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CARISTI'S AND DOWNING-KIRK'S FIXED POINT THEOREMS ON BIPOLAR METRIC SPACES

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Abstract. In this paper, we give some new fixed point results over bipolar metric spaces, that extend Caristi's and Downing-Kirk's fixed point theorems.

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1. Introduction

Caristi's fixed point theorem [10] is one of the most studied and applied generalizations of Banach fixed point theorem, and it has been subject to many generalizations [6, 12, 13, 18, 21, 25, 26, 29], which in particular include the result known as Downing-Kirk's fixed point theorem [14]. Also it has been studied on various distance spaces such as partial metric spaces [2, 3, 17, 28], pseudometric spaces [11], quasicone metric spaces [1], modular ultrametric spaces [4], Polish spaces [8], Kasahara spaces [9], generalized Kasahara spaces [15] and metric spaces with a graph [5].

On the other side, in 2016, the notion of bipolar metric was introduced as a generalized distance, which provides a framework to study distances between dissimilar objects [22] and Banach's and Kannan's fixed point theorems were generalized for bipolar metric spaces. Some recent study have been carried on this area, including coupled fixed point theorems [16, 23], tripled fixed point theorems [27], common fixed point theorems [19], fixed point theorems for convex Reich contractions [7], hybrid pair of mappings [20] and multivalued mappings [24] with some applications to nonlinear mapping theory, integral equations and homotopy theory [19, 20, 25].

In this paper, we state and prove various extensions of Caristi's and Downing-Kirk's fixed point theorems on bipolar metric spaces and illustrate these results.

2. Preliminaries

Here we quickly reintroduce bipolar metric spaces and related notations used throughout the paper. More detailed explanations can be found in [22].

Let X and Y be nonempty sets and $d : X \times Y \to \mathbb{R}^+$ be a function satisfying the properties

(B0) $d(x, y) = 0$ implies $x = y$,

(B1) $x = y$ implies $d(x, y) = 0$,

(B2) $d(u, v) = d(v, u)$,

(B3) $d(x, y) \leq d(x, y') + d(x', y') + d(x', y)$,

for all $x, x' \in X$, $y, y' \in Y$ and $u, v \in X \cap Y$, where $\mathbb{R}^+$ is the set of non-negative real numbers. Then, we call d a bipolar metric on (X, Y), and the triple (X, Y, d) is called a bipolar metric space.

The notation $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$ means that X_1, Y_1, X_2, Y_2 are sets and $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is a function such that $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$. In this case f is called a covariant map, or map for short, from the pair (X_1, Y_1) to (X_2, Y_2). On the other hand, if $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$, then we use the notation $f : (X_1, Y_1) \mathbin{\nearrow\!\!\!\!\!\searrow} (X_2, Y_2)$, or alternatively $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$, and call f a contravariant map from (X_1, Y_1) to (X_2, Y_2). To emphasize bipolar metric space structures if needed, we write $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \mathbin{\nearrow\!\!\!\!\!\searrow} (X_2, Y_2, d_2)$. Also, taking into account that every metric space (X, d) can be regarded as a bipolar metric space (X, X, d), instead of $f : (X_1, Y_1, d_1) \rightrightarrows (X, X, d)$ we shortly write $f : (X_1, Y_1, d_1) \rightrightarrows (X, d)$ or $f : (X_1, Y_1) \rightrightarrows X$.

A covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is said to be left-continuous at $x_0 \in X_1$ if and only if for every $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$, such that $d_1(x_0, y) < \delta$ implies $d_2(f(x_0), f(y)) < \varepsilon$ for all $y \in Y_1$. It is right-continuous at $y_0 \in Y_1$ if and only if for every $\varepsilon > 0$, there exists a $\delta = \delta(y_0, \varepsilon) > 0$, such that $d_1(x, y_0) < \delta$ implies $d_2(f(x), f(y_0)) < \varepsilon$ for all $x \in X_1$. f is called continuous, if it is left-continuous at each $x \in X_1$ and right-continuous at each $y \in Y_1$.

In a bipolar metric space (X, Y, d), points of X, Y and $X \cap Y$ are respectively called left points, right points and central points. A sequence of left (right) points is called a left (right) sequence.

In the context of bipolar metric spaces, the generic term "sequence" is used only for a left or a right sequence, that is (u_n) is called a sequence on a bipolar metric space (X, Y, d) if and only if (u_n) is a sequence on the set $X \cup Y$ and either $\{u_n : n \in \mathbb{N}\} \subseteq X$ or $\{u_n : n \in \mathbb{N}\} \subseteq Y$. A left sequence (x_n) is said to be convergent to a right point y, iff $\lim\limits_{n\to\infty} d(x_n, y) = 0$; and a right sequence (y_n) converges to a left point x provided that $\lim\limits_{n\to\infty} d(x, y_n) = 0$. It was shown in [22] that a covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is continuous iff $(u_n) \to u$ on (X_1, Y_1, d_1) implies $(f(u_n)) \to f(u)$ on (X_2, Y_2, d_2). Without additional conditions, limit of a sequence on a bipolar metric space, need not to be unique.

A sequence (x_n, y_n) on the set $X \times Y$ is called a bisequence on (X, Y, d). If $(x_n) \to y$ and $(y_n) \to x$, then the bisequence (x_n, y_n) is said to be convergent. In particular if $(x_n) \to u$, $(y_n) \to u$ we say that (x_n, y_n) biconverges to u and we use the notation $(x_n, y_n) \rightrightarrows u$.

A bisequence (x_n, y_n) having the property $\lim_{n,m\to\infty} d(x_n, y_m) = 0$ is called a Cauchy bisequence. A bipolar metric space is complete, if every Cauchy bisequence converges on it. It is known that every convergent Cauchy bisequence is biconvergent [22].

3. Main results

Definition 3.1. Let (X, Y, d) be a bipolar metric space. A set $A \subseteq X \cup Y$ is said to be closed provided that every limit of any convergent sequence (u_n) such that $u_n \in A$ for all $n \in \mathbb{N}$, lies in A.

Definition 3.2. Given two bipolar metric spaces (X_1, Y_1, d_1) and (X_2, Y_2, d_2). The graph of a covariant or a contravariant map g from (X_1, Y_1) to (X_2, Y_2) is defined as the set $\{(u, g(u)) : u \in X_1 \cup Y_1\}$. Moreover, consider the bipolar metric space $(X_1 \times X_2, Y_1 \times Y_2, \wp)$, where $\wp : (X_1 \times X_2) \times (Y_1 \times Y_2) \to \mathbb{R}^+$ is defined as $\wp((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$ for all $(x_1, x_2) \in X_1 \times X_2$, $(y_1, y_2) \in Y_1 \times Y_2$ and the bipolar metric space $(X_1 \times Y_2, Y_1 \times X_2, \bar{\wp})$, where $\bar{\wp} : (X_1 \times Y_2) \times (Y_1 \times X_2) \to \mathbb{R}^+$ is defined as $\bar{\wp}((x_1, y_2), (y_1, x_2)) = \wp((x_1, x_2), (y_1, y_2))$. A covariant map $g : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is said to have closed graph, if its graph is closed in the bipolar metric space $(X_1 \times X_2, Y_1 \times Y_2, \wp)$, and a contravariant map $f : (X_1, Y_1, d_1) \nearrow\!\!\!\!\searrow (X_2, Y_2, d_2)$ has closed graph if its graph is closed in $(X_1 \times Y_2, Y_1 \times X_2, \bar{\wp})$.

From Definitions 3.1 and 3.2, we deduce that a covariant or contravariant map g has closed graph, if and only if $(u_n) \to u$ and $(g(u_n)) \to v$ imply $v = g(u)$.

Definition 3.3. Let (X, Y, d) be bipolar metric space.

A map $\varphi : (X, Y) \rightrightarrows C \subseteq \mathbb{R}^+$ is said to be lower semicontinuous, if $(u_n) \to u$ on (X, Y, d) implies $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$.

Theorem 3.4. *Let (X, Y, d) be a complete bipolar metric space, $\varphi : (X, Y) \rightrightarrows \mathbb{R}^+$ be lower semicontinuous. If a contravariant map $T : (X, Y) \nearrow\!\!\!\!\searrow (X, Y)$ satisfies the inequalities*

$$d(x, Tx) \le \varphi(x) - \varphi(Tx), \text{ for all } x \in X, \tag{3.1a}$$

$$d(Ty, y) \le \varphi(y) - \varphi(Ty), \text{ for all } y \in Y, \tag{3.1b}$$

then it has a fixed point.

Proof. Define the sets

$$P(x) = \{y \in Y : d(x, y) \le \varphi(x) - \varphi(y)\} \tag{3.2a}$$

$$R(y) = \{x \in X : d(x, y) \le \varphi(y) - \varphi(x)\} \tag{3.2b}$$

for each $x \in X$ and each $y \in Y$. These sets are nonempty, since $Tx \in P(x)$ by (3.1a) and $Ty \in R(y)$ by (3.1b) for each $x \in X, y \in Y$. So we can define the nonnegative real numbers

$$\alpha(x) = \inf\{\varphi(y) : y \in P(x)\} \tag{3.3a}$$

$$\beta(y) = \inf\{\varphi(x) : x \in R(y)\}. \tag{3.3b}$$

for each $x \in X, y \in Y$. Then by (3.1a) and (3.1b), we have

$$0 \leq \alpha(x) \leq \varphi(Tx) \leq \varphi(x) - d(x, Tx) \leq \varphi(x) \tag{3.4a}$$
$$0 \leq \beta(y) \leq \varphi(Ty) \leq \varphi(y) - d(Ty, y) \leq \varphi(y). \tag{3.4b}$$

Let $x_1 \in X$. For each positive integer k, we select

$$y_k \in P(x_k), \text{ such that } \varphi(y_k) \leq \alpha(x_k) + \frac{1}{k}, \tag{3.5a}$$
$$x_{k+1} \in R(y_k), \text{ such that } \varphi(x_{k+1}) \leq \beta(y_k) + \frac{1}{k}. \tag{3.5b}$$

Thus we have a bisequence (x_n, y_n) on (X, Y, d). Then,

$$d(x_k, y_k) \leq \varphi(x_k) - \varphi(y_k) \tag{3.6a}$$
$$d(x_{k+1}, y_k) \leq \varphi(y_k) - \varphi(x_{k+1}) \tag{3.6b}$$

and

$$\alpha(x_k) \leq \varphi(y_k) \leq \alpha(x_k) + \frac{1}{k} \tag{3.7a}$$
$$\beta(y_k) \leq \varphi(x_{k+1}) \leq \beta(y_k) + \frac{1}{k} \tag{3.7b}$$

by (3.5a), (3.5b), (3.3a) and (3.3b). Moreover (3.6a) and (3.6b) imply that

$$\varphi(x_{k+1}) \leq \varphi(y_k) \leq \varphi(x_k) \tag{3.8}$$

for each positive integer k. Now we have non-increasing sequences $(\varphi(x_n))$, $(\varphi(y_n))$, bounded by 0 by the definition of φ, and hence they converge. Let $c \in \mathbb{R}$ be the limit of $(\varphi(x_n))$. Inequalities (3.7a) and (3.8) yield

$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \varphi(y_n) = \lim_{n\to\infty} \alpha(x_n) = c. \tag{3.9}$$

Since $(\varphi(x_n))$ and $(\varphi(y_n))$ are non-increasing, for every given positive integer k, we can find a $n_0 = n_0(k) \in \mathbb{N}$ such that $\varphi(x_n) \leq c + \frac{1}{k}$ and $\varphi(y_n) \leq c + \frac{1}{k}$ for all integers $n \geq n_0$. In particular for all integers $m, n \geq n_0$

$$c \leq \varphi(x_n) \leq c + \frac{1}{k} \text{ and } c \leq \varphi(y_n) \leq c + \frac{1}{k} \tag{3.10}$$

gives

$$|\varphi(x_n) - \varphi(y_n)| \leq \frac{1}{k}. \tag{3.11}$$

If $m \geq n$, we have

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) + \cdots + d(x_m, y_m) \\ &\leq \varphi(x_n) - \varphi(y_n) + \varphi(y_n) - \varphi(x_{n+1}) + \cdots + \varphi(x_m) - \varphi(y_m) \\ &= \varphi(x_n) - \varphi(y_m) \leq \frac{1}{k} \end{aligned} \tag{3.12}$$

by (3.6a), (3.6b) and (3.11), and similarly, if $n \geq m$,

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_{n-1}) + d(x_{n-1}, y_{n-1}) + \cdots + d(x_{m+1}, y_m) \\ &\leq \varphi(y_{n-1}) - \varphi(x_n) + \varphi(x_{n-1}) - \cdots + \varphi(y_m) - \varphi(x_{m+1}) \\ &= \varphi(y_m) - \varphi(x_n) \leq \frac{1}{k}. \end{aligned} \tag{3.13}$$

Thus $\lim\limits_{m,n\to\infty} d(x_n, y_m) = 0$, and (x_n, y_n) is a Cauchy bisequence on the complete bipolar metric space (X, Y, d), so it converges and hence biconverges to a point $u \in X \cap Y$.

Since $\lim\limits_{m\to\infty} y_m = u$, also $\lim\limits_{m\to\infty} d(x_n, y_m) = d(x_n, u)$ for each positive integer n, and in particular $\liminf\limits_{m\to\infty} d(x_n, y_m) = d(x_n, u)$, and by Definition 3.3, we have

$$\begin{aligned} \varphi(u) &\leq \liminf_{m\to\infty} \varphi(y_m) \\ &\leq \liminf_{m\to\infty} [\varphi(x_n) - d(x_n, y_m)] \\ &\leq \varphi(x_n) - d(x_n, u), \end{aligned} \tag{3.14}$$

that is

$$d(x_n, u) \leq \varphi(x_n) - \varphi(u), \tag{3.15}$$

which yields $u \in P(x_n)$, hence $\alpha(x_n) \leq \varphi(u)$ by (3.3a), and since $(\alpha(x_n))$ is a non-increasing sequence converging to c by (3.9), (3.7a) and (3.8), we have $c \leq \varphi(u)$. Moreover by (3.9),

$$\varphi(u) \leq \liminf_{m\to\infty} \varphi(y_m) = \lim_{m\to\infty} \varphi(y_m) = c. \tag{3.16}$$

and so $\varphi(u) = c$. On the other hand, since $u \in X \cap Y$ and by (3.15) and (3.1a),

$$\begin{aligned} d(x_n, Tu) &\leq d(x_n, u) + d(u, u) + d(u, Tu) \\ &= d(x_n, u) + d(u, Tu) \\ &\leq \varphi(x_n) - \varphi(u) + \varphi(u) - \varphi(Tu) \\ &= \varphi(x_n) - \varphi(Tu), \end{aligned} \tag{3.17}$$

which imply that $Tu \in P(x_n)$, hence $\alpha(x_n) \leq \varphi(Tu)$ by (3.3a), and non-increasingly convergence of (α_n) gives $c \leq \varphi(Tu)$, while (3.1a) implies $\varphi(Tu) \leq \varphi(u) = c$. Then $\varphi(Tu) = \varphi(u) = c$, and by (3.1a), we have $d(u, Tu) = 0$, which means that u is a fixed point of T. □

Example 3.5. Consider the triple $(\mathbb{C}, \mathbb{D}, d)$, where $\mathbb{C}$ is the set of complex numbers, $\mathbb{D}$ is the set of dual numbers and $d : \mathbb{C} \times \mathbb{D} \to \mathbb{R}^+$ be defined as

$$d(a_1 + ib_1, a_2 + \varepsilon b_2) = |a_1 - a_2| + |b_1| + |b_2|$$

for all $a_1 + ib_1 \in \mathbb{C}$ and $a_2 + \varepsilon b_2 \in \mathbb{D}$ where i is the imaginary unit, ε is the dual unit and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then $(\mathbb{C}, \mathbb{D}, d)$ is a complete bipolar metric space. Define the function $\varphi : (\mathbb{C}, \mathbb{D}) \rightrightarrows \mathbb{R}^+$ as $\varphi(a + ib) = \varphi(a + \varepsilon b) = 3\,(|a| + |b|)$. Given a contravariant map $T:(\mathbb{C},\mathbb{D})\mathrel{\nearrow\mkern-18mu\searrow}(\mathbb{C},\mathbb{D})$, with $T(a+ib)=\frac{b}{2}+\varepsilon\frac{a}{2}$ for all $a+ib\in\mathbb{C}$ and $T(a+\varepsilon b)=\frac{b}{2}+i\frac{a}{2}$ for all $a+\varepsilon b\in\mathbb{D}$. Then, we observe that

$$d(a+ib,T(a+ib))=|a-\tfrac{b}{2}|+|b|+|\tfrac{a}{2}|\leq\tfrac{3}{2}(|a|+|b|)=\varphi(a+ib)-\varphi(T(a+ib))$$
$$d(T(a+\varepsilon b),a+\varepsilon b)=|\tfrac{b}{2}-a|+|\tfrac{a}{2}|+|b|\leq\tfrac{3}{2}(|a|+|b|)=\varphi(a+\varepsilon b)-\varphi(T(a+\varepsilon b))$$

Thus by Theorem 3.4, T has a fixed point. We observe that, $0\in\mathbb{C}\cap\mathbb{D}=\mathbb{R}$ is a fixed point of T.

Now we give two generalizations of Downing-Kirk's fixed point theorem [14] on bipolar metric spaces.

Corollary 3.6. *Let (X,Y,d) and (X',Y',d') be complete bipolar metric spaces and $S:(X,Y,d)\rightrightarrows(X',Y',d')$ be a covariant map with closed graph. Given a contravariant map $T:(X,Y)\mathrel{\nearrow\mkern-18mu\searrow}(X,Y)$ and a lower semi-continuous function $\varphi:(S(X),S(Y))\rightrightarrows\mathbb{R}^+$, such that there exists a constant $\lambda>0$ satisfying*

$$\max\{d(x,Tx),\lambda d'(Sx,STx)\}\leq\varphi(Sx)-\varphi(STx),\ \textit{for all } x\in X,\tag{3.19a}$$
$$\max\{d(Ty,y),\lambda d'(STy,Sy)\}\leq\varphi(Sy)-\varphi(STy),\ \textit{for all } y\in Y.\tag{3.19b}$$

Then T has a fixed point.

Corollary 3.6 can be deduced from Theorem 3.4 by letting $e:X\times Y\to\mathbb{R}^+$, $e(x,y)=\max\{d(x,y),\lambda d'(Sx,Sy)\}$, which gives a complete bipolar metric on (X,Y), and defining a lower semi-continuous mapping $\psi:(X,Y)\rightrightarrows\mathbb{R}^+$ with $\psi(z)=\phi(Sz)$ for each $z\in X\cup Y$.

Another corollary of Theorem 3.4 can be given as follows.

Corollary 3.7. *Let (X,Y,d) and (X',Y',d') be complete bipolar metric spaces and $S:(X,Y,d)\mathrel{\nearrow\mkern-18mu\searrow}(X',Y',d')$ be a covariant map with closed graph. Given a contravariant map $T:(X,Y)\mathrel{\nearrow\mkern-18mu\searrow}(X,Y)$, a lower semi-continuous function $\varphi:(S(X),S(Y))\rightrightarrows\mathbb{R}^+$ and a constant $\lambda>0$ such that*

$$\max\{d(x,Tx),\lambda d'(STx,Sx)\}\leq\varphi(Sx)-\varphi(STx),\ \textit{for all } x\in X,\tag{3.20a}$$
$$\max\{d(Ty,y),\lambda d'(Sy,STy)\}\leq\varphi(Sy)-\varphi(STy),\ \textit{for all } y\in Y.\tag{3.20b}$$

Then T has a fixed point.

Example 3.8. Let $L_n(\mathbb{R})$ and $U_n(\mathbb{R})$ be the sets of all $n\times n$ lower triangular matrices over $\mathbb{R}$ and all $n\times n$ upper triangular matrices over $\mathbb{R}$, respectively. Define a function $d:L_n(\mathbb{R})\times U_n(\mathbb{R})\to\mathbb{R}^+$ as

$$d(A,B)=\sum_{i=1}^{n}|a_{ii}-b_{ii}|+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|a_{ij}|+|b_{ji}|$$

for all $A\in L_n(\mathbb{R})$, $B\in U_n(\mathbb{R})$, where $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$. Then $(L_n(\mathbb{R}),U_n(\mathbb{R}),d)$ becomes a complete bipolar metric space. Also, given a contravariant map $T:(L_n(\mathbb{R}),U_n(\mathbb{R}))\mathrel{\nearrow\mkern-18mu\searrow}(L_n(\mathbb{R}),U_n(\mathbb{R}))$ defined as

$$TM=\left(\frac{m_{ji}+\delta_{ij}}{i+j}\right)_{n\times n}\quad\text{for all } M=(m_{ij})_{n\times n}\in L_n(\mathbb{R})\cup U_n(\mathbb{R})$$

where δ_{ij} denotes the Kronecker delta, and a covariant map $S : (L_n(\mathbb{R}), U_n(\mathbb{R}), d) \rightrightarrows (\mathbb{R}, e)$ defined as

$$SM = \sum_{i,j=1}^{n} \left| m_{ij} - \delta_{ij} \frac{m_{ij}+1}{i+j} \right| \quad \text{for all } M = (m_{ij})_{n\times n} \in L_n(\mathbb{R}) \cup U_n(\mathbb{R}),$$

where e is the Euclidean metric on $\mathbb{R}$, considered as a bipolar metric on $(\mathbb{R}, \mathbb{R})$. Noting that $(M_n) \to N$ on $(L_n(\mathbb{R}), U_n(\mathbb{R}), d)$ and $(SM_n) \to v$ on $\mathbb{R}$ implies $SN = v$, we observe that S has closed graph, and taking into account that $S(L_n(\mathbb{R}) \cup U_n(\mathbb{R})) = \mathbb{R}^+$, we consider the lower semi-continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $\varphi(x) = 2x+1$.

Now, we get for all $A = (a_{ij})_{n\times n} \in L_n(\mathbb{R})$

$$\begin{aligned} d(A, TA) &= d\left(\left(a_{ij}\right)_{n\times n}, \left(\frac{a_{ji}+\delta_{ij}}{i+j}\right)_{n\times n} \right) \\ &= \sum_{i=1}^{n} \left| a_{ii} - \frac{a_{ii}+\delta_{ii}}{i+i} \right| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |a_{ij}| + \left| \frac{a_{ij}+\delta_{ji}}{j+i} \right| \\ &= \sum_{i=1}^{n} \left| a_{ii} - \frac{a_{ii}+1}{2i} \right| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |a_{ij}| + \left| \frac{a_{ij}}{i+j} \right| \\ &= \sum_{i=1}^{n} \left| \frac{2i-1}{2i} a_{ii} - \frac{1}{2i} \right| + \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} \frac{i+j+1}{i+j} |a_{ij}| \end{aligned}$$

since $a_{ij} = 0$ for $i < j$, and

$$\begin{aligned} \varphi(SA) - \varphi(STA) &= 2 \sum_{i,j=1}^{n} \left| a_{ij} - \delta_{ij} \frac{a_{ij}+1}{i+j} \right| - 2 \sum_{i,j=1}^{n} \left| \frac{a_{ji}+\delta_{ij}}{i+j} - \delta_{ij} \frac{\frac{a_{ji}+\delta_{ij}}{i+j}+1}{i+j} \right| \\ &= 2 \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} \frac{i+j-1}{i+j} |a_{ij}| + 2 \sum_{i=1}^{n} \frac{2i-1}{2i} \left(\left| \frac{2i-1}{2i} a_{ii} - \frac{1}{2i} \right| \right). \end{aligned}$$

Since $i, j \geq 1$, $d(A, TA) \leq \varphi(SA) - \varphi(STA)$ and in particular $\varphi(SA) - \varphi(STA) \geq 0$. Moreover, we have $d(SA, STA) = |SA - STA| = \frac{1}{2}(\varphi(SA) - \varphi(STA))$, so that $\max\{d(A, TA), \lambda e(SA, STA)\} \leq \varphi(SA) - \varphi(STA)$ for all $A \in L_n(\mathbb{R})$ and $\lambda = 2 > 0$. Similarly, it can be shown that $\max\{d(TB, B), \lambda e(STB, SB)\} \leq \varphi(SB) - \varphi(STB)$ for all $B \in U_n(\mathbb{R})$. In this way, Corollary 3.6 implies that T has a fixed point. It can be observed that $C = \left(\frac{\delta_{ij}}{i+j-1}\right)_{n\times n} \in L_n(\mathbb{R}) \cap U_n(\mathbb{R})$ is such a point.

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