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SOLVABILITY OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER IN REFLEXIVE BANACH SPACE

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Abstract. In this work, we are concerned with weak and pseudo solutions for some initial value problems of fractional order and their corresponding functional integral equation of fractional order. These initial value problems includes many initial value problems that arise in nonlinear analysis and its applications.
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1. Introduction and Preliminaries

Let E be a reflexive Banach space with norm $\|\cdot\|$ and its dual E^*. Let $L^1(I)$ be the space of Lebesgue integrable functions on an interval I.
Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm $\|.\|_0$.
The study of weak solutions for Cauchy problems in reflexive Banach spaces was initiated by, among others, Szep [26], Chow and Schur[3]. However, if E is nonreflexive Banach space the situation is quite different (see [1] and [18]).
In 2005, the existence of weakly differentiable solutions for the initial value problem

$$x'(t) = f(t, D^\gamma x(t)), x(0) = x_0, t \in [0, 1] \tag{1.1}$$

in reflexive Banach spaces has been considered, for the first time, by Salem and El-Sayed [25].
For further existence results of solutions for some integral and differential equations

in Banach spaces (see [5], [4], [12], [10] [17] and [19]-[26] and references therein).
In this paper, we shall study the existence of weak and pseudo solutions for the initial value problem

$$\begin{aligned} &\frac{dx}{dt} = f(t, g(t, D^{\gamma}x(t))),\ \gamma \in (0,1),\ t \in I = [0,T] \\ &x(0) = x_0. \end{aligned} \tag{1.2}$$

which generalizes various known results (see [5],[6], [7],[9], [27] and [25]).
Our consideration will be discussed in reflexive Banach space using O'Regan fixed point theorem, we present two approaches under two sequences of assumptions on f and g.
Now, let us recall the following basic definitions and propositions [25], [26] which will be needed further on.

Proposition 1.1. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 1.2. *Let E be a normed space with $x_0 \neq 0$. Then there exists a $\phi \in E^*$ with $\|\phi\| = 1$ and $\phi(x_0) = \|x_0\|$.*

Definition 1.3. A function $x(.)$ is said to be pseudo-differentiable on I to a function $y(.)$ if for every $\phi \in E^*$, there exists a null set $N(\phi)$ (i.e., N is depending on ϕ and $mes(N(\phi) = 0)$ such that the real function $t \to \phi(x(t))$ is differentiable a.e. on I and

$$\phi(x'(t)) = \phi(y(t)), t \in I \backslash N(\phi).$$

The function $y(.)$ is called a pseudo-derivative of $x(.)$

Proposition 1.4. [23] *Let $x(.) : I \to E$ be a weakly measurable function.*

(*A*) *If $x(.)$ is Pettis integrable on I, then the indefinite Pettis integral*

$$y(t) = \int_0^t x(s)ds, t \in I,$$

is absolutely continuous on I and $x(.)$ is a pseudo-derivative of $y(.)$.

(*B*) *If $y(.)$ is an absolutely continuous function on I and it has a pseudo-derivative $x(.)$ on I, then $x(.)$ is Pettis integrable on I and*

$$y(t) = y(0) + \int_0^t x(s)ds, t \in I.$$

Lemma 1.5. [20], [23] *The integral of weakly continuous (Pettis integrable)function is weakly (absolutely continuous and pseudo) differentiable with respect to the right end-point of the integration interval and its weak (pseudo) derivative equals the integrand at that point.*

Definition 1.6. [24] Let $x : I \to E$. The fractional Pettis integral of x of order $\alpha > 0$ is defined by

$$I^{\alpha}x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, t > 0.$$

In the above definition the sign " $\int$ " denotes the Pettis integral.

Salem and Cichoń [24] observed that such an integral

$$I^{\alpha}x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

is a convolution of a function $h(\tau) = \tau^{\alpha-1}/\Gamma(\alpha)$ for $\tau > 0, h(\tau) = 0$ for $\tau \leq 0$, and the function $(\widetilde{x})(t) = x(t)$ for $t \in I$, where $(\widetilde{x})(t) = 0$ outside the interval I. Note that Pettis integrability of $x(t)$ implies Pettis integrability of $x(t+h)(h > 0)$ and $x(-t)$, so the convolution of Pettis integrable function with real-valued function h can be properly defined. We start with an obvious observation that for $\phi \in E^*$

$$\phi(I^{\alpha}x(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(x(s))ds. \tag{1.3}$$

For more properties of fractional order integral in Banach spaces (see [25] and [24]).

Also, we have the following fixed point theorem, due to O'Regan, in reflexive Banach space (see [21]).

Theorem 1.7. *(O'Regan fixed point theorem) Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

2. Existence theorems

In the light of two sequences of assumptions on the functions f and g, we proceed to prove two existence theorems of solutions to the initial value problem (1.2).

The question of proving the existence of solutions to the initial value problem (1.2) reduces to proving the existence of solutions to a functional integral equation of fractional order or proving the existence of solutions to a coupled system.

Consider the initial value problem (1.2). Operating by $I^{1-\gamma}$ on both sides we obtain

$$D^{\gamma}x(t) = I^{1-\gamma}f(t, g(t, D^{\gamma}x(t)).$$

Let $D^{\gamma}x(t) = u(t) \in C[I, E]$, then we obtain

$$x(t) = x_0 + I^{\gamma}u(t) = x_0 + I^1 f(t, g(t, u(t))), \tag{2.1}$$

where u is the solution of the functional integral equation

$$u(t) = I^{\alpha}f(t, g(t, u(t))), t \in I = [0, T], \alpha = 1 - \gamma. \tag{2.2}$$

So, we have proved the following lemma.

Lemma 2.1. *The solution of the problem (1.2), if it exists, then it can be represented by the solution of the nonlinear functional integral equation (2.2), this solution is given by (2.1).*

2.1. **Coupled system approach.** In this subsection, we prove an existence result in a reflexive Banach space for the functional integral equation (2.2) which can be rewritten as a coupled system. Some existence results for coupled systems of integral and differential equations in reflexive Banach space are proved in [13]-[11].

Let $f, g : I \times E \to E$ satisfy the following assumptions:

(i) For each $t \in I, f(t, .)$ is weakly sequentially continuous.

(ii) For each $v \in C[I, E], f(., v(.))$ is weakly measurable on I.

(iii) For each $t \in I, g(t, .)$ is weakly Lipschitz in u with Lipschitz constant $K < 1$.

(iv) For each $u \in C[I, E], g(., u(.))$ is continuous on I.

(v) There exist a constant M_1 such that $\|f(t, u)\| \leq M_1$.

Now, let $v(t) = g(t, u(t)), t \in I$, then the nonlinear functional integral equation (2.2) can be written in the form of a coupled system

$$u(t) = I^{\alpha} f(t, v(t)), t \in I. \tag{2.3}$$

$$v(t) = g(t, u(t)), t \in I. \tag{2.4}$$

Let X be the class of all ordered pairs $(u, v), u, v \in C[I, E]$ with the norm

$$||(u, v)|| = ||u|| + ||v||.$$

Definition 2.2. By a solution to (2.3)-(2.4) we mean a pair of functions $(u, v) \in X$, $u, v \in C[I, E]$ which satisfies the coupled system (2.3)-(2.4). This is equivalent to find $(u, v) \in X, u, v \in C[I, E]$ with

$$\phi(u(t)) = \phi\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s)) ds\right), \ t \in I$$
$$\phi(v(t)) = \phi(g(t, u(t))), t \in I$$

Now, we can prove the following theorem.

Theorem 2.3. *Let assumptions (i)-(v) be satisfied, then the coupled system (2.3)-(2.4) has at least one solution* $(u, v) \in X, u, v \in C[I, E]$.

Proof. Define the operator A by

$$A(u, v)(t) = A(u(t), v(t)) = (A_1 v(t), A_2 u(t))$$

where

$$\begin{aligned} A_1 v(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s)) ds, t \in I \\ A_2 u(t) &= g(t, u(t)), t \in I. \end{aligned}$$

For any $v \in C[I, E]$, since $f(., v(.))$ is weakly measurable on I and $\|f(t, v)\| \leq M_1$, then $\phi(f(., v(.)))$ is Lebesgue integrable on I for all $\phi \in E^*$. As a consequence of (1.3) we have

$$\phi(I^{\alpha} f(t, v(t))) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s))) ds$$

is Lebesgue integrable on I for all $\phi \in E^*$, which implies that the fractional Pettis integrability of the function f. Thus the operator A_1 makes sense.
Define the set Q by

$$Q_r = \left\{(u,v) \in X, u,v \in C[I,E] : \|(u,v)\|_0 \leq r, r = \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{M_2}{1-K}\right\}.$$

For notational purposes

$$\|x\|_0 = \sup_{t\in I} \|x(t)\|.$$

The remainder of the proof will be given in four steps.

Step 1: The operator A maps Q_r into itself.
Let $(u,v) \in Q$ then using proposition 1.2 we have,

$$\begin{aligned} \|A_1 v(t)\| &= \phi(A_1 v(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,v(s)))ds \\ &\leq M_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \leq \frac{M_1 t^\alpha}{\Gamma(\alpha+1)} \leq \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} = r_1, \end{aligned}$$

and

$$\begin{aligned} \|A_2 u(t)\| &= \phi(A_2 u(t)) = \phi(g(t,u(t))) \\ &\leq \phi(g(t,0)) + K\phi(u(t)) \\ &\leq \phi(g(t,0)) + K\|u(t)\| \\ &\leq M_2 + K\|u\|_0 \\ &\leq M_2 + Kr_2, r_2 = \frac{M_2}{1-K} \end{aligned}$$

where $M_2 = \sup\{\phi(g(t,0)) : t \in I\}$.

$$\begin{aligned} \|A(u,v)(t)\| &= \|(A_1 v(t), A_2 u(t))\| \\ &= \|A_1 v(t)\| + \|A_2 u(t)\| \\ &\leq \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{M_2}{1-K}. \end{aligned}$$

Then

$$\|A(u,v)\|_0 = \sup_{t\in I} \|A(u,v)(t)\| \leq r, r = \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{M_2}{1-K}.$$

Hence $A(u,v) \in Q_r$, this means that $AQ_r \subset Q_r$, i.e., $A : Q_r \to Q_r$ and AQ_r is uniformly bounded.

Step 2: $AQ_r(t)$ is relatively weakly compact in E.
Note that Q_r is nonempty, closed, convex and uniformly bounded subset of X.
According to proposition 1.1, AQ_r is relatively weakly compact in X implies $AQ_r(t)$ is relatively weakly compact in E for each $t \in I$.

Step 3: The operator A maps X into itself.
First, we shall prove that $A_1, A_2 : C[I,E] \to C[I,E]$.
Let $t_1, t_2 \in I, t_2 > t_1$, without loss of generality, assume that

$$A_1 v(t_2) - A_1 v(t_1) \neq 0$$

then there exists $\phi \in E^*$ with $\|\phi\| = 1$ and

$$\begin{aligned}
\|A_1 v(t_2) - A_1 v(t_1)\| &= \phi(A_1 v(t_2) - A_1 v(t_1)) \\
&= \phi(I^\alpha f(t_2, v(t_2)) - I^\alpha f(t_1, v(t_1))) \\
&= \Big| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s))) ds \\
&- \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s))) ds \Big| \\
&= \Big| \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s))) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s))) ds \\
&- \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s))) ds \Big| \\
&\leq \int_0^{t_1} \Big| \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \Big| \mid \phi(f(s, v(s))) \mid ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \mid \phi(f(s, v(s))) \mid ds \\
&\leq M_1 \int_0^{t_1} \Big| \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \Big| ds \\
&+ M_1 \int_{t_1}^{t_2} \Big| \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \Big| ds \\
&\leq \frac{M_1}{\Gamma(\alpha+1)} (\mid t_2^\alpha - t_1^\alpha \mid + 2(t_2 - t_1)^\alpha)
\end{aligned}$$

Now, for $A_2 : C[I,E] \to C[I,E]$.
Without loss of generality, assume that $A_2 u(t_2) - A_2 u(t_1) \neq 0$

$$\begin{aligned}
A_2 u(t_2) - A_2 u(t_1) &= g(t_2, u(t_2)) - g(t_1, u(t_1)) \\
&= g(t_2, u(t_2)) - g(t_1, u(t_1)) - g(t_2, u(t_1)) + g(t_2, u(t_1)).
\end{aligned}$$

Then

$$\begin{aligned}
\|A_2 u(t_2) - A_2 u(t_1)\| &= \phi(A_2 u(t_2) - A_2 u(t_1)) \\
&\leq \phi(g(t_2, u(t_2)) - g(t_2, u(t_1))) + \phi(g(t_2, u(t_1)) - g(t_1, u(t_1))) \\
&\leq K\phi(u(t_2) - u(t_1)) + \phi(g(t_2, u(t_1)) - g(t_1, u(t_1))) \\
&\leq K\|u(t_2) - u(t_1)\| + \|g(t_2, u(t_1)) - g(t_1, u(t_1))\|.
\end{aligned}$$

Now, we shall prove that $A : X \to X$.

$$\begin{aligned}
\|A(u,v)(t_2) - A(u,v)(t_1)\| &= \|(A_1v(t_2), A_2u(t_2)) - (A_1v(t_1), A_2u(t_1))\| \\
&= \|(A_1v(t_2) - A_1v(t_1), A_2u(t_2) - A_2u(t_1))\| \\
&= \|A_1v(t_2) - A_1v(t_1)\| + \|A_2u(t_2) - A_2u(t_1))\| \\
&\leq \frac{M_1}{\Gamma(\alpha+1)}(\mid t_2^\alpha - t_1^\alpha \mid + 2(t_2 - t_1)^\alpha) \\
&+ K\|u(t_2) - u(t_1)\| + \|g(t_2, u(t_1)) - g(t_1, u(t_1))\|.
\end{aligned}$$

Step 4: The operator A is weakly sequentially continuous.
The two sequences in $C[I, E]$ $\{u_n(t)\}$ and $\{v_n(t)\}$ which converge weakly to $u(t), v(t)$ respectively for all $t \in I$. Since $g(t, u(t))$ is weakly Lipschitz $\Rightarrow$ $g(t, u(t))$ is weakly continuous $\Rightarrow$ $g(t, u(t)), f(t, v(t))$ are weakly sequentially continuous in the second argument (assumption (i)), then $f(t, v_n(t)), g(t, u_n(t))$ are convergent weakly to $f(t, v(t)), g(t, u(t))$ respectively. Hence $\phi(f(t, v_n(t))), \phi(g(t, u_n(t)))$ are convergent strongly to $\phi(f(t, v(t))), \phi(g(t, u(t)))$ respectively. Using assumption (v) and applying Lebesgue dominated convergence theorem for Pettis integral ([16]), then we have

$$\begin{aligned}
\phi\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v_n(s))ds\right) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v_n(s)))ds \\
&\to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, v(s)))ds.
\end{aligned}$$

Thus $\phi(A_1v_n(t)) \to \phi(A_1v(t))$ and $\phi(A_2u_n(t)) \to \phi(A_2u(t))$, i.e.,

$$\|A_1v_n(t)\| \to \|A_1v(t)\| \text{ and } \|A_2u_n(t)\| \to \|A_2u(t)\|.$$

Therefore,

$$\begin{aligned}
\|A(u_n, v_n)(t)\| &= \|(A_1v_n(t), A_2u_n(t))\| \\
&= \|A_1v_n(t)\| + \|A_2u_n(t)\| \\
&\to \|A_1v(t)\| + \|A_2u(t)\| \\
&\to \|(A_1v(t), A_2u(t))\| \\
&\to \|A(u,v)(t)\|.
\end{aligned}$$

This means that $A : Q_r \to Q_r$ is weakly sequentially continuous.
Since all conditions of Theorem 1.7 are satisfied, then the operator A has at least one solution $(u, v) \in Q_r$. □

2.1.1. ***Initial Value Problem***. Here, we shall study existence theorems of weak solutions and pseudo-solutions for the initial value problem (1.2).
As particular cases of Theorem 2.3, we can obtain existence theorems of weak and pseudo solutions for the initial value problem (1.2).

Definition 2.4. A function $x : I \to E$ is said to be a pseudo-solution of (1.2) if

(*a*) $x(.)$ is absolutely continuous,

(*b*) $x(0) = x_0$,
(*c*) $f(t, v(t))$ is a pseudo-derivative of $x(t)$.

Theorem 2.5. *Let assumptions of Theorem 2.3 be satisfied, then the initial value problem (1.2) has a pseudo-solution.*

Proof. Since $f(t, v(t))$ is Pettis integrable and weakly measurable function, then the solution

$$x(t) = x_0 + I^1 f(t, v(t)), t \in I$$

of (1.2) is absolutely continuous. Also $x(0) = x_0$, then

$$x'(t) = f(t, v(t)).$$

□

Now, we can deduce the existence of pseudo solutions of the initial value problem

$$x'(t) = f(t, D^\gamma x(t)), x(0) = x_0, t \in [0, 1] \tag{2.5}$$

Corollary 2.6. *Let assumptions of Theorem 2.3 be satisfied with*

$$g(t, D^\gamma x(t)) = D^\gamma x(t),$$

then the initial value problem (2.5) has pseudo solutions $x \in C[I, E]$.

Consider the following assumption

(i^*) $f : I \times E \to E$ is weakly-weakly continuous.

Remark 2.7. It is obvious that if f satisfies the assumption (i^*), then it satisfies assumptions $(i), (ii)$ and (v).

Therefore, we have the following theorem

Theorem 2.8. *Let assumptions* (i^*) *and* $(iii) - (iv)$ *be satisfied, then the initial value problem (1.2) has weakly differentiable solutions* $x \in C[I, E]$.

Proof. From Theorem 2.3 and Lemma 2.1, the solution of the initial value problem (1.2) is given by

$$x(t) = x_0 + I^1 f(t, v(t)),$$

where $v(t) = g(t, u(t)), u(t) = D^\gamma x(t)$ is given by (2.2). Since f is weakly continuous in t, then the integral of f is weakly differentiable with respect to the right end point of the integration interval and its derivative equals the integrand at that point ([20]), therefore $x(.)$ is weakly differentiable and

$$x'(t) = f(t, v(t)) = f(t, g(t, D^\gamma x(t))).$$ □

Taking $g(t, D^\gamma x(t)) = D^\gamma x(t)$, we obtain the following corollary which is proved in [25].

Corollary 2.9. *Let assumptions* (i^*) *and* (v) *of Theorem 2.3 be satisfied with* $g(t, D^\gamma x(t)) = D^\gamma x(t)$, *then the initial value problem (2.5) has weakly differentiable solution* $x \in C[I, E]$.

2.2. **Functional equation approach.** Now, equation (2.2) will be investigated under the following:
Let $f, g : I \times E \to E$ satisfy the assumptions:

(I) for each $t \in I, f(t, .)$ is weakly sequentially continuous;
(II) for each $x \in C[I, E], f(., x(.))$ is weakly measurable on I;
(III) for each $t \in I, g(t, .)$ is weakly Lipschitz in u with Lipschitz constant K;
(IV) for each $u \in C[I, E], g(., u(.))$ is weakly continuous on I;
(V) there exist a function $a : I \to R^+, a \in L^1(I)$ and a positive constant b such that $\|f(t, u)\| \leq a(t) + b\|u\|$, for all $t \in I$ and $u \in E$;
(VI) for any $\beta < \alpha, I^\beta a(t) \leq M$.

Definition 2.10. By a weak solution to (2.2) we mean a function $u \in C[I, E]$ which satisfies the integral equation (2.2). This is equivalent to find $u \in C[I, E]$ with

$$\phi(u(t)) = \phi(I^\alpha f(t, g(t, u(t)))), t \in I, 0 < \alpha < 1.$$

is satisfied for all $\phi \in E^*$.

Now, we shall prove the following existence theorem

Theorem 2.11. *Let assumptions (I)-(VI) be satisfied, then the functional integral equation of fractional order (2.2) has at least one weak solution $u \in C[I, E]$.*

Proof. Let A be an operator defined by

$$Au(t) = I^\alpha f(t, g(t, u(t))), t \in [0, T], 0 < \alpha < 1$$

and define the set Q by

$$Q_r = \{u \in C[I, E] : \|u\|_0 \leq r\}.$$

We shall show that A satisfies the assumptions of Theorem 1.7. The proof will be given in four steps.

Step 1: The operator A maps Q_r into itself.
Let $u \in Q_r$, then by using Proposition 1.2 we get

$$\begin{aligned}
\|Au(t)\| &= \phi(Au(t)) = \phi(I^\alpha f(t, g(t, u(t)))) \\
&= I^\alpha \phi(f(t, g(t, u(t))))) \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s, g(s, u(s))))ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid a(s) \mid ds + b \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s, u(s))\| ds \\
&\leq M \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds + b \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [(\phi(g(s, 0)) + K\|u(s)\|)] ds \\
&\leq \frac{MT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + b \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [M_2 + K\|u\|_0] ds \\
&\leq \frac{MT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{b[M_2 + Kr]T^\alpha}{\Gamma(\alpha+1)} \leq r.
\end{aligned}$$

Thus

$$\|Au\|_0 = \sup_{t\in I}\|Au(t)\| \leq r$$

where

$$r = \left(\frac{MT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{bM_2T^{\alpha}}{\Gamma(\alpha+1)}\right)\left(1-\frac{KT^{\alpha}}{\Gamma(\alpha+1)}\right)^{-1}.$$

Hence, $Au \in Q_r$ and hence $AQ_r \subset Q_r$ which proves that $A : Q_r \to Q_r$ and AQ_r is bounded in $C[I, E]$.

Step 2: The operator A maps $C[I, E]$ into itself.
Let $t_1, t_2 \in I, t_2 > t_1$, without loss of generality, assume that $Au(t_2) - Au(t_1) \neq 0$

$$\begin{aligned}
\|Au(t_2) - Au(t_1)\| &= \phi(Au(t_2) - Au(t_1)) \\
&= \phi(I^{\alpha}f(t_2, g(t_2, u(t_2))) - I^{\alpha}f(t_2, g(t_1, u(t_1)))) \\
&= \Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds \\
&- \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds\Big| \\
&= \Big|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds \\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds \\
&- \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds\Big| \\
&= \Big|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds \\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds \\
&- \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(f(s, g(s, u(s))))ds\Big| \\
&\leq \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \mid \phi(f(s, g(s, u(s)))) \mid ds \\
&\leq \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[a(s) \mid +b\|g(s, u(s))\|]ds \\
&\leq \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[a(s) + b[\phi(g(s, 0)) + K\|u(s)\|]]ds \\
&\leq M\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds \\
&+ b[M_2 + K\|u\|_0]\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds
\end{aligned}$$

$$\leq \frac{M(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{b[M_2+K\|u\|_0]}{\Gamma(\alpha+1)}((t_2-t_1)^{\alpha}).$$

Thus

$$\|Au(t_2)-Au(t_1)\| \leq \frac{M(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{b[M_2+K\|u\|_0]}{\Gamma(\alpha+1)}((t_2-t_1)^{\alpha}).$$

This means that $A : C[I,E] \to C[I,E]$.

Step 3: $AQ_r(t)$ is relatively weakly compact in E.
Note that Q_r is nonempty, closed, convex and uniformly bounded subset of Q_r. According to proposition 1.1, AQ_r is relatively weakly compact in Q_r implies $AQ_r(t)$ is relatively weakly compact in E for each $t \in I$.

Step 4: The operator A is weakly sequentially continuous.
Let $\{u_n\}$ be sequence in Q_r converges weakly to u on I, since $f(t,.)$ and $g(t,.)$ are weakly sequentially continuous in the second argument, then $g(t,u_n(t))$ converges weakly to $g(t,u(t))$ and $f(t,g(t,u_n(t)))$ converges weakly to $f(t,g(t,u(t)))$. Thus $\phi(f(t,g(t,u_n(t))))$ converges strongly to $\phi(f(t,g(t,u(t))))$. By applying Lebesgue dominated convergence theorem we have

$$\phi(I^{\alpha}f(t,g(t,u_n(t)))) = I^{\alpha}\phi(f(t,g(t,u_n(t)))) \to I^{\alpha}\phi(f(t,g(t,u(t)))),\ \forall\phi \in E^*,\ t \in I.$$

i.e., $\phi(Au_n(t)) \to \phi(Au(t)), \forall\phi \in E^*, t \in I$.
Since all conditions of Theorem 1.7 are satisfied, then the operator A has at least one fixed point $u \in Q_r$ and the nonlinear functional integral equation of fractional order (2.2) has at least one solution $u \in C[I,E]$. □

2.2.1. ***Initial value problems.*** As done in subsection 2.1.1, we shall study existence theorems of weak solutions and pseudo-solutions for the initial value problem (1.2) as a consequence of Theorem 2.11.

Theorem 2.12. *Let assumptions of Theorem 2.11 be satisfied, then the initial value problem (1.2) has weakly differentiable solutions $x \in C[I,E]$.*

Proof. From Lemma 2.1, the solution of initial value problem (1.2) is given by

$$x(t) = x_0 + I^1 f(t,g(t,u(t)))$$

where u is given by (2.2). Since f is weakly continuous in t, then the integral of f is weakly differentiable with respect to the right end point of the integration interval and its derivative equals the integrand at that point ([20]), therefore $x(.)$ is weakly differentiable and

$$x'(t) = f(t,g(t,u(t))) = f(t,g(t,D^{\gamma}x(t))).$$ □

Corollary 2.13. *Under assumptions of Theorem 2.11, with $g(t,D^{\gamma}x(t)) = D^{\gamma}x(t)$, the equation (2.5) has weakly differentiable solution $x \in C[I,E]$.*

This result generalized the result in [25].

Conclusions

Our results are concerning the existence of solutions in a reflexive Banach spaces for an initial value problem of fractional order which contains various ones, some of them are:

- for real-valued function; the function f is independent of the fractional derivatives and $g(t,x)=x$, then we have the problems studied in, for examples [7] and [27].
- for real-valued function with $\gamma \in (0,1)$ and $g(t,x)=x$ we have the problem studied in [9] with nonlocal and integral condition.
- in abstract spaces with conditions related to the weak topology on a reflexive Banach space E, then we have the problem studied in [25] with $g(t,x)=x$ which was studied for the first time, by Salem and El-Sayed.
- in abstract spaces with conditions related to the weak topology on E and the function f is independent of the fractional derivatives and $g(t,x)=x$, then we have the problem studied in [5] and [6].

which shows how our work generalize many known results in nonlinear analysis.

References

[1] R.P. Agarwal, V. Lupulescu, D. O'Regan, G. Rahman, *Nonlinear fractional differential equations in nonreflexive Banach spaces and fractional calculus*, Advances in Difference Equations, (2015), 2015:112.

[2] J.M. Ball, *Weak continuity properties of mapping and semi-groups*, Proc. Royal Soc.Edinbourgh Sect., **A72**(1973-1974), 275-280.

[3] S.N. Chow, J. Shur, *An existence theorem for ordinary differential equations in Banach spaces*, Bull. Amer. Math. Soc., **77**(1971), 1018-1020.

[4] M. Cichoń, *On bounded weak solutions of a nonlinear differential equations in Banach spaces*, Funct. Approx., **21**(1992), 27-36.

[5] M. Cichoń, *Weak solutions of differential equations in Banach spaces*, Discuss. Math. Differ. Inc., **15**(1995), 5-14.

[6] M. Cichoń, I. Kubiaczyk, *Kneser's theorem for strong, weak and pseudo-solutions of ordinary differential equations in Banach spaces*, Ann. Pol. Math., **52**(1995), 13-21.

[7] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Mathematics, Vol. 596, Springer, Berlin 1977.

[8] J. Diestel, J.J. Uhl Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[9] A.M.A. El-Sayed, E.O. Bin-Taher, *Nonlocal and integral conditions problems for a multi-term fractional-order differential equation*, Miskolc Mathematical Notes, **15**(2014), no. 2, 439-446.

[10] A.M.A. El-Sayed, N.F.M. El-Haddad, *Weak solutions of a functional integral equation in reflexive Banach space*, Nonlinear Analysis and Differential Equations, **4**(2016), no. 7, 305-313.

[11] A.M.A. El-Sayed, W.G. El-Sayed, A.A.H. Abd El-Mowla, *A coupled system of functional differential equations in reflexive Banach spaces*, International Journal of Pure and Applied Mathematics, **113**(2017), no. 1, 35-42.

[12] A.M.A. El-Sayed, H.H.G. Hashem, *Weak maximal and minimal solutions for Hammerstein and Urysohn integral equations in reflexive Banach spaces*, Differential Equation and Control Processes, **4**(2008), 50-62.

[13] A.M.A. El-Sayed, H.H.G. Hashem, *Coupled systems of Hammerstein and Urysohn integral equations in reflexive Banach spaces*, Differential Equations and Control Processes, **1**(2012), 1-12.

[14] A.M.A. El-Sayed, H.H.G. Hashem, *Coupled systems of integral equations in reflexive Banach spaces*, Acta Mathematica Scientia, **32B**(5)(2012), 1-8.
[15] A.M.A. El-Sayed, H.H.G. Hashem, *A coupled system of fractional order integral equations in reflexive Banach spaces*, Commentationes Mathematicae, **52**(2012), 21-28.
[16] R.F. Geitz, *Pettis integration*, Proc Amer. Math. Soc., **82**(1981), 81-86.
[17] H.H.G. Hashem, *Weak solutions of differential equations in Banach space*, Journal of Fractional Calculus and its Applications, **3**(2012), 1-9.
[18] H.H.G. Hashem, Y. Khouni, *A survey on existence results of some differential and integral equations in abstract spaces*, Journal of Fractional Calculus and its Applications, **8**(2017), 126-141.
[19] I. Kubiaczyk, *On fixed point theorem for weakly sequentially continuous mappings*, Discuss. Math. Differ. Incl., **15**(1995), 15-20.
[20] A.R. Mitchell, Ch. Smith, *An existence theorem for weak solutions of differential equations in Banach spaces*, in: Nonlinear Equations in Abstract Spaces, V. Lakshmikantham (ed.), 1978, 387-403.
[21] D. O'Regan, *Fixed point theory for weakly sequentially continuous mapping*, Math. Comput. Modelling, **27**(1998), 1-14.
[22] D. O'Regan, *Weak solutions of ordinary differential equations in Banach spaces*, Appl. Math. Lett., **12**(1999), 101-105.
[23] B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc., **44**(1938), 277-304.
[24] H.A.H. Salem, M. Cichoń, *On solutions of fractional order boundary value problems with integral boundary conditions in Banach spaces*, Journal of Function Spaces and Applications, **2013**(2013), Article ID 428094, 13 pages.
[25] H.A.H. Salem, A.M.A. El-Sayed, *Weak solution for fractional order integral equations in reflexive Banach spaces*, Math. Slovaca, **55**(2005), 169-181.
[26] A. Szep, *Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*, Studia Sci. Math. Hungar., **6**(1971), 197-203.
[27] S. Szulfa, *On the existence of solutions of differential equations in Banach spaces*, Bull. Acad. Polan. Sci. Ser. Sci. Math., **30**(1982), 507-514.

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