur et al. [23] proposed a new iterative process as follows:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T\left((1-\alpha_n)x_n + \alpha_n z_n\right), \\ x_{n+1} = Ty_n, n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\alpha_n, \beta_n \in (0,1)$.

With the help of an example of Suzuki generalized nonexpansive mapping, they proved that the iterative process (1.5) is better than all of the Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iterative process.

Very recently in 2018, Ullah and Arshad [24] introduced M iterative process as follows:

$$\begin{cases} x_1 \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = Tz_n, \\ x_{n+1} = Ty_n, n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\alpha_n \in (0,1)$.

Ullah and Arshad [24] proved the M iterative process is better than the leading two step S and leading three step Picard-S iterative process for Suzuki generalized nonexpansive mappings. The purpose of this paper is to prove some weak and strong convergence results for a mapping with the $(B_{\gamma,\mu})$ condition, using the iteration process (1.6).

Following are some basic facts, definitions and results needed in the sequel.

Definition 1.1. [6] A Banach space X is called uniformly convex if for every real number $\varepsilon \in (0, 2]$, one can find a real number $\lambda > 0$ such that for each two points x, y of X, one has

$$\left.\begin{array}{r} ||x|| \leq 1 \\ ||y|| \leq 1 \\ ||x-y|| > \varepsilon \end{array}\right\} \Longrightarrow ||x+y|| \leq 2(1-\lambda).$$

Definition 1.2. Let X be a Banach space, $\emptyset \neq C \subseteq X$ and $\{x_n\}$ be a bounded sequence in X. Fix $x \in X$, then we denote and define

(i) asymptotic radius of $\{x_n\}$ at x by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} ||x - x_n||;$$

(ii) asymptotic radius of $\{x_n\}$ relative to C by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\};$$

(iii) asymptotic center of $\{x_n\}$ relative to C by

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

One of the celebrated and well known property of the set $A(C, \{x_n\})$ is that it is always a singleton set if the underlying space is uniformly convex. By [2, 22], $A(C, \{x_n\})$ is nonempty convex whenever C is weakly compact and convex.

Definition 1.3. [15] A Banach space X is said to be endowed with Opial's property if for every given sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, one has

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - w|| \text{ for every } w \in X - \{x\}.$$

The well known examples of Banach spaces having Opial's property are l^p spaces $(1 < p < \infty)$ and Hilbert spaces.

Now we recall the definition of condition (I), which is essentially due to Sentor and Dotson [20].

Definition 1.4. Let C be a nonempty subset of a Banach space X. A self mapping T of C is said to satisfy condition (I) if one can find a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$, $\psi(z) > 0$ for all $z > 0$ and $||x - Tx|| \geq \psi(d(x, F(T)))$ for each $x \in C$, where $d(x, F(T))$ denotes distance of x from $F(T)$.

Lemma 1.1. [16] *Let C be a nonempty subset of a Banach space X and $T : C \to C$ satisfies $(B_{\gamma,\mu})$ condition. If q is a fixed point of $T : C \to C$, then for each $x \in C$*

$$||q - Tx|| \leq ||q - x||.$$

Theorem 1.1. [16] *Let C be a nonempty subset of a Banach space X. Let $T : C \to C$ satisfies condition $(B_{\gamma,\mu})$. If $\{x_n\}$ is sequence in X such that*

(i) *$\{x_n\}$ converges weakly to s,*

(ii) *$\lim_{n\to\infty} ||Tx_n - x_n|| = 0$,*

then $Ts = s$.

We need the following useful lemma from [19].

Lemma 1.2. *Let X be a uniformly convex Banach space and $0 < p \leq \alpha_n \leq q < 1$ for every $n \in \mathbb{N}$. If $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that*

$$\limsup_{n\to\infty} ||x_n|| \leq c, \quad \limsup_{n\to\infty} ||y_n|| \leq c$$

and

$$\lim_{n\to\infty} ||\alpha_n x_n + (1-\alpha_n)y_n|| = c$$

for some $c \geq 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

2. Main results

We begin this section with a key lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (1.6), then $\lim_{n\to\infty} ||x_n - q||$ exists for each $q \in F(T)$.*

Proof. Let $q \in F(T)$. By Lemma 1.1, we have

$$\begin{aligned} ||z_n - q|| &= ||(1-\alpha_n)x_n + \alpha_n T x_n - q|| \\ &\leq (1-\alpha_n)||x_n - q|| + \alpha_n ||Tx_n - q|| \\ &\leq (1-\alpha_n)||x_n - q|| + \alpha_n ||x_n - q|| \\ &\leq ||x_n - q||, \end{aligned}$$

and

$$\begin{aligned} ||y_n - q|| &= ||Tz_n - q|| \\ &\leq ||z_n - q||. \end{aligned}$$

This implies that,

$$\begin{aligned} ||x_{n+1} - q|| &= ||Ty_n - q|| \leq ||y_n - q|| \\ &\leq ||z_n - q|| \leq ||x_n - q||. \end{aligned}$$

Thus $\{||x_n - q||\}$ is bounded and nonincreasing and hence $\lim_{n\to\infty} ||x_n - q||$ exists for each $q \in F(T)$.

Now, we prove the following theorem which is important in the proof of convergence theorems.

Theorem 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a mapping satisfying the $(B_{\gamma,\mu})$ condition. If $\{x_n\}$ is a sequence generated by (1.6). Then, $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$.*

Proof. Suppose that $F(T) \neq \emptyset$ and $q \in F(T)$. Then, by Lemma 2.1, $\lim_{n\to\infty} ||x_n - q||$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} ||x_n - q|| = c. \tag{2.1}$$

By the proof of Lemma 2.1 together with (2.1), we have

$$\limsup_{n\to\infty} ||z_n - q|| \leq \limsup_{n\to\infty} ||x_n - q|| = c. \tag{2.2}$$

By Lemma 1.1, we have

$$\limsup_{n\to\infty} ||Tx_n - q|| \leq \limsup_{n\to\infty} ||x_n - q|| = c. \tag{2.3}$$

Again by the proof of Lemma 2.1, together with (2.1), we have

$$c = \liminf_{n\to\infty} ||x_{n+1} - q|| \leq \liminf_{n\to\infty} ||z_n - q||. \tag{2.4}$$

From (2.2) and (2.4), we obtain

$$c = \lim_{n\to\infty} ||z_n - q||. \tag{2.5}$$

From (2.5), we have

$$c = \lim_{n\to\infty} ||z_n - q|| = \lim_{n\to\infty} ||(1-\alpha_n)(x_n - q) + \alpha_n(Tx_n - q)||.$$

Hence,

$$c = \lim_{n\to\infty} ||(1-\alpha_n)(x_n - q) + \alpha_n(Tx_n - q)||. \tag{2.6}$$

Now from (2.1), (2.3) and (2.6) together with Lemma 1.2, we obtain

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0.$$

Conversely, let $q \in A(C, \{x_n\})$. Proceeding as in the converse part of Theorem 3.17 of [16], it can be shown that $Tq \in A(C, \{x_n\})$. Since X is uniformly convex Banach space, so $Tq = q$.

By using Theorem 2.1, we have the following weak convergence theorem.

Theorem 2.2. *Let C a nonempty closed convex subset of a uniformly convex Banach space X having Opial property. If $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. Then $\{x_n\}$ generated by (1.6) converges weakly to an element of $F(T)$.*

Proof. By Theorem 2.1, $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0.$$

Since X is uniformly convex, X is reflexive. So, subsequence $\{x_{n_i}\}$ of $\{x_n\}$ exists such that $\{x_{n_i}\}$ converges weakly to some $s_1 \in C$. By Theorem 1.1, we have $s_1 \in F(T)$. It is sufficient to show that $\{x_n\}$ converges weakly to s_1. In fact, if $\{x_n\}$ does not converges weakly to s_1. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $s_2 \in C$ such that $\{x_{n_j}\}$ converges weakly to s_2 and $s_2 \neq s_1$.

Again by Theorem 1.1, $s_2 \in F(T)$. By Lemma 2.1 and Opial property, we have

$$\begin{aligned}
\lim_{n\to\infty} ||x_n - s_1|| &= \lim_{i\to\infty} ||x_{n_i} - s_1|| \\
&< \lim_{i\to\infty} ||x_{n_i} - s_2|| \\
&= \lim_{n\to\infty} ||x_n - s_2|| \\
&= \lim_{j\to\infty} ||x_{n_j} - s_2|| \\
&< \lim_{j\to\infty} ||x_{n_j} - s_1|| \\
&= \lim_{n\to\infty} ||x_n - s_1||.
\end{aligned}$$

This is a contradiction. So, $s_1 = s_2$. Thus, $\{x_n\}$ converges weakly to $s_1 \in F(T)$.

Theorem 2.3. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (1.6). Then $\{x_n\}$ converges to an element of $F(T)$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. The necessity is obvious.
Conversely, suppose that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0 \text{ and } q \in F(T).$$

By Lemma 2.1, $\lim_{n\to\infty} ||x_n - q||$ exists for each $q \in F(T)$. By assumption, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Hence for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for each $n \geq k_0$,

$$d(x_n, F(T)) < \frac{\varepsilon}{2}$$

$$\Rightarrow \inf\{||x_n - q|| : q \in F(T)\} < \frac{\varepsilon}{2}.$$

In particular

$$\inf\{||x_{k_0} - q|| : q \in F(T)\} < \frac{\varepsilon}{2}.$$

Therefore there exists $q \in F(T)$ such that

$$||x_{k_0} - q|| < \frac{\varepsilon}{2}.$$

Now for $k, n \geq k_0$,

$$\begin{aligned} ||x_{n+k} - x_n|| &\leq ||x_{n+k} - q|| + ||x_n - q|| \\ &\leq ||x_{k_0} - q|| + ||x_{k_0} - q|| \\ &= 2||x_{k_0} - q|| < \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. As C is closed subset of a Banach space X, so there exists a point $p \in C$ such that

$$\lim_{n\to\infty} x_n = p.$$

Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(p, F(T)) = 0$. This shows that $p \in F(T)$.

We now prove the following theorem using condition (I).

Theorem 2.4. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ satisfies the $(B_{\gamma,\mu})$ condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (1.6). Then $\{x_n\}$ converges strongly to an element of $F(T)$ provided that T satisfies the condition (I).*

Proof. From Theorem 2.1, it follows that

$$\liminf_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{2.7}$$

Since T satisfies the condition (I), we have

$$||x_n - Tx_n|| \geq \psi(d(x_n, F(T))).$$

From (2.7), we have
$$\liminf_{n\to\infty} \psi(d(x_n, F(T))) = 0.$$
Since ψ is a nondecreasing function with $\psi(0) = 0$ and $\psi(z) > 0$ for each $z > 0$, we have
$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$
Therefore all the assumptions of Theorem 2.3 are satisfied and so $\{x_n\}$ converges strongly to an element of $F(T)$.

3. Example

For numerical interpretation of our results, we first construct an example of mapping which satisfies $(B_{\gamma,\mu})$ condition but not the (C) condition. We then use this example to compare the numerical efficiency of M iteration process with the leading Picard-S, Thakur and S iteration process.

Example 3.1. Define $T : [0, 1] \to [0, 1]$ by
$$Tx := \begin{cases} 0 & \text{for} \quad x \in \left[0, \frac{1}{500}\right) \\ \frac{1}{2}x & \text{for} \quad x \in \left[\frac{1}{500}, 1\right]. \end{cases}$$
If $x = \frac{1}{900}$ and $y = \frac{1}{500}$, then T does not satisfy condition C. Because in this case
$$\frac{1}{2}|x - Tx| = \frac{1}{1800} < \frac{1}{1125} = |x - y|$$
and
$$|Tx - Ty| = \frac{1}{1000} > \frac{1}{1125} = |x - y|.$$
On the other hand, if $\gamma = 1$ and $\mu = \frac{1}{2}$, then T satisfies condition $(B_{\gamma,\mu})$. We divide the proof as follows.

Case (a): When $x, y \in [0, \frac{1}{500})$, we have
$$(1 - \gamma)|x - y| + \mu(|x - Ty| + |y - Tx|) \geq 0 = |Tx - Ty|.$$

Case (b): When $x, y \in [\frac{1}{500}, 1]$, we have
$$\begin{aligned} (1 - \gamma)|x - y| + \mu(|x - Ty| + |y - Tx|) &= \frac{1}{2}(|x - Ty| + |y - Tx|) \\ &= \frac{1}{2}\left(\left|x - \frac{y}{2}\right| + \left|y - \frac{x}{2}\right|\right) \\ &\geq \frac{1}{2}\left(\left|\left(x - \frac{y}{2}\right) - \left(y - \frac{x}{2}\right)\right|\right) \\ &= \frac{1}{2}\left(\left|\left(x + \frac{x}{2}\right) - \left(y + \frac{y}{2}\right)\right|\right) \\ &= \frac{3}{4}|x - y| \geq \frac{1}{2}|x - y| \\ &= |Tx - Ty|. \end{aligned}$$

Case (c): When $x \in [\frac{1}{500}, 1]$ and $y \in [0, \frac{1}{500})$, we have

$$\begin{aligned}(1-\gamma)|x-y| + \mu(|x-Ty| + |y-Tx|) &= \frac{1}{2}(|x-Ty| + |y-Tx|) \\ &= \frac{1}{2}\left(|x-0| + \left|y - \frac{x}{2}\right|\right) \\ &= \frac{1}{2}|x| + \frac{1}{2}\left|y - \frac{x}{2}\right| \\ &\geq \frac{1}{2}|x| = |Tx - Ty|.\end{aligned}$$

Case (d): When $y \in [\frac{1}{500}, 1]$ and $x \in [0, \frac{1}{500})$, we have

$$\begin{aligned}(1-\gamma)|x-y| + \mu(|x-Ty| + |y-Tx|) &= \frac{1}{2}(|x-Ty| + |y-Tx|) \\ &= \frac{1}{2}\left(|x - \frac{y}{2}| + |y-0|\right) \\ &= \frac{1}{2}|x - \frac{y}{2}| + \frac{1}{2}\,|y| \\ &\geq \frac{1}{2}|y| = |Tx - Ty|.\end{aligned}$$

Hence, T satisfies the $(B_{1,\frac{1}{2}})$ condition and $q = 0$ is a fixed point of T. Let

$$\alpha_n = n^{\frac{1}{3}} \text{ and } \beta_n = \sqrt{\frac{2n}{4n+5}}.$$

We obtained the influence of initial point for the S (1.3), Picard-S (1.4), Thakur (1.5) and M (1.6) iteration. Tables 1 and 2 shows that the rate of convergence of iteration process (1.4) and (1.5) is almost the same. Note: Items in bold show that the M iteration (1.6) converges faster than others.

Table 1. Influence of initial points for various iteration processes

Number of iterations required to obtain fixed point.				
Initial points	S (1.3)	Picard-S (1.4)	Thakur (1.5)	M (1.6)
0.10	6	3	3	**3**
0.15	6	4	4	**3**
0.20	6	4	4	**3**
0.50	7	4	4	**4**
0.80	8	5	5	**4**
0.90	8	5	5	**4**

Table 2. Influence of parameters: comparison of various iteration processes

Iterations	Initial points					
	0.1	0.15	0.2	0.5	0.8	0.9
For $\alpha_n = \frac{n}{(n+1)^{\frac{10}{9}}}, \beta_n = \frac{1}{(n+3)^{\frac{2}{3}}}$						
S	6	7	7	9	9	9
Thakur	4	4	4	5	5	5
Picard-S	4	4	4	5	5	5
M	**3**	**3**	**4**	**4**	**4**	**4**
for $\alpha_n = 1 - \frac{1}{\sqrt{5n+3}}, \beta_n = \frac{1}{n^3}$						
S	6	7	7	9	9	10
Thakur	3	4	4	5	5	5
Picard-S	3	4	4	5	5	5
M	**3**	**3**	**3**	**4**	**4**	**4**
for $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{\sqrt{n+24}}$						
S	7	7	8	9	10	10
Thakur	4	4	4	5	5	5
Picard-S	4	4	4	5	5	5
M	**3**	**3**	**3**	**4**	**4**	**4**
for $\alpha_n = \sqrt{\frac{n}{4n+3}}, \beta_n = \frac{1}{(4n+9)^{\frac{3}{4}}}$						
S	7	7	8	9	10	9
Thakur	4	4	4	5	5	5
Picard-S	4	4	4	5	5	5
M	**3**	**4**	**4**	**4**	**5**	**5**

Acknowledgment. We are very much thankful to the reviewers for their constructive comments and suggestions which have been useful for the improvement of this paper.

References

[1] M. Abbas, T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesnik, **66**(2014), no. 2, 223-234.

[2] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications Series*, Topological Fixed Point Theory and Its Applications, vol. 6, Springer, New York, 2009.

[3] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., **8**(2007), no. 1, 61-79.

[4] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci., USA, **54**(1965), 1041-1044.

[5] R. Chugh, V. Kumar, S. Kumar, *Strong convergence of a new three step iterative scheme in Banach spaces*, Amer. J. Comp. Math., **2**, 345-357.

[6] J.A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc., **40**(1936), 396-414.

[7] D. Gohde, *Zum Prinzip der Kontraktiven Abbildung*, Math. Nachr., **30**(1965), 251-258.

[8] F. Gursoy, V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, (2014), arXiv:1403.2546v2.

[9] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**(1974), 147-150.

[10] I. Karahan, M. Ozdemir, *A general iterative method for approximation of fixed points and their applications*, Adv. Fixed Point Theory, **3**, 510-526.
[11] S.H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl., **69**(2013), 10, pp.
[12] W.A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly, **72**(1965), 1004-1006.
[13] W.R. Mann, *Mean value methods in iterations*, Proc. Amer. Math. Soc., **4**(1953), 506-510.
[14] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251**(2000), 217-229.
[15] Z. Opial, *Weak and strong convergence of the sequence of successive approximations for non-expansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 591-597.
[16] B. Patir, N. Goswami, V.N. Mishra, *Some results on fixed point theory for a class of generalized nonexpansive mappings*, Fixed Point Theory Appl., **19**(2018), 18 pp.
[17] W. Phuengrattana, S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comp. App. Math., **235**(2011), 3006-3014.
[18] D.R. Sahu, A. Petruşel, *Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces*, Nonlinear Anal., T.M.A., **74**(2011), 6012-6023.
[19] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43**(1991), 153-159.
[20] H.F. Sentor, W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **44**(1974), 375-380.
[21] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340**(2008), 1088-1095.
[22] W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma Publishers, Yokohoma, 2000.
[23] B.S. Thakur, D. Thakur, M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, App. Math. Comp., **275**(2016), 147-155.
[24] K. Ullah, M. Arshad, *Numerical reckoning fixed points for Suzuki's Generalized nonexpansive mappings via new iteration process*, Filomat, **32**(2018), no. 1, 187-196.

Received: November 21, 2019; Accepted: June 23, 2020.