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WEAK ASYMPTOTIC STABILITY FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH FINITE DELAYS

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Abstract. This paper investigates a class of semilinear fractional differential equations with finite delays. Based on the α-resolvent theory, the fixed point theory for condensing maps and the local estimates of solutions, we prove the existence of solutions to the suggested system when the nonlinear part is superlinear. In the case, the nonlinear part is sublinear we study the weak asymptotic stability of the zero solution by applying a new Halanay type inequality. An application to a class of partial differential equations will be given.
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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. We consider the following problem

$$\frac{d}{dt}(\mu_\alpha * [u - u(0)])(t) = Au(t) + f(t, u_t), t > 0; \tag{1.1}$$

$$u(s) = \varphi(s), s \in [-h, 0], \tag{1.2}$$

where $\mu_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ for $\alpha \in (0,1)$, the state function u takes values in X with the history state $u_t \in C([-h,0];X)$ defined by $u_t(s) = u(t+s), s \in [-h,0]$, A is a closed linear operator on X, and the nonlinear function f is defined on $[0,T] \times C([-h,0];X)$. Here $\mu_\alpha * v$, for $v \in L^1_{loc}(\mathbb{R}^+;X)$, denotes the Laplace convolution, i.e.,

$$(\mu_\alpha * v)(t) = \int_0^t \mu_\alpha(t-s)v(s)ds.$$

Equation (1.1) is known as a fractional differential equation (FrDE) with Caputo's fractional derivative of order α.

Nonlocal differential equations like (1.1) have recently been proved to be valuable tools in mathematical physics to model dynamic processes in materials with memory. They are also employed to describe anomalous diffusion processes (see an explanation in, e.g., [12, 14]). As mentioned in [12], by replacing μ_α with another locally integrable kernel, one can use the linear part of (1.1) to express many processes involving subdiffusion, superdiffusion and ultraslow-diffusion.

Stability is the first of all the considered questions in the system analysis and synthesis of modern control theory. In this paper, we adopt the following concept of weak asymptotic stability. Denote $\mathcal{C}_h = C([-h,0];X)$ the space of continuous functions on $[-h,0]$ with values in X. Then $\mathcal{C}_h$ is a Banach space endowed with the norm

$$\|x\|_{\mathcal{C}_h} = \sup_{s\in[-h,0]} \|x(s)\|.$$

Let $\Sigma(\varphi)$ be the solution set of (1.1)-(1.2) with respect to the initial datum φ. Assume that $0 \in \Sigma(0)$, that is (1.1) admits zero solution. The zero solution of (1.1) is said to be weak asymptotically stable if

(1) It is stable, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi|_{\mathcal{C}_h} < \delta$ then $|u_t|_{\mathcal{C}_h} < \varepsilon$ for all $u \in \Sigma(\varphi)$;
(2) It is weak attractive, i.e., for each $\varphi \in \mathcal{C}_h$, there exists $u \in \Sigma(\varphi)$ such that $|u_t|_{\mathcal{C}_h} \to 0$ as $t \to +\infty$.

We refer to [4, 10] for recent studies related to weak asymptotic stability for differential equations.

Let us give a short description on our work. We prove the existence result by using the fixed point theory for condensing maps, which requires the nonlinearity function f satisfy a regular property expressed by the Hausdorff measure of noncompactness. It should be noted that, in our setting, the function f may have a superlinear growth. To analyze the weak asymptotic stability of solutions, we first prove a new Halanay type inequality. This inequality will be used to prove the stability of solutions. Then we employ the fixed point argument for condensing maps to show the weak attractivity.

The rest of our work is organized as follows. In the next section, we recall some notions and facts on the fractional resolvent theory given in [13] and the measure of noncompactness proposed in [8]. We also give a representation for solutions of (1.1) in the form of variation of constant formula. Section 3 is devoted to the existence results in case the nonlinearity has a superlinear growth. In Section 4, we give the main result, where some sufficient conditions for weak asymptotic stability of solutions will be shown. The last section presents an application of the obtained results to a class of partial differential equations.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. In the sequel, we denote by $C([0,T];X)$ the space of continuous functions on $[0,T]$ with values in X, and by $L^p(0,T;X), p \geq 1$, the space of functions on $[0,T]$ taking values in X, which are p-th power integrable in the sense of Bochner. A subset $D \subset L^p(0,T;X)$ is said to be *integrably bounded* if there exists $\nu \in L^p(0,T) := L^p(0,T;\mathbb{R}^+)$ such that

$$\forall f \in D,\ \|f(t)\| \leq \nu(t) \text{ for a.e. } t \in [0,T].$$

In our presentation, the notation $\mathcal{L}(X)$ stands for the Banach space of bounded linear operators on X. For brevity, we also use $\|\cdot\|$ for the norm in $\mathcal{L}(X)$. A family

$\{V(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is said to be norm-continuous if the map $t \mapsto V(t) \in \mathcal{L}(X)$ is continuous on $(0, \infty)$.

2.1. **Resolvents.** Consider the equation

$$\frac{d}{dt}(\mu_\alpha * [u - u(0)])(t) = Au(t) + f(t, u_t), t > 0, \tag{2.1}$$

where $\mu_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ and A is the generator of a C_0-semigroup $S(\cdot)$ such that

$$\|S(t)\| \leq M, \ \forall t \geq 0.$$

Using [19, Lemma 3.1], we have

$$u(t) = \mathcal{S}_\alpha(t)u(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, u_s)ds, t \geq 0, \tag{2.2}$$

where

$$\mathcal{S}_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta)S(t^\alpha\theta)x\, d\theta, \tag{2.3}$$

$$\mathcal{P}_\alpha(t)x = \alpha\int_0^\infty \theta\phi_\alpha(\theta)S(t^\alpha\theta)x\, d\theta, \forall x \in X, \tag{2.4}$$

with ϕ_α being a probability density function defined on $(0, \infty)$, that has the expression

$$\phi_\alpha(\theta) = \frac{1}{\alpha\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin n\pi\alpha, \ \theta \in (0, \infty).$$

Based on (2.2), one has the following definition of integral solutions for (1.1).

Definition 2.1. A function $x \in C([-h, T]; X)$ is called an integral solution of problem (1.1)- (1.2) on the interval $[-h, T]$ iff $u(s) = \varphi(s), s \in [-h, 0]$ and

$$u(t) = \mathcal{S}_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, u_s)\, ds,$$

for any $t \in [0, T]$.

For $\varphi \in \mathcal{C}_h$, we define the space

$$C_\varphi = \{u \in C([0, T]; X) : u(0) = \varphi(0)\}.$$

as a closed subspace of $C([0, T]; X)$. If $v \in C_\varphi$, we have the function $v[\varphi] : [-h, T] \to X$ defined by

$$v[\varphi](t) = \begin{cases} \varphi(t) & \text{if } -h < t \leq 0, \\ v(t) & \text{if } t \in [0, T]. \end{cases}$$

Then, clearly

$$v[\varphi]_t(\theta) = \begin{cases} \varphi(t+\theta), & -h - t < \theta < -t, \\ v(t+\theta), & \theta \in [-t, 0]. \end{cases}$$

Now we consider the operator $\mathcal{F} : C_\varphi \to C_\varphi$ given by

$$\mathcal{F}(v)(t) = \mathcal{S}_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, v[\varphi]_s)ds. \tag{2.5}$$

It is clear that if v is a fixed point of $\mathcal{F}$ then $v[\varphi]$ is an integral solution to (1.1)- (1.2).

Let $E_{\alpha,\beta}$ be the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ z \in \mathbb{R}, \alpha > 0, \beta > 0.$$

We now recall some basic results, which will be used in the sequel.

Lemma 2.1. *Assume that $k : [0, T] \to \mathbb{R}^+$ is a continuous and nondecreasing function. Then, the function $\Sigma : [0, T] \to \mathbb{R}^+$ defined by*

$$\Sigma(k)(t) = \int_0^t (t-s)^{\alpha-1} k(s) ds$$

is also continuous and nondecreasing.

Proof. Let $\epsilon > 0$, we have

$$\begin{aligned}
\Sigma(k)(t+\epsilon) - \Sigma(k)(t) &= \int_0^{t+\epsilon} (t+\epsilon-s)^{\alpha-1} k(s) ds - \int_0^t (t-s)^{\alpha-1} k(s) ds \\
&= \int_0^{\epsilon} (t+\epsilon-s)^{\alpha-1} k(s) ds + \int_{\epsilon}^{t+\epsilon} (t+\epsilon-s)^{\alpha-1} k(s) ds \\
&\quad - \int_0^t (t-s)^{\alpha-1} k(s) ds.
\end{aligned}$$

By putting $y = s - \epsilon$, ones get

$$\begin{aligned}
\Sigma(k)(t+\epsilon) - \Sigma(k)(t) &= \int_0^{\epsilon} (t+\epsilon-s)^{\alpha-1} k(s) ds + \int_0^t (t-y)^{\alpha-1} k(y+\epsilon) dy \\
&\quad - \int_0^t (t-s)^{\alpha-1} k(s) ds \\
&= \int_0^{\epsilon} (t+\epsilon-s)^{\alpha-1} k(s) ds + \int_0^t (t-s)^{\alpha-1} (k(s+\epsilon) - k(s)) ds \\
&\geq 0,
\end{aligned}$$

thanks to the fact that k is a continuous and nondecreasing function. The proof is complete. $\square$

Lemma 2.2. *Assume that A is the generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ in X such that $\|S(t)\| \leq M$ for $t \geq 0$. Then*

i) *$\|\mathcal{S}_\alpha(t)\| \leq M$, $\|\mathcal{P}_\alpha(t)\| \leq \frac{M}{\Gamma(\alpha)}$ for all $t \geq 0$;*

ii) *If $S(t)$ is compact for $t > 0$, then $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are also compact for $t > 0$;*

iii) *If $S(\cdot)$ is norm-continuous, so are $\mathcal{S}_\alpha(\cdot)$ and $\mathcal{P}_\alpha(\cdot)$;*

iv) *If the semigroup $S(\cdot)$ generated by A is exponential stable, i.e., $\|S(t)\| \leq Me^{-\beta t}$ for some $\beta > 0$, then*

$$\begin{aligned}
\|\mathcal{S}_\alpha(t)\| &\leq ME_{\alpha,1}(-\beta t^\alpha), \\
\|\mathcal{P}_\alpha(t)\| &\leq ME_{\alpha,\alpha}(-\beta t^\alpha),
\end{aligned}$$

for all $t \geq 0$.

Proof. The proof of the first and second statements can be found in [19], while the third statement was proved in [16]. The last one was shown in [2]. □

It is known that, if A is a bounded operator then (see, e.g., [9])

$$\mathcal{S}_\alpha(t) = E_{\alpha,1}(t^\alpha A),\ \mathcal{P}_\alpha(t) = E_{\alpha,\alpha}(t^\alpha A),$$

where the series are understood in $\mathcal{L}(X)$.

In our proofs, the following inequalities will be used.

Lemma 2.3. ([18, Corollary 2]) *Suppose that $\beta > 0, b \geq 0$ and σ is a nonnegative, nondecreasing and locally integrable function on $[0,T]$. If v is nonnegative and locally integrable on $[0,T]$ with*

$$v(t) \leq \sigma(t) + b\int_0^t (t-s)^{\beta-1} v(s)ds,\ \forall t \in [0,T],$$

then $v(t) \leq \sigma(t) E_{\beta,1}(b\Gamma(\beta)t^\beta)$ for all $t \in [0,T]$.

Lemma 2.4. ([17, Lemma 2.1]) *If the continuous function $w(t) \geq 0$ for $t \in \mathbb{R}$, and satisfies that*

$$\begin{cases} w(t) & \leq c_1 + c_2 \sup\limits_{t-\tau(t)\leq \xi \leq t} w(\xi), t \in [0,+\infty), \\ w(t) & = |\psi(t)|, t \in [-\sigma, 0], \end{cases}$$

where $\psi(t)$ is a bounded and continuous function and σ is a given positive constant. The coefficients satisfy that $c_1 \geq 0$ and $0 < c_2 < 1$, and $-\sigma \leq t - \tau(t) \leq t$. Let

$$M_0 = \sup_{-\sigma \leq \xi \leq 0} |\psi(\xi)|.$$

Then we have

$$w(t) \leq \frac{c_1}{1-c_2} + M_0, t \geq 0.$$

Further, if $\lim\limits_{t\to+\infty}(t-\tau(t)) = +\infty$, then for any given $\epsilon > 0$, there exists

$$t_* = t_*(M_0, \epsilon) > t_0$$

such that

$$w(t) \leq \frac{c_1}{1-c_2} + \epsilon,\ t \geq t_*.$$

Lemma 2.5. ([7, Lemma 13]) *Let a bounded measurable function $w : [0,T] \to \mathbb{R}$ satisfy the integral inequality*

$$w(t) \leq E_{\alpha,1}(-\eta t^\alpha) w(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\eta(t-s)^\alpha)\left(K + m\, w(s)\right) ds,$$

where $K \geq 0, 0 < m < \eta$. Then

$$w(t) \leq E_{\alpha,1}\left((-\eta+m)t^\alpha\right) w(0) + K\int_0^t (t-s)^\alpha E_{\alpha,\alpha}\left((-\eta+m)(t-s)^\alpha\right) ds.$$

Let $BC(\mathbb{R}^+)$ be the space of continuous and bounded functions on $\mathbb{R}^+$. We are now in a position to prove the following Halanay type inequality, which play an important role in our analysis.

Proposition 2.6 (Halanay type inequality)**.** *Let v be a continuous and nonnegative function satisfying*

$$v(t) \le E_{\alpha,1}(-\eta t^\alpha)v_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\eta(t-s)^\alpha)\left(\kappa + \ell \sup_{\xi\in[s-h,s]} v(\xi)\right)ds,\ t\ge 0,$$
$$v(\xi) = \psi(\xi),\ \xi\in[-h,0],$$

for $\eta>0, \kappa\ge 0, \ell>0$ such that $\kappa+\ell<\eta$, $\psi\in C([-h,0];\mathbb{R}^+)$. Then

$$v(t) \le \frac{\eta-\kappa}{\eta-\kappa-\ell}v_0 + \sup_{\xi\in[-h,0]}\psi(\xi). \tag{2.6}$$

Proof. We first claim that, if $w\in C([-h,\infty);\mathbb{R}^+)$ satisfies

$$w(t) \le a(t) + b\sup_{\xi\in[-h,t]} w(\xi),\ t>0$$
$$w(\xi) = \psi(\xi),\ \xi\in[-h,0],$$

where $a(\cdot)$ is nondecreasing and $0<b<1$, then

$$w(t) \le (1-b)^{-1}a(t) + \sup_{\xi\in[-h,0]}\psi(\xi), \quad \text{for all } t>0. \tag{2.7}$$

The reason of this assertion is similar to that in Lemma 2.4. Using the same arguments as in Lemma 2.5, we get

$$v(t) \le E_{\alpha,1}(-(\eta-\ell)t^\alpha)v_0 + \kappa\sup_{\xi\in[-h,t]} v(\xi)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\eta-\ell)(t-s)^\alpha)ds. \tag{2.8}$$

$$\le E_{\alpha,1}(-(\eta-\ell)t^\alpha)v_0 + \frac{\kappa}{\eta-\ell}\sup_{\xi\in[-h,t]} w(\xi)(1-E_{\alpha,1}(-(\eta-\ell)t^\alpha)) \tag{2.9}$$

$$\le v_0 + \frac{\kappa}{\eta-\ell}\sup_{\xi\in[-h,t]} w(\xi), \tag{2.10}$$

thanks to the fact that $E_{\alpha,1}(-(\eta-\ell)t^\alpha)\le 1$, and the relation

$$\frac{d}{dt}E_{\alpha,1}(-\gamma t^\alpha) = -\gamma t^{\alpha-1}E_{\alpha,\alpha}(-\gamma t^\alpha),\ \forall\gamma>0, t>0.$$

Hence we are able to apply (2.7) with

$$a(t) = v_0,\ b = \frac{\kappa}{\eta-\ell} < 1,$$

to conclude that $v(\cdot)$ satisfies inequality (2.6). The proof is complete. □

2.2. **Measures of compactness and condensing maps.** Let E be a Banach space. Denote by $\mathcal{B}(E)$ the collection of nonempty bounded subsets of E. We will use the following definition of measure of noncompactness (see, e.g. [8]).

Definition 2.2. A function $\psi : \mathcal{B}(E) \to \mathbb{R}^+$ is called a measure of noncompactness *(MNC)* on E if

$$\psi(\overline{\text{co}}\ \Omega) = \psi(\Omega) \text{ for every } \Omega \in \mathcal{B}(E),$$

where $\overline{\text{co}}$ denotes the closure of convex hull in E. An MNC ψ is said to be:

(i) monotone if for each $\Omega_0, \Omega_1 \in \mathcal{B}(E)$ such that $\Omega_0 \subseteq \Omega_1$, we have $\psi(\Omega_0) \leq \psi(\Omega_1)$;
(ii) nonsingular if $\psi(\{a\} \cup \Omega) = \psi(\Omega)$ for any $a \in E, \Omega \in \mathcal{B}(E)$;
(iii) algebraically semi-additive if $\psi(\Omega_0 + \Omega_1) \leq \psi(\Omega_0) + \psi(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{B}(E)$;
(iv) regular if $\psi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC satisfying all properties, is *the Hausdorff* MNC $\chi(\cdot)$ defined as follows

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon-\text{net}\}.$$

We also define two useful MNCs on $E = C([0, T]; X)$.
For given $L > 0$ and $D \subset C([0, T]; X)$, put

$$\omega_T(D) = \sup_{t\in[0,T]} e^{-Lt}\chi(D(t)), \text{ where } D(t) := \{x(t) : x \in D\}, \tag{2.11}$$

$$\text{mod}_T(D) = \lim_{\delta\to 0} \sup_{x\in D} \max_{t,s\in[0,T],|t-s|<\delta} \|x(t) - x(s)\|. \tag{2.12}$$

According to [8, Example 2.1.2, 2.1.4], ω_T and mod_T are MNCs which satisfy all properties stated in Definition 2.2, except for regularity. In addition, for $D \subset C([0, T]; X)$,

- $\omega_T(D) = 0$ iff $D(t)$ is relatively compact for all $t \in [0, T]$;
- $\text{mod}_T(D) = 0$ iff D is equicontinuous.

Let

$$\chi_T(D) = \omega_T(D) + \text{mod}_T(D),$$

then χ_T is a regular MNC on $C([0, T]; X)$.
Indeed, if $\chi_T(D) = 0$ then $\omega_T(D) = \text{mod}_T(D) = 0$. This implies that $D(t)$ is relatively compact for all $t \in [0, T]$ and D is equicontinuous. Hence D is relatively compact due to the Arzelà-Ascoli theorem.

We are now in a position to recall a basic estimate based on the Hausdorff MNC.

Proposition 2.7. ([3]) *Let $D \subset L^1(0, T; X)$ be such that*

(i) *D is integrably bounded,*
(ii) *$\chi(D(t)) \leq q(t)$ for a.e. $t \in [0, T]$, where $q \in L^1(0, T)$. Then*

$$\chi\left(\int_0^t D(s)\, ds\right) \leq 4\int_0^t q(s)\, ds,$$

here $\displaystyle\int_0^t D(s)\, ds = \Big\{\int_0^t \zeta(s)\, ds : \zeta \in D\Big\}.$

In order to prove the solvability of our problem, we make use of the fixed point principle for condensing maps.

Definition 2.3. A continuous map $\mathcal{F} : Z \subseteq E \to E$ is said to be condensing with respect to an MNC ψ ($\psi-$condensing) if for any bounded set $\Omega \subset Z$, the relation

$$\psi(\Omega) \leq \psi(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let ψ be a monotone nonsingular MNC in E. We have the following fixed point principle.

Theorem 2.8. ([8, Corollary 3.3.1]) *Let $\mathcal{M}$ be a bounded convex closed subset of E and let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a ψ-condensing map. Then* $\text{Fix}(\mathcal{F}) := \{x = \mathcal{F}(x)\}$ *is nonempty and compact set.*

3. Existence results

Consider the operator $\mathcal{W} : L^1(0,T;X) \to C([0,T];X)$ given by

$$\mathcal{W}(f)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds. \tag{3.1}$$

Then the solution operator has the following representation

$$\mathcal{F}(v)(t) = \mathcal{S}_\alpha(t)\varphi(0) + \mathcal{W} \circ N_f(v)(t),$$

where $\mathcal{W}$ is defined by (3.1) and $N_f(v)(t) = f(t, v[\varphi]_t)$ for $v \in C_\varphi$. The following results was proved in [11].

Proposition 3.1. *Let $S(\cdot)$ be norm continuous, i.e., the map $t \mapsto S(t)$ is continuous on $(0,\infty)$. The operator $\mathcal{W}$ defined by (3.1) has the following properties:*

(1) *If $\Omega \subset L^1(0,T;X)$ is integrably bounded, then $\mathcal{W}(\Omega)$ is equicontinuous in $C([0,T];X)$. In addition, if A is a generator of a compact C_0-semigroup, then $\mathcal{W}(\Omega)$ is relatively compact in $C([0,T];X)$.*

(2) *If $\{f_n\} \subset L^1(0,T;X)$ is a semicompact sequence (i.e it is integrably bounded and the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact for almost every $t \in [0;T]$) then $\{\mathcal{W}(f_n)\}$ is relatively compact in $C([0,T];X)$, moreover the weak convergence $f_n \rightharpoonup f$ in $L^1(0,T;X)$ implies the strong convergence $\mathcal{W}(f_n) \to \mathcal{W}(f)$ in $C([0,T];X)$.*

Concerning the formulation of problem (1.1)-(1.2), we give the following assumptions.

(**A**) The C_0-semigroup $\{S(t)\}_{t\geq 0}$ generated by A is norm-continuous and globally bounded, i.e., there is $M \geq 1$ such that

$$\|S(t)x\| \leq M\|x\|, \forall t \geq 0, \forall x \in X.$$

(**F**) The nonlinear function $f : [0,T] \times \mathcal{C}_h \to X$ is continuous and satisfies:

(1) the growth condition

$$\|f(t,v)\| \leq m(t)\Psi(\|v\|_{\mathcal{C}_h}), \forall t \in [0,T], v \in \mathcal{C}_h,$$

where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function and $m : [0,T] \to \mathbb{R}^+$ is a continuous function;

(2) if $S(\cdot)$ is non-compact then for any bounded set $\Omega \subset \mathcal{C}_h$, we have

$$\chi(f(t,\Omega)) \le k(t) \sup_{\theta\in[-h,0]} \chi(\Omega(\theta)),$$

where $k \in L^1(0,T)$ is a nonnegative function.

Remark 3.1. Let us mention that the assumption $(\mathbf{F})(2)$ will be satisfied if f is completely continuous or Lipschitzian with constant k (see [1]).

The next lemma will be used to show the condensivity of $\mathcal{F}$.

Lemma 3.2. *Let the hypotheses* $(\mathbf{A})$ *and* $(\mathbf{F})$ *hold. Then*

$$\chi_T(\mathcal{F}(\Omega)) \le \Big(4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} \|\mathcal{P}_\alpha(t-s)\| k(s) ds\Big) \chi_T(\Omega),$$

for all bounded sets $\Omega \subset C_\varphi$.

Proof. Let $\Omega \subset C_\varphi$ be a bounded set. For $v \in \Omega$, we recall that

$$\mathcal{F}(v)(t) = \mathcal{S}_\alpha(t)\varphi(0) + \mathcal{W} \circ N_f(v)(t),$$

where

$$\mathcal{W} \circ N_f(v)(t) = \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s, v[\varphi]_s) ds.$$

By $(\mathbf{F})(1)$, $N_f(\Omega)$ is integrably bounded. Thanks to Proposition 3.1, we have $\mathcal{W} \circ N_f(\Omega)$ is an equicontinuous set in C_φ. Therefore

$$\mathrm{mod}_T(\mathcal{W} \circ N_f(\Omega)) = 0. \tag{3.2}$$

We now evaluate $\omega_T(\mathcal{W} \circ N_f(\Omega))$. If A is a generator of a compact C_0-semigroup, then $\mathcal{W} \circ N_f(\Omega)$ is compact according to Proposition 3.1. This implies

$$\omega_T(\mathcal{W} \circ N_f(\Omega)) = \sup_{t\in[0,T]} \chi(\mathcal{W} \circ N_f(\Omega)(t)) = 0.$$

In the opposite case, using $(\mathbf{F})(2)$, we get

$$\begin{aligned}
\chi(\mathcal{W} \circ N_f(\Omega)(t)) &\le 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \chi(f(s, \Omega[\varphi]_s)) ds \\
&\le 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| k(s) \sup_{\theta\in[-h,0]} \chi(\Omega[\varphi](s+\theta)) ds \\
&\le 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| k(s) \sup_{r\in[0,s]} \chi(\Omega(r)) ds,
\end{aligned}$$

here we use the fact that $\Omega[\varphi](r) = \{\varphi(r)\}$ for $r \in [-h,0]$. It follows that

$$e^{-Lt}\chi(\mathcal{F}(\Omega)(t)) \le 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} \|\mathcal{P}_\alpha(t-s)\| k(s) e^{-Ls} \chi(\Omega(s)) ds.$$

So

$$\omega_T(\mathcal{F}(\Omega)) \le \left(4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} \|\mathcal{P}_\alpha(t-s)\| k(s) ds\right) \omega_T(\Omega). \tag{3.3}$$

Combining (3.2) and (3.3), we have the conclusion of the lemma. □

We now choose L in (2.11) such that

$$4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} \|\mathcal{P}_\alpha(t-s)\| k(s) ds < 1.$$

Then by Lemma 3.2, the solution operator $\mathcal{F}$ is χ_T-condensing. Denote by $|\cdot|_\infty$ the supremum norm in $C([0,T];\mathbb{R})$. The following theorem states the result of solvability for our problem.

Theorem 3.3. *Let the hypotheses* (**A**) *and* (**F**) *hold. If there exists* $R > 0$ *such that*

$$M\|\varphi\|_{\mathcal{C}_h} + M|I_0^\alpha m|_\infty \Psi(\|\varphi\|_{\mathcal{C}_h} + R) \le R, \tag{3.4}$$

then the solution set to (1.1)-(1.2) *is nonempty.*

Proof. In oder to apply Theorem 2.8, it remains to show that $\mathcal{F}(B_R) \subset B_R$, where B_R is the closed ball in C_φ centered at origin with radius R. Let $v \in B_R$, we have

$$\begin{aligned}
\|\mathcal{F}(v)(t)\| &\le \|\mathcal{S}_\alpha(t)\| \|\varphi\|_{\mathcal{C}_h} + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|f(s, v[\varphi]_s)\| ds \\
&\le M\|\varphi\|_{\mathcal{C}_h} + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s) \Psi(\|v[\varphi]_s\|_{\mathcal{C}_h}) ds \\
&\le M\|\varphi\|_{\mathcal{C}_h} + M I_0^\alpha m(t) \Psi(\|\varphi\|_{\mathcal{C}_h} + R).
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\|\mathcal{F}(v)\|_{C_\varphi} &\le M\|\varphi\|_{\mathcal{C}_h} + M|I_0^\alpha m|_\infty \Psi(\|\varphi\|_{\mathcal{C}_h} + R) \\
&\le R.
\end{aligned}$$

The proof is complete. □

Remark 3.2. If Ψ possesses a polynomial growth, then the condition (3.4) takes place provided that $\|\varphi\|_{\mathcal{C}_h}$ as well as $|I_0^\alpha m|_\infty$ are small. In particular, if $\Psi(r) = r^q$ for $q > 1$, then (3.4) is testified with small initial data, that is, the condition on $|I_0^\alpha m|_\infty$ is relaxed.

4. WEAK ASYMPTOTIC STABILITY

In order to analyze the weak asymptotic stability of solutions of (1.1), we replace the hypotheses (**A**), (**F**) by the following ones.

(**A***) The semigroup $S(\cdot)$ generated by A is norm-continuous and there exist $M \ge 1, \beta > 0$ such that

$$\|S(t)x\| \le Me^{-\beta t}\|x\|, \forall t \ge 0, \forall x \in X.$$

(**F***) The nonlinear function $f : [0, +\infty) \times \mathcal{C}_h \to X$ is continuous and satisfies:

(1) the growth condition

$$\|f(t, v)\| \le m(t)\|v\|_{\mathcal{C}_h}, \forall t \ge 0, v \in \mathcal{C}_h,$$

where $m \in L^\infty(\mathbb{R}^+)$ and $\beta - M|m|_\infty > 0$.

(2) there exists a function $k \in L^\infty(\mathbb{R}^+)$ such that, for every bounded set $D \subset \mathcal{C}_h$ we have $\chi(f(t,D)) \leq k(t) \sup_{s\in[-h,0]} \chi(D(s))$ a.e. $t \in \mathbb{R}^+$.

In order to prove the weak attractivity of zero solution, we make use of the following properties

Lemma 4.1. *Let the hypotheses (**A***) and (**F***(1)) holds. Then*

$$\lim_{T\to\infty} \sup_{t\geq T} \int_0^{\sigma t} (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| m(s) ds = 0, \tag{4.1}$$

$$\ell = \sup_{t\geq 0} \int_0^{t} (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| m(s) ds < 1, \tag{4.2}$$

for some $\sigma \in (0,1)$.

Proof. Using Lemma 2.2(iv), we have

$$\begin{aligned}\int_0^{\sigma t} (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| m(s) ds &\leq |m|_\infty M \int_0^{\sigma t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha) ds \\ &= |m|_\infty M \int_{(1-\sigma)t}^{t} \tau^{\alpha-1} E_{\alpha,\alpha}(-\beta\tau^\alpha) d\tau.\end{aligned}$$

Then

$$\begin{aligned}\sup_{t\geq T} \int_0^{\sigma t} (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| m(s) ds &\leq |m|_\infty M \int_{(1-\sigma)T}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\beta\tau^\alpha) d\tau \\ &\to 0 \text{ as } T \to \infty.\end{aligned}$$

This ensure (4.1). To testify (4.2), we see that

$$\begin{aligned}\int_0^{t} (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| m(s) ds &\leq |m|_\infty M \int_0^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha) ds \\ &\leq \frac{|m|_\infty M}{\beta} \left[1 - E_{\alpha,1}(-\beta t^\alpha)\right] < 1,\end{aligned}$$

thanks to (**F***(1)). Thus (4.2) is fulfilled. □

Consider the following function space

$$BC_0(\mathbb{R}^+; X) = \{u \in C([0,+\infty); X) : \lim_{t\to\infty} u(t) = 0\}, \tag{4.3}$$

endowed with the norm

$$\|u\|_{BC} = \sup_{t\geq 0} \|u(t)\|.$$

Then it is easily seen that $BC_0(\mathbb{R}^+; X)$ is a Banach space. We now define an MNC on this space. We make use of the restriction operator $\pi_T : BC_0(\mathbb{R}^+; X) \to$

$C([0,T];X)$ defined by $\pi_T(u) = u|_{[0,T]}$. Let Ω be a bounded set in $BC_0(\mathbb{R}^+;X)$. Put

$$\chi_\infty(\Omega) = \sup_{T>0} \omega_T(\pi_T(\Omega)) + \sup_{T>0} \mathrm{mod}_T(\pi_T(\Omega)), \tag{4.4}$$

$$d_\infty(\Omega) = \lim_{T\to\infty} \sup_{u\in\Omega} \sup_{t\geq T} \|u(t)\|, \tag{4.5}$$

$$\chi^*(\Omega) = \chi_\infty(\Omega) + d_\infty(\Omega), \tag{4.6}$$

where ω_T and mod_T is given by (2.11) and (2.12), respectively. Then one can check that χ_∞, d_∞ and χ^* are monotone, nonsingular MNCs on $BC_0(\mathbb{R}^+;X)$. The following lemma tests the compactness of a subset in $BC_0(\mathbb{R}^+;X)$.

Lemma 4.2. [4] *Let $\Omega \subset BC_0(\mathbb{R}^+;X)$ be a bounded set such that $\chi^*(\Omega) = 0$. Then Ω is relatively compact in $BC_0(\mathbb{R}^+;X)$.*

We consider the solution operator $\mathcal{F}$ on the space

$$\mathcal{BC}_{0,\varphi} = \{v \in BC_0(\mathbb{R}^+;X) : v(0) = \varphi(0)\},$$

with the norm

$$\|v\|_\infty = \sup_{t\geq 0} \|v(t)\|.$$

As a consequence, we have the following result.

Theorem 4.3. *Let the hypotheses (**A***) and (**F***) holds. Then the zero solution of the problem* (1.1)-(1.2) *is weak asymptotically stable.*

Proof. The idea for proof of this theorem is that, we testify the existence of a solution in $\mathcal{BC}_{0,\varphi}$, which implies the weak attractivity of the zero solution. Then the conclusion follows after verifying the stability of this solution. Let us divide the proof into two steps.

Step 1. We prove the existence of a solution in $\mathcal{BC}_{0,\varphi}$. Considering the MNC ω_T defined in (2.11), we choose $L > 0$ such that

$$\ell + 4\sup_{t\geq 0} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} \|\mathcal{P}_\alpha(t-s)\| k(s)ds < 1,$$

where ℓ is defined by (4.2). Let Ω be a bounded set in $\mathcal{BC}_{0,\varphi}$, using the same estimates as in the proof of Lemma 3.2, we have

$$\chi_\infty(\mathcal{F}(\Omega)) \leq \left(4\sup_{t\geq 0} \int_0^t (t-s)^{\alpha-1} e^{-L(t-s)} \|\mathcal{P}_\alpha(t-s)\| k(s)ds\right) \chi_\infty(\Omega). \tag{4.7}$$

We are in a position to estimate $d_\infty(\mathcal{F}(\Omega))$. For $u \in \Omega$, put

$$z = \mathcal{F}(u) \text{ and } R = \sup_{u\in\Omega} \|u\|_{BC} + \|\varphi\|_{\mathcal{C}_h}.$$

Then

$$\|z(t)\| \leq \|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| m(s) \sup_{\rho\in[-h,0]} \|u(s+\rho)\| ds, \forall t \geq 0.$$

For given $T > 0$, there exists $T_1 > T$ such that

$$\sigma t - h \geq T \text{ for all } t > T_1.$$

Thus for $t > T_2 := \sigma^{-1}T_1 > T$, we get

$$\begin{aligned}\|z(t)\| &\leq \|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| + \int_0^{\sigma t}(t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)\sup_{\rho\in[-h,0]}\|u(s+\rho)\|ds \\ &\quad + \int_{\sigma t}^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)\sup_{\rho\in[-h,0]}\|u(s+\rho)\|ds \\ &\leq \|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| + R\int_0^{\sigma t}(t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds \\ &\quad + \sup_{\xi\geq T}\|u(\xi)\|\int_{\sigma t}^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds.\end{aligned}$$

This implies

$$\begin{aligned}\sup_{u\in D}\sup_{t\geq T_2}\|z(t)\| \leq{}& \sup_{t\geq T_2}\|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| \\ &+ R\sup_{t\geq T_2}\int_0^{\sigma t}(t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds \\ &+ \left(\sup_{u\in D}\sup_{\xi\geq T}\|u(\xi)\|\right)\sup_{t\geq T_2}\int_{\sigma t}^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds.\end{aligned}$$

Let $T \to \infty$ then $T_2 \to \infty$ and we obtain

$$d_\infty(\mathcal{F}(\Omega)) \leq \ell \cdot d_\infty(\Omega), \tag{4.8}$$

thanks to the relations (4.1)-(4.2) and the fact that

$$\sup_{t\geq T_2}\|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| \leq M\sup_{t\geq T_2}E_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \to 0 \text{ as } T_2 \to \infty.$$

Combining (4.7)-(4.8), we ensure that the solution operator $\mathcal{F}$ is χ^*-condensing.
It suffices to prove that $\mathcal{F}(\mathbf{B}_R) \subset \mathbf{B}_R$ for some $R > 0$, here $\mathbf{B}_R$ is the closed ball with center at origin and radius R. Assume to the contrary that for each $n \in \mathbb{N}$ there exists $v_n \in \mathcal{BC}_{0,\varphi}$ with $\|v_n\|_\infty \leq n$ such that $\|z_n\|_\infty > n$ for some $z_n = \mathcal{F}(v_n)$. We have

$$z_n(t) = \mathcal{S}_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, v_n[\varphi]_s)ds.$$

Then

$$\begin{aligned}\|z_n(t)\| &\le \|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\|f(s,v_n[\varphi]_s)\|ds \\ &\le \|\mathcal{S}_\alpha(t)\|\|\varphi\|_{\mathcal{C}_h} + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)\|v_n[\varphi]_s\|ds \\ &\le \|\mathcal{S}_\alpha(t)\|\|\varphi\|_{\mathcal{C}_h} \\ &\quad + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)\left(\|\varphi\|_{\mathcal{C}_h} + \sup_{\rho\in[0,s]}\|v_n(\rho)\|\right)ds \\ &\le \|\mathcal{S}_\alpha(t)\|\|\varphi\|_{\mathcal{C}_h} + \|\varphi\|_{\mathcal{C}_h}\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds \\ &\quad + n\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds \\ &= n\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds + G(t),\end{aligned}$$

where

$$G(t) = \|\mathcal{S}_\alpha(t)\|\|\varphi\|_{\mathcal{C}_h} + \|\varphi\|_{\mathcal{C}_h}\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds,$$

which is uniformly bounded, thanks to Lemma 2.2(iv) and the fact that $m \in L^1(\mathbb{R}^+)$. Then

$$1 < \frac{1}{n}\sup_{t\ge 0}\|z_n(t)\| \le \sup_{t\ge 0}\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|m(s)ds + \frac{1}{n}\sup_{t\ge 0} G(t).$$

Passing to the limit in the last inequality as $n \to \infty$, we get a contradiction with (4.2).

Step 2. We check that the zero solution is stable by showing that for all $u \in \Sigma(\varphi)$, $\|u_t\|_\infty \le \Theta\,\|\varphi\|_{\mathcal{C}_h}$ for some $\Theta > 0$. Indeed, the following estimate holds

$$\begin{aligned}\|u(t)\| &\le \|\mathcal{S}_\alpha(t)\|\|\varphi(0)\| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\|f(s,u[\varphi]_s)\|ds \\ &\le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\ &\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)m(s)\sup_{\rho\in[s-h,s]}\|u(\rho)\|ds\end{aligned}$$

Thus

$$\begin{aligned}\frac{\|u(t)\|}{M} &\le E_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)M|m|_\infty \sup_{\rho\in[s-h,s]}\frac{\|u(\rho)\|}{M}ds, \forall t > 0.\end{aligned}$$

Applying Proposition 2.6, we obtain

$$\frac{\|u(t)\|}{M} \le \frac{\beta}{\beta - M|m|_\infty}\|\varphi(0)\| + \frac{1}{M}\sup_{\xi\in[-h,0]}\|\varphi(\xi)\|, \forall t > 0.$$

This implies

$$\|u(t)\| \leq \left(\frac{M\beta}{\beta - M|m|_\infty} + 1\right) \|\varphi\|_{\mathcal{C}_h}, \forall t > 0.$$

Hence

$$\begin{aligned}\|u_t\|_\infty &\leq \|\varphi\|_{\mathcal{C}_h} + \sup_{\xi \in [0,t]} \|u(\xi)\| \\ &\leq \|\varphi\|_{\mathcal{C}_h} + \left(\frac{M\beta}{\beta - M|m|_\infty} + 1\right) \|\varphi\|_{\mathcal{C}_h} \\ &\leq \left(\frac{M\beta}{\beta - M|m|_\infty} + 2\right) \|\varphi\|_{\mathcal{C}_h}, \forall t > 0.\end{aligned}$$

The proof is complete. □

5. Application

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following problem

$$\partial_t^\alpha u(t,x) = \Delta_x u(t,x) + k(t)\tilde{f}(x, u(t-h,x)), t \geq 0, \tag{5.1}$$

$$u(t,x) = 0,\ x \in \partial\Omega, t > 0, \tag{5.2}$$

$$u(s,x) = \varphi(s,x), x \in \Omega, s \in [-h, 0], \tag{5.3}$$

where $\alpha \in (0,1)$, ∂_t^α stands for the Caputo fractional derivative of order α, $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $k \in L^2(\Omega)$.

Let

$$X = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\},$$

endowed with the norm $\|v\| = \sup_{x \in \overline{\Omega}} |v(x)|$. Put $A = \Delta$ with the domain

$$D(A) = \{v \in C_0(\overline{\Omega}) \cap H_0^1(\Omega) : \Delta v \in C_0(\overline{\Omega})\}.$$

We denote the space $\mathcal{C}_h$ by $C([-h,0];X)$ and define $f : \mathbb{R}^+ \times \mathcal{C}_h \to C_0(\overline{\Omega})$ as follows

$$f(t,w)(x) = k(t)\tilde{f}(x, w(-h,x)), w \in \mathcal{C}_h.$$

Then system (5.1)-(5.3) is in the form of the abstract model (1.1)-(1.2). It is known that A generates a contraction C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X, i.e., $\|S(t)\| \leq 1, \forall t \geq 0$ (see [15]). In addition, by [5, Theorem 2.3], $S(\cdot)$ is a compact semigroup. Hence, (**A**) is fulfilled.

It should be noted that, for the contraction semigroup $S(\cdot)$, either $\|S(t)\| = 1$ for all $t \geq 0$ or $S(\cdot)$ is exponentially stable. According to [6, Theorem 4.2.2], we have

$$\|S(t)\| \leq Me^{-\lambda_1 t},\ M = \exp\left(\frac{\lambda_1 |\Omega|^{2/N}}{4\pi}\right),$$

where λ_1 is the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$, that is

$$\lambda_1 = \inf\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} : u \in H_0^1(\Omega), u \neq 0\right\},$$

and $|\Omega|$ is the volume of Ω. Therefore $(\mathbf{A}^*)$ is satisfied.

Regarding the nonlinearity $\tilde{f}: \Omega \times \mathbb{R} \to \mathbb{R}$, we assume that

$$|\tilde{f}(x,z)| \le \kappa(x)|z|, \kappa \in X,$$

for all $x \in \Omega, z \in \mathbb{R}$. Then we have

$$\begin{aligned}\|f(t,w)\| &\le k(t)\|\kappa\|\|w(-h,\cdot)\| \\ &\le k(t)\|\kappa\|\|w\|_{\mathcal{C}_h},\end{aligned}$$

for all $w \in \mathcal{C}_h$.

One can check that f is continuous, thanks to the continuity of $\tilde{f}$. Hence f satisfies $(\mathbf{F}^*)$. Using Theorem 4.3, we conclude that the zero solution to (5.1) is weak asymptotically stable.

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