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SOLUTION TO SECOND ORDER DIFFERENTIAL EQUATIONS VIA F_w-CONTRACTIONS

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Abstract. In this article, we introduce the notions of F-contractions and Hardy-Rogers type F-contractions via w-distances in the backdrop of an orthogonal metric space. After this, we prove some fixed point results concerning the said kind of contractions by taking a weaker version of completeness of the underlying space instead of completeness. Further, we employ the results to obtain some existence and uniqueness criteria of the solution(s) to a certain type of second order initial value and boundary value problems. Along with these, we illustrate some numerical examples to interpret our achieved fixed point results.
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1. Technical preliminaries

The metric fixed point theory has inspired many researchers since the emergence of *Banach Contraction Principle* in 1922. Therefore, many interesting works are there in existing literature involving various types of contractive conditions and abstract spaces, see [2, 3, 4, 5, 7, 8, 10, 12, 18]. Of late, a very novel generalization of the above-mentioned principle is proposed by Gordji et al. [9]. They firstly came up with the notion of an orthogonal set and further, obtained the extension of Banach fixed point result for such kind of sets. We here put down the formal definition of an orthogonal set.

Definition 2.1. Suppose that X is a non-empty set and $\perp$ is a binary relation on X. If there exists $x_0 \in X$ such that

$$(\forall y \in X)x_0 \perp y \vee (\forall y \in X)y \perp x_0,$$

then the element x_0 is said to be an orthogonal element and $(X, \perp)$ is said to be an orthogonal set (O-set). An O-set can possess more than one orthogonal element.

Definition 2.2. In an orthogonal set $(X, \perp)$, any two elements $x, y \in X$ are called orthogonally related if $x \perp y$.

Suppose that $(X, \perp)$ is an orthogonal set equipped with a metric d on X. Then $(X, \perp, d)$ is termed as an orthogonal metric space (O-metric space). In [9], the authors also introduced the allied notions of sequences, completeness and continuity for such metric structures, which we omit here. In fact, we refer the reader to [1, 19] for more notations, terminologies and results on orthogonal metric setting.

On the other hand, in 1996, Kada et al. [11] proposed the concept of a w-distance in usual metric spaces and derived some interesting results using this notion. The definition of w-distance is as follows.

Definition 2.3. Assume that (X, d) is a metric space. A map $q : X \times X \to [0, \infty)$ is termed as a w-distance when the following hold:

- $(w1)$ $q(x, z) \leq q(x, y) + q(y, z)$ for any $x, y, z \in X$;
- $(w2)$ for every $x \in X$, $q(x, .) : X \to [0, \infty)$ is lower semi-continuous;
- $(w3)$ for each $\epsilon > 0$, there is a $\delta > 0$ such that if $q(z, x) \leq \delta$ and $q(z, y) \leq \delta$, then $d(x, y) \leq \epsilon$.

For further examples and references on such generalized distances, keen readers are referred to [13, 14, 21, 25]. Afterwards, Senapati et al. [22] revised the definition of w-distances for the setting of an orthogonal metric space.

Definition 2.4. Assume that $(X, \perp, d)$ is an O-metric space. Consider the mapping $q : X \times X \to [0, \infty)$ which satisfies the succeeding hypotheses:

- $(w1')$ $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;
- $(w2')$ for each $x \in X$, $q(x, .) : X \to [0, \infty)$ is O-lower semi-continuous;
- $(w3')$ for any $\epsilon > 0$, there is a $\delta > 0$ with $q(z, x) \leq \delta$ and $q(z, y) \leq \delta$ imply that $d(x, y) \leq \epsilon$.

Then q is said to be a w-distance on X. The authors also revised the following lemma in orthogonal metric context.

Lemma 2.5. *Suppose that q is a w-distance defined on an O-metric space $(X, \perp, d)$. Also suppose that (x_n) and (y_n) are two O-sequences in X and $x, y, z \in X$. Let (u_n) and (v_n) be sequences of positive reals which converge to 0. Subsequently, the following hold:*

- (i) *If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq v_n$, then $y = z$. Besides, when $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.*
- (ii) *If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq v_n$, then $y_n \to z$ as $n \to \infty$.*
- (iii) *If $q(x_n, x_m) \leq u_n$ for every $m > n$, then (x_n) is a Cauchy O-sequence in X.*
- (iv) *If $q(x_n, y) \leq u_n$, then (x_n) is a Cauchy O-sequence in X.*

Recently, Wardowski [23] introduced the notion of an F-contraction, in the framework of usual metric space and established a fixed point result involving the said contractions. Here we note down the definition first.

Definition 2.6. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping which holds the following:

$(F1)$ F is strictly increasing, that is, for each $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

$(F2)$ for every sequence (γ_n) of positive reals, $\lim_{n\to\infty} \gamma_n = 0$ if and only if $\lim_{n\to\infty} F(\gamma_n) = -\infty$;

$(F3)$ there is $k \in (0, 1)$ so that $\lim_{\gamma\to 0^+} \gamma^k F(\gamma) = 0$.

We denote the collection all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfies $(F1), (F2)$ and $(F3)$ by $\mathfrak{F}$. Now we recall the following:

Definition 2.7. A self-map f on a metric space (X, d) is said to be an F-contraction if there exists $\tau > 0$ such that for $x, y \in X$

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)),$$

where $F \in \mathfrak{F}$.

Many other results related to the F-contractions can be found in [6, 15, 16, 17, 20, 24] and the references therein.

In the existing literature of the metric fixed point theory, one can observe that the fixed point results related to F-contractions and w-distances require the underlying space to be complete. So naturally a question comes that whether we can obtain some fixed point results concerning F-contractions and w-distances in the context of a metric space which is not necessarily complete. Since an O-complete metric space need not to be complete, results involving F-contractions and w-distances in the context of O-complete metric spaces can provide an affirmative answer to the aforementioned question. Keeping all these facts in mind, we propose the notions of F_w-contractions and Hardy-Rogers type F_w-contractions in the setting of an orthogonal metric space. In addition to that, we establish a pair of fixed point results involving such kind of contractions in O-complete metric spaces. Moreover, we make use of the results to investigate the possibility of existence a unique solution for a class of initial value problems and another class of boundary value problems. Along with these, we construct a couple of non-trivial numerical examples to support our obtained results.

2. Fixed point results

At the beginning of this section, we propose the notions of an orthogonal F_w-contraction and orthogonal F_w-contraction of Hardy-Rogers type in orthogonal metric spaces.

Definition 3.1. Assume that p is a w-distance defined on an orthogonal metric space $(X, \perp, d)$. A self-map T on X is said to be an orthogonal F_w-contraction if there is a function $F \in \mathfrak{F}$ such that for all $x, y \in X$ with $x \perp y$, the following hold:

(i) $p(x, y) = 0 \Rightarrow p(Tx, Ty) = 0$;

(ii) there exists a real number $\tau > 0$ such that

$$p(Tx, Ty) > 0 \Rightarrow \tau + F(p(Tx, Ty)) \leq F(p(x, y)).$$

If the condition (ii) is replaced by

(ii') there exists a number $\tau > 0$ with

$$\tau + F(p(Tx, Ty)) \leq F(\alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty) + \delta p(x, Ty) + Lp(y, Tx))$$

for any two orthogonally related elements $x, y \in X$ with $p(Tx, Ty) > 0$, where $\alpha, \beta, \gamma, \delta$ are positive real numbers such that $\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ and $L \geq 0$, then T is called an orthogonal F_w-contraction of Hardy-Rogers type.

To begin with, we derive the subsequent theorem regarding the existence and uniqueness of a fixed point of orthogonal F_w-contractions.

Theorem 3.2. *Assume that p is a w-distance defined on an O-complete metric space $(X, \perp, d)$. Also let $T : X \to X$ be an O-continuous, $\perp$-preserving, orthogonal F_w-contraction. Then*

(i) *T owns a unique fixed point $\tilde{x}$.*
(ii) *$(T^n x)$ converges to $\tilde{x}$ for every $x \in X$.*

Proof. (i) Suppose that x_0 is an orthogonal element in X. Then we have

$$(\forall y \in X) x_0 \perp y \vee (\forall y \in X) y \perp x_0.$$

Let us now define the sequence (x_n) by setting $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since x_0 is an orthogonal element of X and T is an $\perp$-preserving map, (x_n) is an O-sequence, that is,

$$(\forall n \in \mathbb{N}) x_n \perp x_{n+1} \vee (\forall n \in \mathbb{N}) x_{n+1} \perp x_n.$$

We first assume that

$$p(x_{k-1}, x_k) = 0 \tag{2.1}$$

for some $k \in \mathbb{N}$. Then from condition (i) of Definition 3.1, we have

$$p(x_k, x_{k+1}) = 0. \tag{2.2}$$

By the triangular condition of a w-distance, we have

$$p(x_{k-1}, x_{k+1}) \leq p(x_{k-1}, x_k) + p(x_k, x_{k+1}).$$

Here, using (2.1) and (2.2), it follows that

$$p(x_{k-1}, x_{k+1}) = 0. \tag{2.3}$$

Therefore from (2.1), (2.3) and by Lemma 2.5, we have $x_k = x_{k+1}$, i.e., x_k is a fixed point of T.

Next, we assume that $p(x_{n+1}, x_n) > 0$ for each $n \in \mathbb{N}$. Since, T is an orthogonal F_w-contraction and (x_n) is an O-sequence, it follows that

$$\tau + F(p(x_n, x_{n+1})) \leq F(p(x_{n-1}, x_n)). \tag{2.4}$$

Let $\gamma_n = p(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then (γ_n) is a sequence of non-negative real numbers. Then from (2.4), we have

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq \cdots \leq F(\gamma_0) - n\tau, \tag{2.5}$$

which implies that

$$\lim_{n \to \infty} F(\gamma_n) = -\infty. \tag{2.6}$$

By $(F2)$ and (2.6), we get

$$\lim_{n \to \infty} \gamma_n = 0. \tag{2.7}$$

Again by condition $(F3)$, there is a real number $k \in (0,1)$ with

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{2.8}$$

Now from (2.5), we have

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq \gamma_n^k (F(\gamma_0) - n\tau) - \gamma_n^k F(\gamma_0) = -n\gamma_n^k \tau \leq 0. \tag{2.9}$$

Employing (2.7), (2.8) and (2.9), we can conclude that

$$\lim_{n\to\infty} n\gamma_n^k = 0.$$

Therefore there is a natural number n_0 with

$$\gamma_n < \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_0.$$

Hence the infinite series $\sum_{n=1}^{\infty} \gamma_n$ is convergent. Since p is a w-distance, for all $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &= \gamma_n + \gamma_{n+1} + ... + \gamma_{m-1} < \alpha_n, \end{aligned} \tag{2.10}$$

where $\alpha_n = \sum_{i=n}^{\infty} \gamma_i$. Thus (α_n) is a sequence of positive real numbers which converges to 0. Therefore, (x_n) is a Cauchy O-sequence in X. By the O-completeness of X, there is $\tilde{x} \in X$ with

$$\lim_{n\to\infty} x_n = \tilde{x}.$$

By the O-continuity of T, we get

$$d(\tilde{x}, T\tilde{x}) = \lim_{n\to\infty} d(x_{n+1}, T\tilde{x}) = \lim_{n\to\infty} d(Tx_n, T\tilde{x}) = d(T\tilde{x}, T\tilde{x}) = 0.$$

This shows that $\tilde{x}$ is a fixed point of T. Next, we check for the uniqueness of the fixed point. If possible, let $\tilde{x}$ and $\tilde{y}$ be two distinct fixed points of T. Then we have

$$(x_0 \perp \tilde{x} \wedge x_0 \perp \tilde{y}) \vee (\tilde{x} \perp x_0 \wedge \tilde{y} \perp x_0).$$

As T is $\perp$-preserving, it follows that for every $n \in \mathbb{N}$, we have

$$(T^n x_0 \perp \tilde{x} \wedge T^n x_0 \perp \tilde{y}) \vee (\tilde{x} \perp T^n x_0 \wedge \tilde{y} \perp T^n x_0).$$

Since T is an orthogonal F_w-contraction, we have

$$F(p(T^n x_0, T^n \tilde{x})) \leq F(p(x_0, \tilde{x})) - n\tau. \tag{2.11}$$

Letting n tending to ∞ in (2.11), we get

$$\lim_{n\to\infty} F(p(x_n, \tilde{x})) = -\infty.$$

By $(F2)$, it follows that

$$\lim_{n\to\infty} p(x_n, \tilde{x}) = 0. \tag{2.12}$$

In a similar manner, it can be proved that

$$\lim_{n\to\infty} p(x_n, \tilde{y}) = 0. \tag{2.13}$$

Now we have

$$p(T^n x_0, T^n \tilde{x}) \leq p(T^n x_0, T^n \tilde{x}) + \frac{1}{n} \tag{2.14}$$

and

$$p(T^n x_0, T^n \tilde{y}) \leq p(T^n x_0, T^n \tilde{y}) + \frac{1}{n}. \tag{2.15}$$

Therefore using (2.14), (2.15) and Lemma 2.5, we get $\tilde{x} = \tilde{y}$. This proves the uniqueness of the fixed point.

(ii) Let $x \in X$ be arbitrary. Then we have

$$x_0 \perp x \vee x \perp x_0.$$

Since T is $\perp$-preserving, we have

$$(T^n x_0 \perp T^n x) \vee (T^n x \perp T^n x_0).$$

Again, as T is an orthogonal F_w-contraction, it follows that

$$F(p(T^n x_0, T^n x)) \leq F(p(x_0, x)) - n\tau.$$

Thus we get

$$\lim_{n \to \infty} F(p(T^n x_0, T^n x)) = -\infty,$$

which implies that

$$\begin{aligned} \lim_{n \to \infty} p(T^n x_0, T^n x) &= 0 \\ \Rightarrow \lim_{n \to \infty} p(x_n, T^n x) &= 0. \end{aligned} \tag{2.16}$$

Hence using (2.12), (2.16) and Lemma 2.5, we get $\lim_{n \to \infty} T^n x = \tilde{x}$. Since $x \in X$ was chosen arbitrarily, it follows that the sequence $(T^n x)$ converges to the fixed point for any $x \in X$.

In Theorem 3.2, we can replace the continuity of T by some other suitable condition. We discuss this in the following theorem.

Theorem 3.3. *Let a w-distance p is defined on an O-complete metric space $(X, \perp, d)$. Also suppose that $T : X \to X$ is a $\perp$-preserving, orthogonal F_w-contraction and there exists an orthogonal element $x_0 \in X$ such that the iterative sequence $(T^n x_0)$ converges to $\tilde{x}$ and*

$$(\forall n \in \mathbb{N}) T^n x_0 \perp \tilde{x} \vee (\forall n \in \mathbb{N}) \tilde{x} \perp T^n x_0.$$

Then

(i) T owns a unique fixed point $\tilde{x}$.
(ii) $(T^n x)$ converges to $\tilde{x}$ for every $x \in X$.

Proof. Proceeding in a similar way to that of Theorem 3.2, we obtain that for any orthogonal element x_0 in X, the iterative sequence (x_n) converges to some $\tilde{x} \in X$. Now, by the lower semi-continuity of p and (2.10), we get

$$p(x_{n+1}, \tilde{x}) \leq \liminf_{m \to \infty} p(x_{n+1}, x_{m+1}) < \alpha_{n+1}$$

where $\alpha_{n+1} = \sum_{i=n+1}^{\infty} \gamma_i$. Taking $n \to \infty$ in the above equation and using the fact that the infinite series $\sum_{n=1}^{\infty} \gamma_n$ is convergent, we get

$$p(x_{n+1}, \tilde{x}) = 0.$$

Since $T^n x_0 \perp \tilde{x}$, by $(F1)$ and taking into account the lower semi-continuity of p and (2.10), we get

$$p(T^{n+1}x_0, T\tilde{x}) < p(x_n, \tilde{x}) \leq \liminf_{m\to\infty} p(x_n, x_m) < \alpha_n,$$

where $\alpha_n = \sum_{i=n}^{\infty} \gamma_i$. Since the infinite series $\sum_{n=1}^{\infty} \gamma_n$ is convergent, we have

$$p(x_{n+1}, T\tilde{x}) = 0.$$

Therefore we can conclude that $T\tilde{x} = \tilde{x}$. The rest of the claims follow from Theorem 3.2.

Next, we present some examples in support of Theorem 3.2.

Example 3.4. Let us consider the O-complete orthogonal metric space $(X, \perp, d)$, where $X = (0, \infty)$, $d : X \times X \to \mathbb{R}$ is defined by $d(x, y) = |x-y|$ and a binary relation '$\perp$' is defined as $x \perp y$ if and only if $x + y \geq 2$. We take a w-distance

$$p(x,y) = \begin{cases} x+y, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Also, we take $\tau = \ln 2 > 0$ and $F(x) = \ln x$. Next, we define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 1, & \text{if } x \leq 5; \\ \frac{x}{5}, & \text{if } x > 5. \end{cases}$$

Then clearly T is O-continuous, $\perp$-preserving and also $p(x, y) = 0 \Rightarrow p(Tx, Ty) = 0$. Let $x, y \in X$ be such that $p(Tx, Ty) > 0$. Then the following two cases arise.

Case I: Let $x, y > 5$. Then

$$\begin{aligned} \tau + F(p(Tx, Ty)) &= \ln 2 + \ln\left(\frac{x}{5} + \frac{y}{5}\right) \\ &\leq \ln(x+y) = F(p(x,y)). \end{aligned}$$

Case II: Let $x \leq 5$ and $y > 5$. Then

$$\begin{aligned} \tau + F(p(Tx, Ty)) &= \ln 2 + \ln\left(1 + \frac{y}{5}\right) \\ &\leq \ln y \leq F(p(x,y)). \end{aligned}$$

Thus T is an orthogonal F_w-contraction. Hence all the assumptions of Theorem 3.2 are satisfied and so T has a unique fixed point. Indeed 2 is the only fixed point of T. Further, note that for any $x \in X$, the sequence $(T^n x)$ converges to 2.

Example 3.5. Let us consider an O-complete orthogonal metric space $(X, \perp, d)$ where $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} x + y, \text{ if } x \neq y; \\ 0, \text{ if } x = y. \end{cases}$$

We also define a binary relation '$\perp$' on X such that, for $x, y \in X$, $x \perp y$ if and only if $xy \leq x$ or $xy \leq y$. We now define a w-distance $q : X \times X \to \mathbb{R}$ by $q(x,y) = y$ for each $x, y \in X$. Next, we consider a self-map $T : X \to X$ by

$$Tx = \begin{cases} 0, \text{ if } x = 0; \\ \frac{1}{n+5}, \text{ if } x = \frac{1}{n} \text{ and } n \leq 5; \\ 0, \text{ if } x = \frac{1}{n} \text{ and } n > 5. \end{cases}$$

Note that if $q(x,y) = 0$, then $q(Tx,Ty) = 0$. Next, we suppose that $x \perp y$ and $q(Tx,Ty) > 0$. Then $y = \frac{1}{n}$ for $n \leq 5$. Also, we take $\tau = \ln\left(\frac{3}{2}\right) > 0$ and $F(x) = \ln x$. Therefore,

$$\begin{aligned} \tau + F(q(Tx,Ty)) - F(q(x,y)) &= \ln\big(\frac{3n}{2n+10}\big) \leq 0 \\ \Rightarrow \tau + F(q(Tx,Ty)) &\leq F(q(x,y)). \end{aligned}$$

It is easy to check that T is $\perp$-preserving and O-continuous. Thus we see that all the conditions of Theorem 3.2 hold good. So employing the result, we conclude that, T has a unique fixed point and we can easily check that 0 is the solitary fixed point of the map.

Example 3.6. Let us consider an O-complete orthogonal metric space $([0,1], \perp, d)$, where d is the usual metric on $[0,1]$ and '$\perp$' be an orthogonal relation on $[0,1]$, defined by $u \perp v$ if and only if $uv \leq u \vee v$, $u, v \in [0,1]$. We now define a function $q : [0,1] \times [0,1] \to \mathbb{R}$ by $q(u,v) = \max\{u,v\}$. One can easily check that q is a w-distance on $[0,1]$. Now we consider the function $T : [0,1] \to \mathbb{R}$ defined by

$$T(u) = \frac{u}{3+u}.$$

If $q(u,v) = 0$, then $q(Tu,Tv) = 0$. Let us take $\tau = \ln 3$ and $F(u) = \ln u$. Then clearly $\tau > 0$ and $F \in \mathcal{F}$. Let $u, v \in [0,1]$ be arbitrary with $u \perp v$ and $q(Tu,Tv) > 0$. Then $u > 0$ or $v > 0$. Let $q(u,v) = u$. Then $u > v$ and so $\frac{u}{3+u} > \frac{v}{2+v}$. Therefore, we obtain

$$\tau + F(q(Tu,Tv)) \leq F(q(u,v)).$$

Similarly if $q(u,v) = v$, then we can show that

$$\tau + F(q(Tu,Tv)) \leq F(q(u,v)).$$

Thus T is an orthogonal F_w-contraction. It is quite easy to check that T is $\perp$-preserving and O-continuous. Hence T satisfies all the hypotheses of Theorem 3.2 and therefore T owns a unique fixed point. Indeed 0 is the unique fixed point of T.

Now we proof the following theorem which is an analogous version of Theorem 3.2 involving orthogonal F_w-contraction of Hardy-Rogers type.

Theorem 3.7. *Suppose that p is a w-distance defined on an O-complete metric space $(X, \perp, d)$. Also suppose that $T : X \to X$ is an O-continuous, $\perp$-preserving orthogonal*

F_w-contraction of Hardy-Rogers type. Assume that there exists an orthogonal element $x_0 \in X$ such that $p(x_0, x_0) = 0$. Then T possesses a fixed point.

Proof. Since x_0 be an orthogonal element in X, we have

$$(\forall y \in X)x_0 \perp y \vee (\forall y \in X)y \perp x_0.$$

Now let us define a sequence (x_n) by setting $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since x_0 is an orthogonal element of X and T is an orthogonal preserving map, (x_n) is an O-sequence, which implies,

$$(\forall n \in \mathbb{N})x_n \perp x_{n+1} \vee (\forall n \in \mathbb{N})x_{n+1} \perp x_n.$$

If $p(x_{k+1}, x_k) = 0$ for some $k \in \mathbb{N}$, then proceeding similarly to the arguments of Theorem 3.2, we can show that x_k is a fixed point of T.

Therefore we can assume that $p(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. Since, T is an orthogonal F_w-contraction of Hardy-Rogers type and $x_n \perp x_{n+1}$, we have

$$\begin{aligned}\tau + F(p(x_n, x_{n+1})) \leq &F(\alpha p(x_{n-1}, x_n) + \beta p(x_{n-1}, x_n) + \gamma p(x_n, x_{n+1}) \\ &+ \delta p(x_{n-1}, x_{n+1}) + Lp(x_n, x_n)).\end{aligned} \tag{2.17}$$

Now, as $p(x_0, x_0) = 0$ using (i) of Definition 3.1, we obtain $p(x_n, x_n) = 0$ for each $n \in \mathbb{N}$. Again, since F is strictly increasing, we have

$$\begin{aligned}\tau + F(p(x_n, x_{n+1})) &\leq F(\alpha p(x_{n-1}, x_n) + \beta p(x_{n-1}, x_n) + \gamma p(x_n, x_{n+1}) \\ &\quad + \delta[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]) \\ \Rightarrow \tau + F(p(x_n, x_{n+1})) &\leq F((\alpha + \beta + \delta)p(x_{n-1}, x_n) + (\gamma + \delta)p(x_n, x_{n+1})).\end{aligned} \tag{2.18}$$

As F is strictly increasing, we get

$$\begin{aligned}p(x_n, x_{n+1}) &\leq (\alpha + \beta + \delta)p(x_{n-1}, x_n) + (\gamma + \delta)p(x_n, x_{n+1}) \\ \Rightarrow p(x_n, x_{n+1}) &\leq \frac{(\alpha + \beta + \delta)}{(1 - \gamma - \delta)}p(x_{n-1}, x_n) \\ \Rightarrow p(x_n, x_{n+1}) &\leq p(x_{n-1}, x_n).\end{aligned}$$

Using the above fact in (2.18), we get

$$\tau + F(p(x_n, x_{n+1})) \leq F(p(x_{n-1}, x_n)).$$

Now the remaining portion of the proof is analogous to that of Theorem 3.2 and so excluded.

3. Applications to differential equations

In this section, we consider two different types of second order differential equations, one is an initial value problem and another is a boundary value problem, and by using our established result we try to find the solutions to those.

Firstly, let us consider the following initial value problem:

$$\begin{cases} \frac{d^2y}{dt^2} + k\frac{dy}{dt} = K(t, y(t)), \ o \leq t \leq I; \\ y(0) = 0, y'(0) = a, \end{cases} \tag{3.1}$$

where $K(t, y(t))$ is a continuous function from $[0, I] \times \mathbb{R}^+$ to $\mathbb{R}^+$, for some $I > 0$.

(The above initial value problem exhibits the engineering problem of activation of spring that is affected by an exterior force). It is quite easy to check that the above initial value problem is equivalent to the following integral equation:

$$y(t) = \int_0^t G(t,s)K(s,y(s))ds, \ t \in [0,I], \tag{3.2}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} (t-s)e^{\tau(t-s)}, & \text{if } 0 \le s \le t \le I; \\ 0, & \text{if } 0 \le t \le s \le I, \end{cases}$$

where $\tau > 0$ is a constant, which depends on the value of k in the initial value problem (3.1). Let $X = C([0,I], \mathbb{R}^+)$ be the set of all continuous functions from $[0,I]$ to $\mathbb{R}^+$. For all functions $y(t) \in X$ we define

$$||y||_\tau = \sup_{t \in [0,I]} \{|y(t)|e^{-2t\tau}\}$$

where $\tau > 0$ is a constant. Let us now consider an orthogonal relation on X by $f \perp g$ if and only if $fg \ge 0$, where $f,g \in X$ and a w-distance $p : X \times X \to [0,\infty)$ by

$$p(x,y) = \max\{\|x\|_\tau, \|y\|_\tau\}$$

for all $f,g \in X$. Now we also consider a function $T : X \to X$ by

$$Ty(t) = \int_0^t G(t,s)K(s,y(s))ds > 0$$

for all $y \in X$ and $t \in [0,I]$. It is quite obvious to note that the fixed point of the mapping T will be the solution of the initial value problem (3.1). In the following theorem we prove the existence and uniqueness of the fixed point of T, when T is an orthogonal F_w-contraction.

Theorem 4.1. *Consider the non-linear integral equation* (3.2) *and assume that the following conditions hold:*

(*i*) *K is an increasing function;*
(*ii*) *there exists $\tau > 0$ such that*

$$|K(s,y)| \le \tau^2 e^{-\tau} y,$$

where $s \in [0,I]$ and $y \in \mathbb{R}^+$;

Then there exists a unique solution to the integral equation.

Proof. Let $x, y \in X$ be arbitrary. We have

$$\begin{aligned}|Ty(s)| &\leq \int_0^t G(t,s)|K(s,y(s))|ds \\ &\leq \int_0^t G(t,s)\tau^2 e^{-\tau}|y(s)|ds \text{ [from the condition } (b)] \\ &\leq \int_0^t (t-s)e^{\tau(t-s)}\tau^2 e^{-\tau}e^{2s\tau}||y||_\tau ds \\ &= \tau^2||y||_\tau e^{-\tau+t\tau}\int_0^t (t-s)e^{s\tau}ds \\ &= ||y||_\tau e^{2\tau+t\tau}[1-\tau te^{-t\tau}-e^{-t\tau}].\end{aligned}$$

Since $(1-\tau te^{-t\tau}-e^{-t\tau}) \leq 1$, we have

$$||Ty||_\tau \leq e^{-\tau}||y||_\tau \leq e^{-\tau}\max\{\|x\|_\tau, \|y\|_\tau\}.$$

Similarly we can show that

$$\|Tx\|_\tau \leq e^{-\tau}\max\{\|x\|_\tau, \|y\|_\tau\}.$$

Thus

$$\begin{aligned}&\max\{\|Tx\|_\tau, \|Ty\|_\tau\} \leq e^{-\tau}\max\{\|x\|_\tau, \|y\|_\tau\} \\ &\Rightarrow \tau + \ln(p(Tx,Ty)) \leq \ln(p(x,y)).\end{aligned}$$

Now we take $F \in \mathcal{F}$, defined by $F(\beta) = \ln \beta$, then we obtain

$$\tau + F(p(Tx(t), Ty(t)) \leq F(p(x(t), y(t))).$$

This shows that T is a $\perp$-preserving, O-continuous, orthogonal F_w-contraction. Consequently, from Theorem 3.2, it can be concluded that the self-map has a unique solution, i.e., the integral equation possesses a unique solution.
Subsequently, let us consider the following second order differential equation

$$\begin{cases} y''(t) + y'(t) + f(t, y(t)) = 0, \ 0 \leq t \leq \frac{1}{3} \\ y(0) = a, \ y\left(\frac{1}{3}\right) = b. \end{cases} \tag{3.3}$$

One can comfortably obtain the following theorem which deals with the equivalence between the previous boundary value problem and an integral equation.

Theorem 4.2. *The boundary value problem* (3.3) *is equivalent to the following integral equation:*

$$y(t) = a + 3bt - 3at + \int_0^{\frac{1}{3}} 3ty(x)dx - \int_0^t y(x)dx + \int_0^{\frac{1}{3}} G(x,t)f(x,y(x))dx, \tag{3.4}$$

where

$$G(x,t) = \begin{cases} x - 3xt, & \text{if } x \leq t; \\ t - 3xt, & \text{if } x > t. \end{cases}$$

Now we prove the following result which plays a crucial role in our further findings.

Lemma 4.3. *For the function $G(x,t)$ of Theorem 4.2, we have, $G(x,t) \geq 0$ for all $x, t \in [0, \frac{1}{3}]$ and* $\sup_{t\in[0,\frac{1}{3}]} \int_0^{\frac{1}{3}} G(x,t)dx = \frac{1}{72}.$

Proof. Since $x, t \in \left[0, \frac{1}{3}\right]$, it follows that $1 - 3t \geq 0$ and $1 - 3x \geq 0$ and hence $G(x,t) \geq 0$ for all $x, t \in \left[0, \frac{1}{3}\right]$. Again we can easily calculate that

$$\int_0^{\frac{1}{3}} G(x,t)dx = \frac{t}{6} - \frac{t^2}{2}.$$

Therefore, $\int_0^{\frac{1}{3}} G(x,t)dx$ attains its maximum value at $t = \frac{1}{6}$ and the maximum value is $\frac{1}{72}$.

Theorem 4.4. *Assume that the following conditions hold:*

(*i*) *$f(x, y(x))$ is continuous;*

(*ii*) *$|f(x, y_1(x)) - f(x, y_2(x))| \leq K|y_1(x) - y_2(x)|$ where K is a positive constant such that there exists a positive real number $\alpha > 0$ satisfying $\frac{2}{3} + \frac{K}{72} \leq \alpha < 1$.*

Then the boundary value problem (3.3) *has a unique solution.*

Proof. Let us consider the metric space $C[0, \frac{1}{3}]$ of all real-valued continuous functions defined on $[0, \frac{1}{3}]$ endowed with the sup norm metric d. We define a relation $\perp$ on $C[0, \frac{1}{3}]$ by the following rule: for $x,\ y \in C[0, \frac{1}{3}]$, $x \perp y$ if and only if $x \cdot y$ is continuous on $[0, \frac{1}{3}]$. If we take $x_0(t) = 1$ for each $t \in [0, \frac{1}{3}]$, then $x \perp x_0$ for every $x \in C[0, \frac{1}{3}]$. Therefore, $\left(C[0, \frac{1}{3}], d, \perp\right)$ is an orthogonal metric space. Also we know that the metric space $\left(C[0, \frac{1}{3}], d\right)$ is a complete metric space. We take $F(x) = \ln(x)$ and $\tau = -\ln\alpha$, then clearly $F \in \mathcal{F}$ and $\tau > 0$. Next, we consider a function $T : C[0, \frac{1}{3}] \to C[0, \frac{1}{3}]$ by

$$Ty(t) = a + 3bt - 3at + \int_0^{\frac{1}{3}} 3ty(x)dx - \int_0^t y(x)dx + \int_0^{\frac{1}{3}} G(x,t)f(x, y(x))dx \quad (3.5)$$

for all $y(t) \in C[0, \frac{1}{3}]$, where $G(x,t)$ is given by Theorem 4.2. Then, it is clear that to find a solution of boundary value problem (3.3) is equivalent to find a fixed point of T. Now for any $y_1,\ y_2 \in C[0, \frac{1}{3}]$ and $t \in [0, \frac{1}{3}]$, we have

$$\begin{aligned}
Ty_1(t) - Ty_2(t) &= \int_0^{\frac{1}{3}} 3t(y_1(x) - y_2(x))dx - \int_0^t (y_1(x) - y_2(x))dx \\
&+ \int_0^{\frac{1}{3}} G(x,t)(f(x, y_1(x)) - f(x, y_2(x)))dx \\
\Rightarrow |Ty_1(t) - Ty_2(t)| &\leq \int_0^{\frac{1}{3}} 3t|(y_1(x) - y_2(x))|dx + \int_0^t |(y_1(x) - y_2(x))|dx \\
&+ \int_0^{\frac{1}{3}} G(x,t)|(f(x, y_1(x)) - f(x, y_2(x)))|dx \\
&\leq d(y_1, y_2) \int_0^{\frac{1}{3}} 3tdx + d(y_1, y_2) \int_0^t dx \\
&+ \int_0^{\frac{1}{3}} KG(x,t)|y_1(x) - y_2(x)|dx \\
&\leq td(y_1, y_2) + td(y_1, y_2) + Kd(y_1, y_2) \int_0^{\frac{1}{3}} G(x,t)dx.
\end{aligned}$$

Therefore,

$$\sup_{0\le t\le \frac{1}{3}} |Ty_1(t) - Ty_2(t)| \le \sup_{0\le t\le \frac{1}{3}} 2td(y_1,y_2) + Kd(y_1,y_2) \sup_{0\le t\le \frac{1}{3}} \int_0^{\frac{1}{3}} G(x,t)dx.$$

Using Lemma 4.3 in above equation, we get

$$\begin{aligned} d(Ty_1,Ty_2) &\le \left(\frac{2}{3}+\frac{K}{72}\right) d(y_1,y_2)\\ \Rightarrow d(Ty_1,Ty_2) &\le \alpha d(y_1,y_2). \end{aligned} \tag{3.6}$$

Now we define a function $p : C[0,\frac{1}{3}] \times C[0,\frac{1}{3}] \to \mathbb{R}$ by $p(x,y) = d(x,y)$ for all $x,y \in C[0,\frac{1}{3}]$. Then p is a w-distance on $C[0,\frac{1}{3}]$ and $p(x,y) = 0 \Rightarrow p(Tx,Ty) = 0$. Let $y_1(t), y_2(t) \in C[0,\frac{1}{3}]$ be arbitrary with $y_1 \perp y_2$ and $p(Ty_1,Ty_2) > 0$. Then using (3.6), we get

$$\begin{aligned} p(Ty_1,Ty_2) &\le \alpha p(y_1,y_2)\\ \Rightarrow -\ln(\alpha) + \ln(p(Ty_1,Ty_2)) &\le \ln(p(y_1,y_2))\\ \Rightarrow \tau + F(p(Ty_1,Ty_2)) &\le F(p(y_1,y_2)). \end{aligned}$$

Further, T is $\perp$-preserving and O-continuous. Thus we see that all the conditions of Theorem 3.2 hold good and so by that theorem, T has a unique fixed point in $C[0,\frac{1}{3}]$ and hence the boundary value problem (3.3) has a unique solution.

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