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FIXED POINT THEOREMS IN r-NORMED AND LOCALLY r-CONVEX SPACES AND APPLICATIONS

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Abstract. In this paper we prove some new fixed point theorems in r-normed and locally r-convex spaces. Our conclusions generalize many well-known results and provide a partial affirmative answer to Schauder's conjecture. Based on the obtained results, we prove the analogue of a Von Neumann's theorem in locally r-convex spaces. In addition, an application to game theory is presented.
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1. Introduction and preliminaries

Throughout this paper X is a vector space over the field of real or complex numbers $\mathbb{K}$ with the origin 0. Let $0 < r \leq 1$. A mapping $p : X \to \mathbb{R}$ is called an r-seminorm if it satisfies the requirements :

(i) $p(x) \geq 0$, for all $x \in X$,
(ii) $p(\lambda x) = |\lambda|^r p(x)$, for all $x \in X$, $\lambda \in \mathbb{K}$,
(iii) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$.

An r-seminorm is called an r-norm if $x = 0$ whenever $p(x) = 0$. A vector space with a specific r-norm is called an r-normed space. The r-norm of an element $x \in X$ will usually be denoted by $\|x\|_r$. If $r = 1$, X is a usual normed space. If X is an r-normed space, then (X, d_r) is a metric linear space with a translation invariant metric d_r such that $d_r(x, y) = \|x - y\|_r$ for each $x, y \in X$. We point out that r-normed spaces are very important in the theory of topological vector spaces. Specifically, a Hausdorff topological vector space is locally bounded if and only if it is an r-normed space for some r-norm $\|.\|_r$, $0 < r \leq 1$ (see [13, p.114]). Examples of r-normed spaces include the $L^r(\mu)$-spaces and Hardy spaces H^r, $0 < r < 1$, endowed with their usual r-norms. As stressed in [9], there is no open convex non-void subset in $L^r[0,1]$, $(0 < r < 1)$ except $L^r[0,1]$ itself. This example leads us to conclude that r-normed spaces with $0 < r < 1$ are not necessarily locally convex. However, we know that every r-normed space is locally r-convex (see Section 3). In [25, Theorem 2.13], the authors proved the following interesting fixed point result of Schauder type in a complete r-normed space.

Theorem 1.1. *Let $(X, \|.\|_r)$ be a complete r-normed space and C be a compact s-convex subset of X, where $0 < s \leq r \leq 1$. If $T : C \to C$ is continuous, then there exists $z \in C$ such that $Tz = z$.*

We should emphasize that the completeness of the space in not required in the original result of Schauder in a normed space [1, Theorem 4.14, p.38]. So, it is quite natural to ask whether this condition could be removed from Theorem 1.1. It should also be mentioned that Theorem 1.1 does not include the case of convex sets in r-normed spaces with $0 < r < 1$.

Somewhat later, Alghamdi et al. [2], took an interesting step in the same direction by proving the analogue of Krasnosel'skii's and Sadovskii's fixed point theorems for s-convex sets in complete r-normed spaces. Further progress was achieved in [24] where the results of [2, 25] were greatly improved. On the other hand, Bayoumi [5, Theorem 34, p.64] proved the following Schauder's type fixed point theorem for r-convex sets in locally r-convex F-spaces (with $0 < r < 1$). Recall that an F-space is a topological vector space whose topology is induced by a complete translation invariant metric (see [21, p.9]).

Theorem 1.2. *If K is a nonempty compact r-convex set in a locally r-convex F-space X $(0 < r < 1)$ and $f : K \to K$ is continuous, then f has at least one fixed point in K.*

We point out that Theorem 1.2 requires metrizability and completeness of the space and does not include the case of non-convex sets in usual locally convex spaces $(r = 1)$.

The main purpose of this paper is to extend and improve the aforementioned results and to establish some new fixed point theorems on s-convex sets in r-normed spaces and locally r-convex spaces. We will also provide a partial affirmative answer to Schauder's conjecture (see Section 2).

The paper is arranged as follows. In Section 2, we prove some fixed point theorems for single-valued mappings defined on s-convex subsets of r-normed spaces. In Section 3, we provide some fixed point theorems for single-valued mappings on s-convex subsets of locally r-convex spaces. In Section 4, we present some fixed point theorems for multi-valued mappings in s-convex subsets of r-normed and locally r-convex spaces $(0 < r \leq 1,\ 0 < s \leq 1)$. Finally, Section 5 is devoted to applications.

2. Fixed point theorems in r-normed spaces

In this section we shall prove that the completeness of the space in Theorem 1.1 is redundant and could be removed. This result is of fundamental importance for our subsequent analysis. Before making a formal statement of the principal theorem of this section, we present an example of an r-normed space which is not complete.

Example 2.1. Let $E = \mathcal{C}([0,1],\mathbb{R})$ be the vector space of real valued continuous functions on the compact interval $[0,1]$ endowed with the r-norm $\|.\|_r$ defined by

$$\|f\|_r = \int_0^1 |f(t)|^r dt$$

for all $f \in \mathcal{C}([0,1],\mathbb{R})$, where $0 < r \leq 1$. In order to show that this space is not complete, we consider the sequence $(f_n)_{n\geq 2}$ of functions of E defined by

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \leq t < \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}\left(t - \frac{1}{2} + \frac{1}{n}\right) & \text{if } \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n}, \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} < t \leq 1. \end{cases}$$

Clearly,

$$\|f_m - f_n\|_r = \frac{2^{1-r}}{r+1}\left(\frac{1}{n} - \frac{1}{m}\right)^r \frac{1}{n^{1-r}} \leq \frac{2^{1-r}}{r+1}\frac{1}{n},$$

for each $m \geq n \geq 2$. Hence, $\|f_m - f_n\|_r \to 0$ as $m, n \to \infty$. This shows that $(f_n)_{n\geq 2}$ is a Cauchy sequence in E. We claim that $(f_n)_{n\geq 2}$ admits no $\|.\|_r$-limit in E. Indeed, let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{1}{2} \\ \frac{1}{2} & \text{for } t = \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < t \leq 1. \end{cases}$$

Obviously, $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in [0,1]$.
This implies $\lim_{n\to\infty} |f_n(t) - f(t)|^r dt = 0$ for all $t \in [0,1]$. Since $|f_n(t) - f(t)|^r \leq 2^r$ for all $t \in [0,1]$, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^1 |f_n(t) - f(t)|^r dt = 0.$$

Consequently, the sequence $(f_n)_n$ converges in the $\|.\|_r$-norm to $f \in L^r\big([0,1],\mathbb{R}\big) \setminus \mathcal{C}\big([0,1],\mathbb{R}\big)$. Since the $\|.\|_r$-limit is unique, then $(f_n)_n$ has no $\|.\|_r$-limit in $\mathcal{C}\big([0,1],\mathbb{R}\big)$.

Now, we recall the definition of r-convex and absolutely r-convex sets.

Definition 2.1. Let $0 < r \leq 1$. A subset A of a vector space X is called :

(i) r-convex if $\alpha\, x + \beta\, y \in A$ for all $x,\, y \in A$ and all $\alpha,\, \beta \geq 0$ with $\alpha^r + \beta^r = 1$.
(ii) Absolutely r-convex if $\alpha\, x + \beta\, y \in A$ for all $x,\, y \in A$ and all $\alpha,\, \beta \in \mathbb{R}$ with $|\alpha|^r + |\beta|^r \leq 1$.

Remarks 2.1.

(i) A subset A of a vector space X is 1-convex if and only if it is convex.
(ii) An absolutely r-convex set is r-convex and contains 0.
(iii) A subset A of X is absolutely r-convex if and only if it is r-convex and balanced (see [4, Lemma 4.1.4, p.176]).

The following lemma was stated in [20, Remark 2.1]. For the reader's convenience we give a proof here.

Lemma 2.1. *Let A be an r-convex subset of a vector space X with $0 < r < 1$. Then, $\alpha\, x \in A$ for any $x \in A$ and any $0 < \alpha \leq 1$.*

Proof. We will use mathematical induction to prove that the statement $P(n)$ given by "for all $x \in A$ and all $\alpha \in \left[2^{(n+1)\left(1-\frac{1}{r}\right)}, 2^{n\left(1-\frac{1}{r}\right)}\right]$ we have $\alpha\, x \in A$" is true for all integers $n \geq 0$. First, we verify that the base case $P(0)$ is true. To do so, let $x \in A$ and $\alpha \in [\beta, 1]$ with $\beta = 2^{1-\frac{1}{r}}$. Since $[\beta, 1]$ is the range of the continuous function $t \mapsto t^{\frac{1}{r}} + (1-t)^{\frac{1}{r}}$ defined on $[0, 1]$, then there exists $t \in [0, 1]$ such that $\alpha = t^{\frac{1}{r}} + (1-t)^{\frac{1}{r}}$. The r-convexity of the set A implies that $\alpha\, x = t^{\frac{1}{r}}\, x + (1-t)^{\frac{1}{r}}\, x \in A$. Thus, the base case $P(0)$ has been verified. Next, we perform the inductive step. Assume that $P(n)$ is true for some integer $n \geq 0$. We will use this to show that $P(n+1)$ is true. To this end, take $x \in A$ and $\alpha \in \left[2^{(n+2)\left(1-\frac{1}{r}\right)}, 2^{(n+1)\left(1-\frac{1}{r}\right)}\right]$. From $P(0)$ we know that $\beta x \in A$. Keeping in mind that $\alpha\beta^{-1} \in \left[2^{(n+1)\left(1-\frac{1}{r}\right)}, 2^{n\left(1-\frac{1}{r}\right)}\right]$ we infer from our inductive assumption that $\alpha x = \alpha\beta^{-1}(\beta x) \in A$. Thus, we have shown our statement $P(n+1)$ to be true and thus our inductive step is complete. Taking into account the fact that $(0, 1] = \bigcup_{n\geq 0} \left[2^{(n+1)\left(1-\frac{1}{r}\right)}, 2^{n\left(1-\frac{1}{r}\right)}\right]$ we get the desired result. $\square$

The following lemma is crucial for our purposes.

Lemma 2.2. *Let A be a subset of a vector space X.*

(i) *If A is convex and $0 \in A$, then A is s-convex for any real $s \in (0, 1]$.*
(ii) *If A is r-convex for some $r \in (0, 1)$, then A is s-convex for any $s \in (0, r]$.*
(ii) *If A is absolutely r-convex for some $r \in (0, 1]$, then it is absolutely s-convex for any $s \in (0, r]$.*

Proof.

(i) Assume that A is a convex subset of X with $0 \in A$ and take a real $s \in (0, 1]$. We show that A is s-convex. To see this, let $x, y \in A$ and α, $\beta > 0$ such that $\alpha^s + \beta^s = 1$. Since A is convex then, $\dfrac{\alpha}{\alpha+\beta}\, x + \dfrac{\beta}{\alpha+\beta}\, y \in A$. Keeping in mind that $0 < \alpha + \beta \leq \alpha^s + \beta^s = 1$, we deduce that
$$\alpha\, x + \beta\, y = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}\, x + \frac{\beta}{\alpha+\beta}\, y\right) + (1-\alpha-\beta)\, 0 \in A.$$

(ii) Now, assume that A is r-convex for some $r \in (0, 1)$ and pick up any real $s \in (0, r]$. We show that A is s-convex. To do this, let $x, y \in A$ and α, $\beta > 0$ such that $\alpha^s + \beta^s = 1$. First notice that $0 < \alpha^{\frac{r-s}{r}} \leq 1$ and $0 < \beta^{\frac{r-s}{r}} \leq 1$ imply $\alpha^{\frac{r-s}{r}}\, x \in A$ and $\beta^{\frac{r-s}{r}}\, y \in A$. From the r-convexity of A and the equality
$$\left(\alpha^{\frac{s}{r}}\right)^r + \left(\beta^{\frac{s}{r}}\right)^r = 1,$$

we deduce

$$\alpha\, x + \beta\, y = \alpha^{\frac{s}{r}}\, \left(\alpha^{\frac{r-s}{r}}\, x\right) + \beta^{\frac{s}{r}}\, \left(\beta^{\frac{r-s}{r}}\, y\right) \in A.$$

(iii) Assume that A is absolutely r-convex for some $r \in (0, 1)$ and take a real $s \in (0, r]$. If $x,\, y \in A$ and $\alpha,\, \beta \in \mathbb{R}$ satisfy $|\alpha|^s + |\beta|^s \leq 1$, then

$$|\alpha|^r + |\beta|^r \leq |\alpha|^s + |\beta|^s \leq 1.$$

Taking into account the absolute r-convexity of the set A, we infer that $\alpha\, x + \beta\, y \in A$. This proves that A is absolutely s-convex. □

Now, we are in a position to state and prove the main result of this section.

Theorem 2.1. *Let $(X, \|.\|_r)$ be an r-normed space and K be a compact s-convex subset of X, where $0 < r \leq 1$ and $0 < s \leq 1$. Let $T : K \to K$ be a continuous mapping. Then T has at least one fixed point in K.*

Proof. To prove Theorem 2.1, we distinguish three cases.

First case: Assume that $0 < r \leq 1$ and $0 < s \leq r$. Notice first that X is a metric space with the metric d_r defined by $d_r(x,\, y) = \|x - y\|_r$ for all $x,\, y \in X$. Let $\widehat{X}$ be a completion of the metric space $(X,\, d_r)$. Then there exists a linear isometric embedding $i : X \to \widehat{X}$ with $i(X)$ dense in $\widehat{X}$ (see [14, p.41, p.69 and p.70]). Define $\widehat{T} : i(K) \to i(K)$ by $\widehat{T}\big(i(x)\big) = i\big(T(x)\big)$, $x \in K$. The continuity of i and T imply the continuity of $\widehat{T}$ on $i(K)$. The s-convexity of $i(K)$ follows easily from the fact that i is linear and K is s-convex. Also, the compactness of K and the continuity of i imply the compactness of $i(K)$. Hence, by Theorem 1.1, there exists $x \in K$ such that $\widehat{T}\big(i(x)\big) = i(x)$. Thus, $i\big(T(x)\big) = i(x)$ and so $T(x) = x$.

Second case: Assume that $0 < r < 1$ and $r < s < 1$. Since K is s-convex then, according to Lemma 2.2, K is r-convex. The result then follows from an application of the first case.

Third case: Assume that $0 < r < 1$ and $s = 1$. Let us fix any element $x_0 \in K$ and let $K_0 = \{x - x_0 : x \in K\}$. Plainly, the set K_0 is compact convex and $0 \in K_0$. Referring to Lemma 2.2 we see that K_0 is r-convex. Define the map $T_0 : K_0 \to K_0$ by $T_0(x - x_0) = T(x) - x_0$. It is easy to verify that the result of the first case may be applied to T_0, giving thus an element $x \in K$ such that $T_0(x - x_0) = x - x_0$, that is $T(x) = x$. □

Remarks 2.2.

(i) Theorem 2.1 extends [25, Theorem 2.13]. In our considerations, the space is not necessarily complete and the real s may be greater than r.

(iii) The set of fixed points of T is compact.

As a convenient specialization of Theorem 2.1, we obtain the following sharpening of Schauder's fixed point theorem in normed spaces. We particularly show that Schauder's statement remains valid for both convex and s-convex sets (with $0 < s \leq 1$) in normed spaces.

Corollary 2.1. *Let K be a nonempty compact s-convex set in a normed space X for some $0 < s \leq 1$ and let $T\colon K \to K$ be a continuous mapping. Then T has at least one fixed point in K.*

Proof. Apply Theorem 2.1 with $r = 1$. □

Another consequence of Theorem 2.1 is the following.

Corollary 2.2. *Let K be a nonempty compact convex set in an r-normed space X for some $0 < r \leq 1$ and let $T\colon K \to K$ be a continuous mapping. Then T has at least one fixed point in K.*

Proof. Apply Theorem 2.1 with $s = 1$. □

Now we are in a position to state the following interesting fixed point result in locally bounded Hausdorff topological vector spaces. Recall that a topological vector space is said to be locally bounded if it contains a bounded neighborhood of the origin.

Theorem 2.2. *Every nonempty compact convex subset K of a Hausdorff locally bounded topological vector space X has the fixed point property, that is, every continuous function $T\colon K \to K$ has at least a fixed point in K.*

Proof. Since X is a Hausdorff locally bounded topological vector space then, according to [13, p.114], X is an r-normed space for some r-norm $\|.\|_r$, with $0 < r \leq 1$. The result follows from Corollary 2.2. □

Remark 2.3. Schauder's conjecture which states that every compact convex subset of a topological vector space has the fixed point property is one of the most resistant open problems in the fixed point theory of non-locally convex topological vector spaces. This problem is still open despite great efforts by topologists for more than half a century. Up to now only some partial answers to Schauder's problem have been obtained, see for instance [15, 17, 18, 19]. In Theorem 2.2, we provide another partial affirmative answer to Schauder's conjecture. More precisely, we prove that every nonempty compact convex subset of a Hausdorff locally bounded topological vector space X has the fixed point property. Our result particularly yields that compact convex sets in L^r, ℓ^r or Hardy spaces H^r with $0 < r < 1$, have the fixed point property.

3. Fixed point theorems in locally r-convex spaces

In this section, we prove some fixed point theorems in locally r-convex spaces. Specifically, we shall use the results of the previous section to extend and improve Theorem 1.2. In our considerations, the underlying space is neither metrizable nor complete. Also, our result includes as a special case the well known Schauder-Tychonov fixed point theorem for both convex and s-convex sets $(0 < s < 1)$ in usual locally convex spaces $(r = 1)$, a fact that does not follow from Theorem 1.2. We carried out substantial modifications and changes on an ingenious proof of Schauder-Tychonov fixed point theorem presented in [6] to apply in this setting. The proof is technical and leads to the complete analogue of the Schauder-Tychonov theorem for s-convex sets in locally r-convex spaces.

We first recall some basic definitions and facts concerning locally r-convex spaces. Let $0 < r \leq 1$. A topological vector space is said to be locally r-convex space if it has a basis $\mathcal{U}$ of neighborhoods of 0 whose members are absolutely r-convex (see [13, p.108]).

Now, we make a short note about the so-called *Minkowsky r-functionals.* Let A be an r-convex and absorbing subset of a vector space X with $0 < r \leq 1$. Define $p_A : X \to \mathbb{R}$ by

$$p_A(x) = \inf \left\{ t > 0;\, x \in t^{\frac{1}{r}}\, A \right\},\, x \in X.$$

It is easy to check that p_A is subadditive and

$$p_A(\lambda\, x) = \lambda^r\, p_A(x),$$

for all $\lambda > 0$ and $x \in X$. If, in addition, A is absolutely r-convex, then p_A is an r-seminorm. Also, we plainly have $p_B \leq p_A$ for any two r-convex and absorbing subsets A and B of X with $A \subset B$. Further details concerning Minkowsky r-functionals are provided in [4, p.179] and [25, Lemma 1.5].

It is well-known [5, p.52] that if X is a locally r-convex space with a basis $\mathcal{U}$ of absolutely r-convex neighborhoods of the origin, then the topology of X is generated by the directed family of continuous r-seminorms $\mathcal{P} = (p_U)_{U \in \mathcal{U}}$.

Before we state the main results of this section, we give an example of a locally r-convex space. Let $(E, \|.\|_r)$ be an r-normed space and let $X = \mathcal{C}(\mathbb{R}, E)$ be the vector space of E-valued continuous mappings on the real field $\mathbb{R}$. For any integer $n \geq 1$, the mapping p_n defined on X by:

$$p_n(f) = \sup_{-n \leq t \leq n} \|f(t)\|_r,$$

for each $f \in X$ is an r-seminorm on X. It is easily seen that the family $(p_n)_{n \geq 1}$ of r-seminorms produces a metrizable locally r-convex topology on X with a metric d defined by:

$$d(f,\, g) = \sum_{n=1}^{\infty} 2^{-n}\, \min\big(1, p_n(f-g)\big), \quad f, g \in X.$$

Proposition 3.1. *X is complete if and only if E is complete.*
Proof. Assume that E is complete and let $(f_k)_{k \geq 1}$ be a Cauchy sequence in X. For each $n \geq 1$, let $f_{k,n}$ denote the restriction of f_k to $[-n, n]$. Clearly, $(f_{k,n})_{k \geq 1}$ is a Cauchy sequence in the complete r-normed space $X_n = \mathcal{C}([-n, n], E)$, and converges to some $g_n \in X_n$. Note that for each integer $n \geq 1$, g_n is the restriction of g_{n+1} to $[-n,\, n]$. Let $f \in X$ defined by $f(x) = g_n(x)$ for all $x \in [-n,\, n]$ and all $n \geq 1$. It is easy to check that f is well defined and $(f_k)_{k \geq 1}$ converges to f in X. We conclude that X is a complete metrizable locally r-convex space.
Conversely, we suppose that X is complete. Let $(x_k)_k$ be a Cauchy sequence in E. The sequence $(f_k)_k$ in X defined by:

$$f_k(t) = x_k \text{ for all } t \in \mathbb{R},\, k \in \mathbb{N},$$

is a Cauchy sequence in X that converges to some $f \in X$. Hence

$$\|x_k - \ f(0)\|_r \leq p_1(x_k - f) \to 0.$$

Consequently, E is complete. □

Now, let $B_p = \{x \in X : p(x) < 1\}$ and $B'_p = \{x \in X : p(x) \leq 1\}$, where p is a given r-seminorm. Then,

(i) B_p and B'_p are absorbing absolutely r-convex subsets of X.
(ii) $p_{B_p} = p_{B'_p} = p$.

The assertion (i) being obvious, we prove (ii). To do this, let $x \in X$. Then,

$$\begin{aligned} p_{B_p}(x) &= \inf\left\{t > 0 : t^{-\frac{1}{r}} x \in B_p\right\} \\ &= \inf\left\{t > 0 : p\big(t^{-\frac{1}{r}} x\big) < 1\right\} \\ &= \inf\{t > 0 : p(x) < t\} \\ &= p(x). \end{aligned}$$

Consequently, $p_{B_p} = p$. A similar reasoning yields $p_{B'_p} = p$.

With these preliminaries, we can proceed to the following Lemma.

Lemma 3.1. *Let p be an r-seminorm on X and A be an absorbing absolutely r-convex subset of X with $0 \in A$ $(0 < r \le 1)$. Then $p = p_A$ if and only if:*

$$B_p \subset A \subset B'_p. \tag{3.1}$$

Proof. Suppose that $p = p_A$. The inclusion $A \subset B'_p$ being obvious, let us prove that $B_p \subset A$. To do so, let $x \in B_p$ and take a real t such that $p_A(x) < t < 1$. Hence, $x \in t^{\frac{1}{r}} A$ and so $t^{-\frac{1}{r}} x \in A$. Thus, $x = t^{\frac{1}{r}}\left(t^{-\frac{1}{r}} x\right) + (1-t)^{\frac{1}{r}} 0 \in A$. Consequently, $B_p \subset A$. Conversely, assume that (3.1) holds. Then,

$$p = p_{B'_p} \le p_A \le p_{B_p} = p.$$

Consequently, $p = p_A$. □

Recall that a topological vector space X is said to be quasi-complete if every bounded closed subset of X is complete. It is well known [22] that every quasi-complete topological vector space is sequentially complete.

Now, we state and prove a Mazur type result in locally r-convex spaces. This result extends [5, Proposition 1, p.52] and [25, Lemma 2.1].

Theorem 3.1 (Mazur type). *Let $0 < r \le 1$ and $0 < s \le r$. Let X be a locally r-convex space and let A be a totally bounded subset of X. Then $\overline{co_s}(A)$ is totally bounded. If, in addition, X is quasi-complete, then $\overline{co_s}(A)$ is compact.*

Proof. Since the closure of a totally bounded set is totally bounded (see [22, p.25]), it suffices to show that $co_s(A)$ is totally bounded. To see this, let V be a neighborhood of the origin and let U be an absolutely r-convex neighborhood of the origin such that $U + U \subset V$. Since A is totally bounded and U is s-convex then, there exists $x_1, \dots, x_n \in X$ such that $A \subset \{x_1, \dots, x_n\} + U$, implying

$$co_s(A) \subset co_s(x_1, \dots, x_n) + U. \tag{3.2}$$

The compactness of $co_s(x_1, \dots, x_n)$ ensures that there exists $y_1, \dots, y_m \in X$ such that

$$co_s(x_1, \dots, x_n) \subset \{y_1, \dots, y_m\} + U.$$

Using (3.2) we arrive at

$$co_s(A) \subset co_s(x_1, \dots, x_n) + U \subset \{y_1, \dots, y_m\} + U + U \subset \{y_1, \dots, y_m\} + V.$$

If X is quasi-complete, then $\overline{co_s}(A)$ is totally bounded and complete. By virtue of [13, Corollary, p.65] we have that $\overline{co_s}(A)$ is compact. □

Let X and Y be two topological vector spaces. A mapping $T : D \subset X \to Y$ is said to be uniformly continuous if for every neighborhood V of 0 in Y, there exists a neighborhood U of 0 in X such that:

$$\big(x,\, y \in D,\, x - y \in U\big) \implies Tx - Ty \in V.$$

The following lemma will be quite useful below.

Lemma 3.2. [6, p.36] *Let X, Y be topological vector spaces and K be a compact subset of X. Let $T : K \to Y$ be a continuous mapping. Then T is uniformly continuous.*

Definition 3.1. Let X be a topological vector space and K be a compact subset of X. Let $T : K \to K$ be a continuous mapping and $\mathcal{S}$, $\mathcal{S}'$ be two sets of continuous r-seminorms on X. We say that $\mathcal{S}'$ dominates $\mathcal{S}$ with respect to T if:

(a) For each $q \in \mathcal{S}'$ and $x \in K$, we have $q(x) \le 1$,

(b) for any $p \in \mathcal{S}$ and $\varepsilon > 0$, there exist $q \in \mathcal{S}'$ and $\alpha > 0$ such that

$$q(y - x) < \alpha \Rightarrow p\big(Ty - Tx\big) < \varepsilon,$$

for all x, $y \in K$.

If $\mathcal{S}' = \mathcal{S}$, we say that $\mathcal{S}$ is self-dominating.
If $\mathcal{S} = \{p\}$, we say that $\mathcal{S}'$ dominates p.

The following result is of fundamental importance for our subsequent analysis.

Theorem 3.2. *Let X be a locally r-convex space ($0 < r \le 1$) and K be a compact subset of X. Let $T : K \to K$ be a continuous mapping and p_0 be a continuous r-seminorm on X. Then, there exists an r-seminorm q on the vector subspace span(K) of X satisfying:*

(i) *$p_0(x) \le q(x)$ for all $x \in$ span(K).*

(ii) *q is continuous on $K - K$.*

(iii) *K is compact with respect to the topology generated by the r-seminorm q.*

(iv) *T is uniformly continuous in K with respect to q i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$[x,\, y \in K,\, q(y - x) < \delta] \implies q\big(Ty - Tx\big) < \varepsilon.$$

Proof. Since p_0 is bounded on K then, without loss of generality, we may assume that $p_0(x) \le 1$ for all $x \in K$. We shall construct a countable self-dominating set $\mathcal{S}_\infty$ of continuous r-seminorms such that $p_0 \in \mathcal{S}_\infty$. To do this, let p be a continuous r-seminorm. For any integer $n \ge 1$, the set $V_n = \left\{x \in X;\, p(x) < \dfrac{1}{n}\right\}$ is a neighborhood of 0 in X. Referring to Lemma 3.2, we see that there exists an open absolutely r-convex neighborhood U_n of 0 in X such that, for each x, $y \in K$, we have

$$y - x \in U_n \Rightarrow p\big(Ty - Tx\big) < \frac{1}{n}.$$

Clearly, the r-seminorm p_{U_n} is continuous and we have

$$U_n = \{x \in X;\, p_{U_n}(x) < 1\}.$$

Multiplying p_{U_n} by an appropriate constant $\alpha_n > 0$, we obtain a continuous r-seminorm q_n satisfying:

- For each $x \in K,\ q_n(x) \leq 1$,
- For each $x,\, y \in K,\ q_n(y-x) < \alpha_n \Rightarrow p\big(Ty - Tx\big) < \dfrac{1}{n}$.

The set $\{q_n;\, n \geq 1\}$ of continuous r-seminorms is countable and dominates p. Accordingly, for any countable set $\mathcal{S}$ of continuous r-seminorms, there exists a countable set $\mathcal{S}'$ of continuous r-seminorms that dominates $\mathcal{S}$. The set $\{p_0\}$ is then dominated by a countable set $\mathcal{S}_1$, $\mathcal{S}_1$ is dominated by a countable set $\mathcal{S}_2$ and so on. Accordingly, $\mathcal{S}_\infty = \{p_0\} \cup \bigcup_{n=1}^{\infty} \mathcal{S}_n$ is a countable self dominating set of continuous r-seminorms such that $p(x) \leq 1$ for each $p \in \mathcal{S}$ and $x \in K$. Let $\mathcal{S}_\infty = \{p_n : n \geq 0\}$ and consider

$$q(x) = \sum_{n=0}^{\infty} 2^{-n}\, p_n(x),\ x \in span(K). \tag{3.3}$$

Note that $\{p_n(x) : n \in \mathbb{N}\}$ is bounded for each $x \in span(K)$. Particularly, $p_n(x) \leq 2$ for all $x \in K-K$ and all $n \in \mathbb{N}$. Since the series (3.3) converges on $span(K)$, then q is an r-seminorm on $span(K)$ satisfying (i).

Now we show that q is continuous on $K-K$. To see this, let $\varepsilon > 0$. Then, there is an integer N such that $2^{-N} < \dfrac{\varepsilon}{8}$. For each $x, y \in K-K$, we have

$$q(y-x) \leq \sum_{n=0}^{N} 2^{-n}\, p_n(y-x) + \sum_{n=N+1}^{\infty} 2^{-n+2} < \sum_{n=0}^{N} 2^{-n}\, p_n(y-x) + \frac{\varepsilon}{2}.$$

Since the r-seminorm $\sum_{n=0}^{N} 2^{-n}\, p_n$ is continuous at 0 then, there is an open neighborhood U of 0 such that:

$$z \in U \Rightarrow \sum_{n=0}^{N} 2^{-n}\, p_n(z) < \frac{\varepsilon}{2}.$$

Hence,

$$[x, y \in K-K,\ y-x \in U] \Rightarrow q(y-x) < \varepsilon.$$

This proves (ii).

Now, since q is continuous on $K-K$ then, for any $x \in K$ and for any $\rho > 0$ the set

$$B(x,\, \rho) = \{y \in K \,:\, q(y-x) < \rho\},$$

is an open subset of K in the topology τ_K on $span(K)$ induced by the topology of X. Accordingly, each open subset of K in the topology induced by τ_q (topology defined by q on $span(K)$) is an open subset of K in the topology induced by τ_K. In other words, the topology on K generated by the r-seminorm q is weaker than the topology on K induced by the initial topology on X. Consequently, K is compact with respect the topology generated by q. This proves (iii).

Now, let $\varepsilon > 0$ and choose an integer N such that $2^{-N} < \dfrac{\varepsilon}{4}$. For $x,\, y \in K$, we have

$$\sum_{n=N+1}^{\infty} 2^{-n}\, p_n(y-x) \leq \sum_{n=N+1}^{\infty} 2^{-n+1} = 2.\, 2^{-N} < \frac{\varepsilon}{2},$$

and

$$q(y-x) \leq \sum_{n=0}^{N} 2^{-n}\, p_n(y-x) + \frac{\varepsilon}{2}.$$

Also, Tx, $Ty \in K$ implies

$$q\big(Ty - Tx\big) \leq \sum_{n=0}^{N} 2^{-n}\, p_n\big(Ty - Tx\big) + \frac{\varepsilon}{2}. \tag{3.4}$$

Since $\mathcal{S}_\infty$ is self dominating, then for each integer n, there exists an integer k_n and a real $\alpha_n > 0$ such that for each x, $y \in K$, we have

$$p_{k_n}(y-x) < \alpha_n \Rightarrow p_n\big(Ty - Tx\big) < \frac{\varepsilon}{4}.$$

Put $N' = \max\{k_0, \ldots, k_N\}$ and $\alpha = 2^{-N'} \min\{\alpha_0, \ldots, \alpha_N\}$. Since $p_k \leq 2^{N'}\, q$ for all $k \leq N'$, we have,

$$q(y-x) < \alpha \Rightarrow p_{k_n}(y-x) < \alpha_n,$$

and

$$q(y-x) < \alpha \Rightarrow p_n(Ty - Tx) < \frac{\varepsilon}{4},$$

for any x, $y \in K$ and $n \leq N$. Using (3.4) we get

$$q(y-x) < \alpha \Rightarrow q\big(Ty - Tx\big) < \varepsilon,$$

for each x, $y \in K$. As a result, the r-seminorm q satisfies (iv). □

Theorem 3.3. *Let X be a locally r-convex space and K be a compact s-convex subset of X for some $r, s \in (0, 1]$. Let $T : K \to K$ be a continuous mapping. Then for all continuous r-seminorm p on X, there exists a $x \in K$ such that $p(Tx - x) = 0$.*

Proof. Without loss of generality, we may assume that $span(K) = X$. Let p be a continuous r-seminorm on X. In view of Theorem 3.2, there is an r-seminorm q satisfying conditions (i), (ii), (iii), (iv) where $p_0 = p$. Let $N = \{x \in X : \; q(x) = 0\}$ be the kernel of q. Then the quotient space X/N equipped with the r-norm $\|\widetilde{x}\|_r = q(x)$ where $\widetilde{x} = x + N$, is an r-normed space. Taking into account the fact that the quotient map $x \mapsto \widetilde{x}$ is a continuous linear homeomorphism from X endowed with topology induced by q to X/N endowed with the r-norm topology, we deduce that $\widetilde{K} = \{\widetilde{x} : \; x \in K\}$ is a compact s-convex subset of the r-normed space X/N.
Now, let x, $y \in K$ such that $\widetilde{x} = \widetilde{y}$. For each $\varepsilon > 0$, there is a $\delta > 0$ satisfying (iv). Since $q(y-x) = 0 < \delta$ then, $q\big(Ty - Tx\big) < \varepsilon$. Hence, $q\big(Ty - Tx\big) = 0$ and so $\widetilde{T(x)} = \widetilde{T(y)}$. Accordingly, the mapping $\widetilde{T} : \widetilde{K} \to X/N$ defined by $\widetilde{T}(\widetilde{x}) = \widetilde{T(x)}$ makes sense. Taking into account that T is uniformly continuous on K with respect to q, we infer that $\widetilde{T}$ maps continuously $\widetilde{K}$ into itself. Invoking Theorem 2.1, we deduce that there is a $x \in K$ such that $\widetilde{T}(\widetilde{x}) = \widetilde{x}$. Thus, $T(x) - x \in N$ and therefore $q\big(T(x) - x\big) = 0$. Keeping in mind that $T(x) - x \in K - K$ and $p \leq q$ on $span(K)$, we conclude that $p\big(T(x) - x\big) = 0$. □

Now, we are ready to state the main result of this section.

Theorem 3.4 (Schauder-Tychonov type). *Let X be a Hausdorff locally r-convex space and K be a compact s-convex subset of X for some $r, s \in (0, 1]$. Let $T : K \to K$ be a continuous mapping. Then T has at least one fixed point in K.*

Proof. Let $\mathcal{P}_r$ denote the set of all continuous r-seminorms on X. From Theorem 3.3, we know that for every $p \in \mathcal{P}_r$ there exists $x_p \in K$ such that $p\big(T(x_p) - x_p\big) = 0$. The compactness of K ensures the existence of a subnet $(x_{p_i})_{i \in I}$ of the net $(x_p)_{p \in \mathcal{P}_r}$ converging to some $x^* \in K$. Thus, $T(x_{p_i}) \to T(x^*)$ and therefore

$$T(x_{p_i}) - x_{p_i} \to T(x^*) - x^*.$$

Let $p \in \mathcal{P}_r$. Then there exists $i_0 \in I$ such that $p \leq p_i$ for all $i \geq i_0$. Hence,

$$p\big(T(x_{p_i}) - x_{p_i}\big) \leq p_i\big(T(x_{p_i}) - x_{p_i}\big) = 0,$$

and therefore, $p\big(T(x_{p_i}) - x_{p_i}\big) = 0$ for all $i \geq i_0$. Consequently,

$$p\big(T(x^*) - x^*\big) = \lim_i p\big(T(x_{p_i}) - x_{p_i}\big) = 0.$$

Since p is arbitrary we have $T(x^*) = x^*$. □

Remarks 3.1.

(i) Theorem 3.4 is a sharpening of [5, Theorem 34, p.64], [6, Theorem 3.2, p.40] and [8, Theorem 2.3].

(ii) It is worthwhile to mention that we can remove the completeness condition from Theorem 1.2, via a completion procedure as in Theorem 2.1. However, this approach fails to remove metrizability condition and to cover the case $r = 1$.

(iii) It might be noted that a locally convex space is locally r-convex for any $0 < r \leq 1$. This is a straightforward consequence of Lemma 2.2.

As a direct consequence of Theorem 3.4, we obtain the following sharpening of the Schauder-Tychonov fixed point theorem in locally convex spaces. We particularly show that Schauder-Tychonov's statement remains valid for both convex and s-convex sets (with $0 < s \leq 1$) in locally convex spaces. This result does not follow from Theorem 1.2.

Theorem 3.5. *Let K be a nonempty compact s-convex set ($0 < s \leq 1$) in a Hausdorff locally convex space X and let $T : K \to K$ be a continuous mapping. Then T has at least one fixed point in K.*

Proof. Apply Theorem 3.4 with $r = 1$. □

Likewise, we can derive from Theorem 3.4 the following interesting result which extends Theorem 2.2.

Theorem 3.6. *Let K be a nonempty compact convex set in a Hausdorff locally r-convex space X ($0 < r \leq 1$) and let $T : K \to K$ be a continuous mapping. Then T has at least one fixed point in K.*

Proof. Apply Theorem 3.4 with $s = 1$. □

Another consequence of Theorem 3.4 is the following sharpening of [8, Theorem 2.4].

Theorem 3.7 (Schauder-Tychonov type). *Let X be a quasi-complete Hausdorff locally r-convex space, C a closed s-convex subset of X and $T : C \to C$ a continuous* *compact mapping (i.e.* $\overline{T(C)}$ *is compact) where* $0 < s \leq r \leq 1$*. Then* T *has a fixed point in* C*.*

Proof. Let $K = \overline{co_s}\big(T(C)\big)$. Invoking Theorem 3.1, we deduce that K is a compact s-convex subset of C. Since $K \subset C$ then $T(K) \subset T(C) \subset K$. Applying Theorem 3.4 we get a fixed point for T. □

We now consider the case of non-self mappings. In this context, we present the following Rothe type fixed point theorem in locally r-convex spaces.

Theorem 3.8 (Rothe type). *Let* $0 < s \leq 1$ *and* $0 < r \leq 1$*. Let* X *be a Hausdorff locally* r*-convex space and* K *be a compact* s*-convex subset of* X *such that* $0 \in int(K)$*, where* $int(K)$ *is the interior of* K*. Let* $T : K \to X$ *be a continuous mapping such that* $T(\partial K) \subset K$*, where* ∂K *is the boundary of* K*. Then* T *has a fixed point in* K*.*

Proof. Since $0 \in int(K)$, then the Minkowsky s-functional p_K of K is continuous on X (see [25, Lemma 1.5]). Define the mapping $f : X \to X$ by

$$f(x) = \frac{x}{\max\left(1, p_K(x)^{\frac{1}{s}}\right)} \text{ for } x \in X.$$

It is readily verified that f is continuous on X and $f(X) \subset K$.

The mapping $T_1 = f \circ T : K \to K$ being continuous, Theorem 3.4 ensures the existence of $x^* \in K$ such that $T_1(x^*) = x^*$. If $x^* \in int(K)$, then

$$1 > p_K(x^*) = p_K\big(T_1(x^*)\big) = \frac{p_K\big(T(x^*)\big)}{\max\big(1, p_K\big(T(x^*)\big)\big)}.$$

Thus, $p_K\big(T(x^*)\big) < 1$ and therefore $Tx^* \in K$. Consequently,

$$x^* = T_1(x^*) = f\big(T(x^*)\big) = T(x^*).$$

If $x^* \in \partial K$, then

$$x^* = T_1(x^*) = \frac{T(x^*)}{\max\left(1, p_K\big(T(x^*)\big)^{\frac{1}{s}}\right)}, \tag{3.5}$$

and therefore

$$1 = p_K(x^*) = \frac{p_K\big(T(x^*)\big)}{\max\big(1, p_K\big(T(x^*)\big)\big)}. \tag{3.6}$$

From our hypotheses we know that $T(x^*) \in K$ (since $x^* \in \partial K$) and so $p_K\big(T(x^*)\big) \leq 1$. Using (3.6) we get $p_K\big(T(x^*)\big) = 1$. Going back to (3.5), we infer that $T(x^*) = x^*$. This completes the proof. □

4. Fixed point theorems for multivalued mappings

In this section, we shall prove some fixed point theorems for multi-valued operators in r-normed and locally r-convex spaces. Before proving our main results, we give some useful definitions. Let X and Y be two topological Hausdorff spaces and $T : X \to 2^Y$ be a multi-valued operator. We say that T is upper semi-continuous at $x \in X$ if for every open set V which contains Tx, there exists a neighborhood U of x such that $T(U) \subset V$. If T is upper semi-continuous at every $x \in X$ then we say that T is upper semi-continuous. Note that T is upper semi-continuous if for every open subset V of Y, there exists an open subset U of X such that $T(U) \subset V$. The operator T is said to have closed graph if $G_T = \{(x, y) \in X \times Y : y \in Tx\}$ is a closed subset of the product space $X \times Y$.

Now, we state the main result of this section.

Theorem 4.1. *Let X be an r-normed space and C be a compact s-convex subset of X ($0 < s \leq 1$, $0 < r \leq 1$). If $T : C \to 2^C$ is upper semi-continuous and Tx is nonempty closed and s-convex for every $x \in X$. Then there exists $x_0 \in C$ such that $x_0 \in Tx_0$.*

Proof. Since C is compact then it is totally bounded. Thus, for any integer $n \geq 1$, there are $x_{1,n}, \ldots, x_{k_n,n}$ in C such that $C \subset \bigcup_{i=1}^{k_n} B\left(x_{i,n}, \frac{1}{n}\right)$. Now, for $i = 1, \ldots, k_n$ and $x \in C$, set:

$$\varphi_{i,n}(x) = \max\left\{\frac{1}{n} - \|x - x_{i,n}\|_r, 0\right\}.$$

Clearly, $\varphi_{i,n}$ is a continuous functional on C and $\sum_{i=1}^{k_n} \varphi_{i,n}(x) > 0$ for all $x \in C$. Let $y_{i,n}$ be fixed in $T(x_{i,n})$ and $T_n : C \to C$ defined by:

$$T_n(x) = \sum_{i=1}^{k_n} \left(\frac{\varphi_{i,n}(x)}{\varphi_n(x)}\right)^{\frac{1}{s}} y_{i,n}, \ x \in C,$$

where $\varphi_n(x) = \sum_{i=1}^{k_n} \varphi_{i,n}(x)$.

Invoking Theorem 2.1, we deduce that the continuous function $T_n : C \to C$ has a fixed point in C, say x_n. By extracting a subsequence, if necessary, we may assume that $(x_n)_n$ converges to some $x^* \in C$. For each $\varepsilon > 0$, the subset $U_\varepsilon = Tx^* + B(0, \varepsilon)$ is an open neighborhood of the closed subset Tx^*. Hence,

$$\bigcap_{\varepsilon > 0} U_\varepsilon = \overline{Tx^*} = Tx^*. \tag{4.1}$$

The upper semi-continuity of T ensures the existence of $\delta > 0$ such that $T\big(B(x^*, \delta)\big) \subset U_\varepsilon$. Since $x_n \to x^*$, there is a positive integer $N > \frac{2}{\delta}$ such that $x_n \in B\left(x^*, \frac{\delta}{2}\right)$ for all $n \geq N$. Let $n \geq N$ and $1 \leq i \leq k_n$. Then, $\|x_n - x_{i,n}\|_r < \frac{1}{n}$ whenever $\varphi_{i,n}(x_n) > 0$. Thus, $x_{i,n} \in B(x^*, \delta)$. Hence, $y_{i,n} \in Tx_{i,n} \subset T\big(B(x^*, \delta)\big) \subset U_\varepsilon$, and therefore $y_{i,n} \in U_\varepsilon$. Since Tx^* and $B(0, \varepsilon)$ are s-convex, then according to [13, Propostion 3, p.102] we have that U_ε is s-convex. Consequently,

$$x_n = T_n x_n = \sum_{i=1}^{k_n} \left(\frac{\varphi_{i,n}(x)}{\varphi_n(x)}\right)^{\frac{1}{s}} y_{i,n} \in U_\varepsilon.$$

Letting $n \to \infty$, we obtain $x^* \in \overline{U_\varepsilon} \subset U_{2\varepsilon}$ and so, $x^* \in \bigcap_{\varepsilon > 0} U_\varepsilon$. Going back to (4.1) we get $x^* \in Tx^*$. □

Remark 4.1. Theorem 4.1 improves [25, Theorem 2.15].

Before to state the next result, we recall the following lemma.

Lemma 4.1. [8, Proposition 2.5] *Let X, Y be topological spaces and $f : X \to 2^Y$ a set-valued mapping.*

(a) *If Y is regular, f is upper semi-continuous and for every $x \in X$ the set $f(x)$ is non empty and closed, then f has a closed graph.*

(b) *Conversely, if the space Y is compact Hausdorff and f is with closed graph, then f is upper semi-continuous.*

Theorem 4.2. *Let X be an r-normed space and C a compact s-convex subset of X ($0 < s \leq 1$, $0 < r \leq 1$). If $T : C \to 2^C$ is a mapping with closed graph and Tx is nonempty and s-convex for every $x \in C$, then there exists $z \in C$ such that $z \in Tz$.*

Proof. Keeping in mind that T has a closed graph, it is readily verified that $T(x)$ is closed for any $x \in C$. According to Lemma 4.1, T is upper semi-continuous. The result follows from Theorem 4.1. □

Remark 4.2. Theorem 4.2 is a sharpening of [25, Theorem 2.16].

Theorem 4.3. *Let X be a Hausdorff locally r-convex space and C be a compact s-convex subset of X, where $0 < r \leq 1$ and $0 < s \leq 1$. If $T : C \to 2^C$ is upper semi-continuous mapping such that Tx is nonempty closed s-convex for every $x \in C$, then T has a fixed point in C.*

Proof. We shall use some ideas from [10]. Let $\mathcal{U}$ be a basis of absolutely r-convex open neighborhoods of the null element 0 that generates the locally r-convex topology of X. Let $U \in \mathcal{U}$ be fixed and let $V \in \mathcal{U}$ such that $\overline{V} \subset U$. Since C is compact, there are $x_1, \ldots, x_n \in C$ such that $C \subset \{x_1, \ldots, x_n\} + V$. Let $K = co_s(x_1, \ldots, x_n)$ the s-convex hull of $\{x_1, \ldots, x_n\}$. It is easily seen that $K \subset C$ and K is a compact s-convex subset of the finite-dimensional topological vector space $span\{x_1, \ldots, x_n\}$. Let $T_U : K \to 2^K$ be the mapping defined by:

$$T_U x = \big(Tx + \overline{V}\big) \cap K,$$

for all $x \in K$. It is easy to check that $T_U x$ is a closed s-convex subset of K. We show that T_U is upper semi-continuous. To see this, let $x_0 \in K$ and V_0 an open set such that $T_U x_0 \subset V_0$. Since $T_U x_0$ is compact, then there is $W_0 \in \mathcal{U}$ such that $T_U x_0 + W_0 \subset V_0$ $\big($ see [21, Theorem 1.10, p.10]$\big)$. Keeping in mind the closedness of $Tx_0 + \overline{V}$, the use of [10, Lemma 1] ensures the existence of $W_1 \in \mathcal{U}$ such that:

$$\Big(\big(Tx_0 + \overline{V}\big) + W_1\Big) \cap (K + W_1) \subset T_U x_0 + W_0.$$

Since T is upper semi-continuous then, there exists an open neighborhood U_0 of x_0 such that $Tx \subset Tx_0 + W_1$ for all $x \in C \cap U_0$. Thus,

$$Tx + \overline{V} \subset \big(Tx_0 + \overline{V}\big) + W_1,$$

for all $x \in K \cap U_0$. Therefore,

$$T_U x \subset \Big(\big(Tx_0 + \overline{V}\big) + W_1\Big) \cap K \subset \Big(\big(Tx_0 + \overline{V}\big) + W_1\Big) \cap \big(K + W_1\big) \subset T_U x_0 + W_0.$$

Consequently,

$$T_U x \subset T_U x_0 + W_0.$$

Since $T_U x_0 + W_0 \subset V_0$, then $T_U x \subset V_0$. This proves that T_U is upper semi-continuous. According to Theorem 4.2, there is $x_U \in K$ such that $x_U \in T_U x_U \subset Tx_U + \overline{V}$. Therefore, $x_U \in C$ and $x_U \in Tx_U + U$.

Notice that $(x_U)_{U \in \mathcal{U}}$ is a net of elements of the compact set C. Thus, there is a subnet $(x_{U_i})_{i \in I}$ that converges to some element $x^* \in C$. We show that x^* is fixed point of T. Indeed, let $U \in \mathcal{U}$. Then, there exists $V \in \mathcal{U}$ such that $V + V + V \subset U$. Since T is upper semi-continuous at x^*, there is $W \in \mathcal{U}$ such that $W \subset V$ and $Tx \subset Tx^* + V$ for each $x \in x^* + W$. Furthermore, the subnet $(x_{U_i})_{i \in I}$ converges to x^*. Hence, there exists $i_0 \in I$ such that:

$$i \in I,\, i \geq i_0 \Rightarrow x_{U_i} \in x^* + W.$$

Accordingly,

$$i \in I,\, i \geq i_0 \Rightarrow Tx_{U_i} \subset Tx^* + V.$$

Also, there is a $j_0 \in I$ such that:

$$i \in I,\, i \geq j_0 \Rightarrow U_i \subset V.$$

Consequently,

$$i \in I,\, i \geq j_0 \Rightarrow x_{U_i} \in Tx_{U_i} + U_i \subset Tx_{U_i} + V.$$

Let $k \in I$ such that $k \geq i_0$ and $k \geq j_0$. Then $x^* \in x_{U_k} + W$ (since $-W = W$), $x_{U_k} \in Tx_{U_k} + V$ and $Tx_{U_k} \subset Tx^* + V$. Therefore,

$$x^* \in Tx_{U_k} + V + W \subset Tx^* + V + V + V \subset Tx^* + U.$$

As a result, $x^* \in Tx^* + U$. Consequently,

$$x^* \in \bigcap_{U \in \mathcal{U}} (Tx^* + U) = \overline{Tx^*}.$$

Since Tx^* is closed, then $x^* \in Tx^*$. This completes the proof. □

Remark 4.3.

(1) Theorem 4.2 extends [8, Theorem 2.6], [10, Theorem 1] and [12, Theorem].
(2) The mapping T in Theorem 4.2 is upper semi-continuous, compact-valued and X is Hausdorff. Invoking [3, Corollary 14.47, p.483], we conclude that the set of fixed points of T is compact.

Corollary 4.1. *Let X be a Hausdorff locally r-convex space and C a compact s-convex subset of X, where $0 < r \leq 1$ and $0 < s \leq 1$. If $T : C \to 2^C$ has closed graph and Tx is nonempty s-convex for every $x \in X$, then T has a fixed point in C.*

Proof. The result follows from Theorem 4.3 on the basis of Lemma 4.1. □

Remark 4.4. Corollary 4.1 is the sharpening of [3, Corollary 14.50, p.484].

5. Applications

In an attempt to illustrate our results, we shall give two applications. The first one concerns a Von Neumann's result [23](see also [7, Corollary 16.4, p.75] and [16, Corollary, p.69]). The second one is about a problem arising in game theory.

5.1. **On a Von Neumann's result.**

Theorem 5.1. *Let $X_1, \dots, X_k$ be Hausdorff locally r-convex spaces for some $0 < r \leq 1$. Let $C_1, \dots, C_k$ be nonempty compacts s-convex subsets of $X_1, \dots, X_k$, respectively, where $0 < s \leq 1$ and $k \geq 1$. Let $M_1, \dots, M_k$ be nonempty closed subsets of $C = C_1 \times \cdots \times C_k$, such that*

$$T_i(x) = \{t \in C_i : (x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_k) \in M_i\}$$

is nonempty and s-convex for any $x = (x_1, \dots, x_k) \in C$ and $1 \leq i \leq k$. Then

$$\bigcap_{i=1}^{k} M_i \neq \emptyset.$$

Proof. Consider the mapping $T : C \to 2^C$ defined by

$$T(x) = T_1(x) \times \cdots \times T_k(x)$$

for all $x \in C$. Let $\mathcal{P}_i$ denote the family of continuous r-seminorms that generates the topology of X_i for $1 \leq i \leq k$.
Let $p_1 \in \mathcal{P}_1, \dots, p_k \in \mathcal{P}_k$ and let $p : X_1 \times \cdots \times X_k \to [0, \infty)$ defined by

$$p(x) = \max_{1 \leq i \leq k} p_i(x_i)$$

for all $x = (x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$. Obviously, each p is an r-seminorm on $X_1 \times \cdots \times X_k$ and their family $\mathcal{P}$ generates a product topology on $X_1 \times \cdots \times X_k$ that makes $X_1 \times \cdots \times X_k$ a Hausdorff locally r-convex space. It is easily seen that the set $C = C_1 \times \cdots \times C_k$ is a compact s-convex subset of $X_1 \times \cdots \times X_k$ and $T(x)$ is a nonempty s-convex set for each $x \in C$. We show that T has a closed graph. To see this, notice first that the mapping $f_i : C^2 \to C$ defined by

$$f_i(x, y) = (x_1, \dots, y_i, \dots, x_k)$$

is continuous, for each $1 \leq i \leq k$, where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ are elements of C. Thus the set

$$f_i^{-1}(M_i) = \{(x, y) \in C^2 : (x_1, \dots, y_i, \dots, x_k) \in M_i\}$$

is closed. Since,

$$y \in T(x) \iff \big[(x_1, \dots, y_i, \dots, x_k) \in M_i \text{ for all } 1 \leq i \leq k\big],$$

then the graph of T:

$$\begin{aligned} G_T &= \big\{(x, y) \in C^2 : (x_1, \dots, y_i, \dots, x_k) \in M_i \text{ for all } 1 \leq i \leq k\big\} \\ &= \bigcap_{i=1}^{k} f_i^{-1}(M_i) \end{aligned}$$

is closed. Invoking Corollary 4.1, we deduce that there is $x^* = (x_1^*, \dots, x_k^*) \in C$ such that $x^* \in T(x^*) = T_1(x^*) \times \cdots \times T_k(x^*)$ and so $x_i^* \in T_i(x^*)$ for all $1 \leq i \leq k$. Thus,

$$x^* \in \bigcap_{i=1}^{k} M_i \text{ and therefore } \bigcap_{i=1}^{k} M_i \neq \emptyset. \qquad \square$$

Remark 5.1. Theorem 5.1 extends [7, Corollary 16.4, p.75] and [16, Corollary, p.69].

5.2. **On a problem from game theory.**

A game is a triple $(A,\, B,\, G)$, where A and B are two nonempty sets, whose elements are called strategies and $G:\, A\times B\to\mathbb{R}$ is the gain function. The value $G(x,\, y)$ represents the gain of a first player P and the loss of a second player Q when the player P chooses the strategy $x\in A$ and the player Q chooses the strategy $y\in B$. The goal of the player P is to maximize his gain whereas the other player Q chooses a strategy that will a total fiasco for his rival, that is, to choose $x_0\in A$ such that $\inf_{y\in B} G(x_0,y)=\sup_{x\in A}\inf_{y\in B} G(x,y)$. Similarly, the player Q chooses $y_0\in B$ such that $\sup_{x\in A} G(x,y_0)=\inf_{y\in B}\sup_{x\in A} G(x,y)$. Thus,

$$\inf_{y\in B} G(x_0,y)=\sup_{x\in A}\inf_{y\in B} G(x,y)\leq G(x_0,y_0)\leq \sup_{x\in A} G(x,y_0)=\inf_{y\in B}\sup_{x\in A} G(x,y).$$

Note that

$$\sup_{x\in A}\inf_{y\in B} G(x,y)\leq \inf_{y\in B}\sup_{x\in A} G(x,y). \tag{5.1}$$

Let A be a nonempty s-convex subset of a vector space $(0<s\leq 1)$. A mapping $f:\, A\to\mathbb{R}$ is called s-convex if,

$$f\left(\rho^{\frac{1}{s}}\, x+(1-\rho)^{\frac{1}{s}}\, y\right)\leq \rho\, f(x)+(1-\rho)\, f(y),$$

for all $x,\, y\in A$, $0\leq\rho\leq 1$, and f is called s-concave if $-f$ is s-convex.

Now, we are in position to state the following.

Theorem 5.2. *Let X, Y be a two Hausdorff locally r-convex spaces and $A\subset X$, $B\subset Y$ nonempty compact s-convex sets, where $0<r\leq 1$ and $0<s\leq 1$. Let $G:\, A\times B\to\mathbb{R}$ be a continuous function such that:*

(i) *for each $x\in A$, the function $G(x,.):\, B\to\mathbb{R}$ is s-convex,*
(ii) *for each $y\in B$, the function $G(.,y):\, A\to\mathbb{R}$ is s-concave.*

Then:

$$\inf_{y\in B}\sup_{x\in A} G(x,y)=\sup_{x\in A}\inf_{y\in B} G(x,y),$$

and the game $(A,\, B,\, G)$ has a solution.

Proof. Let $\mathcal{P}_X$ and $\mathcal{P}_Y$ denote the family of continuous r-seminorms that generate the topology of X and Y respectively. Let $p\in\mathcal{P}_X$, $p'\in\mathcal{P}_Y$ and let the mapping $q:\, X\times Y\to[0,\,\infty)$ be defined by $q(x,\, y)=\max\big(p(x),\, p'(y)\big)$ for all $(x,\, y)\in X\times Y$. Obviously, q is a continuous r-seminorm on $X\times Y$. The family of such r-seminorms generates the product topology on $X\times Y$.

Let $f:\, A\to\mathbb{R}$ and $g:\, B\to\mathbb{R}$ be the functions defined by:

$$f(x)=\inf_{y\in B} G(x,\, y),\; g(y)=\sup_{x\in A} G(x,\, y).$$

By [8, Lemma 3.3], the functions f and g are continuous. Let

$$A_y=\{x\in A:\, G(x,\, y)=g(y)\}\text{ for all } y\in B$$

and

$$B_x = \{y \in B : G(x, y) = f(x)\} \text{ for all } x \in A.$$

For all $(x, y) \in A \times B$, the sets A_y and B_x are nonempty and closed. We show that they are s-convex. To see this, let $0 \leq \rho \leq 1$, $x_1, x_2 \in A_y$, $y_1, y_2 \in B_x$ and set

$$x' = \rho^{\frac{1}{s}}\, x_1 + (1-\rho)^{\frac{1}{s}}\, x_2,\ y' = \rho^{\frac{1}{s}}\, y_1 + (1-\rho)^{\frac{1}{s}}\, y_2.$$

Since $G(x, .)$ is s-convex and $G(., y)$ is s-concave, then we have:

$$f(x) \leq G(x, y') \leq \rho\, G(x, y_1) + (1-\rho)\, G(x, y_2) = \rho\, f(x) + (1-\rho)\, f(x) = f(x),$$

$$g(y) \geq G(x', y) \geq \rho\, G(x_1, y) + (1-\rho)\, G(x_1, y) = \rho\, g(y) + (1-\rho)\, g(y) = g(y).$$

Hence, $f(x) = G(x, y')$ and $g(y) = G(x', y)$. Therefore $y' \in B_x$ and $x' \in A_y$. This shows that A_y and B_x are s-convex.

Let $C = A \times B$ and define the mapping $T : C \to 2^C$ by $T(x, y) = A_y \times B_x$ for all $(x, y) \in C$. Clearly, $T(x, y)$ is a nonempty closed s-convex subset of C. Now, we claim that the graph $G_T = \{(x, y, z, t) \in C \times C : (z, t) \in T(x, y)\}$ of T is closed. Indeed, let $(x_i, y_i)_{i\in I}$ be a net in C that converges to $(x, y) \in C$, and $(z_i, t_i) \in T(x_i, y_i)$ for each $i \in I$, such that the net $(z_i, t_i)_{i\in I}$ converges to $(z, t) \in C$. Note that:

$$(z_i, t_i) \in T(x_i, y_i) \Longleftrightarrow G(z_i, y_i) = g(y_i),\, G(x_i, t_i) = f(x_i).$$

The continuity of the functions G, f and g yields $G(z, y) = g(y)$ and $G(x, t) = f(x)$. Thus $z \in A_y$, $t \in B_x$ and so $(z, t) \in T(x, y)$. This proves our claim.

Invoking Corollary 4.1, there is $(x_0, y_0) \in C$ such that $(x_0, y_0) \in T(x_0, y_0)$, that is,

$$G(x_0, y_0) = g(y_0) = f(x_0).$$

From (5.1), it follows that

$$G(x_0, y_0) = f(x_0) \leq \sup_{x\in A} \inf_{y\in B} G(x,y) \leq \inf_{y\in B} \sup_{x\in A} G(x,y) \leq g(y_0) = G(x_0, y_0).$$

Accordingly,

$$G(x_0, y_0) = \sup_{x\in A} \inf_{y\in B} G(x,y) = \inf_{y\in B} \sup_{x\in A} G(x,y).$$

This completes the proof. □

Remark 5.2. Theorem 5.2 extends [8, Theorem 3.4], [26, Proposition 9.18, p.461] and [25, Theorem 3.2].

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