Fixed Point Theory, 22(2021), No. 2, 603-624
DOI: 10.24193/fpt-ro.2021.2.40
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINTS AND TOPOLOGICAL PROPERTIES OF EXTENDED QUASI-METRIC SPACES

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Abstract. In this paper, we introduce an extension of metric spaces, which includes S spaces. Some properties of this topological structure are analyzed. Also, a non-compactness measure and condensing correspondences are defined for these type of spaces and natural results are obtained. Moreover, fixed point theorems for functions and correspondences satisfying certain Banach orbital condition are introduced and proved. These results are applied to contractions, which are defined by means of the extended quasi-metric, both for functions and correspondences.
Key Words and Phrases: Quasi-metric spaces, non-compactness measure, fixed point, contractions, correspondences, lower and upper semicontinuity, b-metric spaces.
2020 Mathematics Subject Classification: 47H08, 47H10, 54H15, 54G99.

1. Introduction and background

Metric spaces constitute an appropriate topological structure for various mathematical models and problems in applied sciences. In particular, fixed point theory plays an important role in different branches of mathematics and other sciences. However, some problems demand a more general scenario than the classical metric spaces and some generalizations are needed. To this respect, we refer to [4, 12, 20, 23, 32] as recent references. This is also the case of the quasi-metric spaces (cf. [25, 28], for instance), which are naturally involved in a part of the harmonic analysis related to the theory of spaces of homogeneous type. We refer to [25] as a reference about this theory. These spaces are also known as b-metric spaces (see [5, 11], for instance) and extensions of these ones are recently considered in [23, 32]. In this work, we study a variant of the quasi-metric spaces, which consists of relaxing the condition of symmetry in the bivariate function defining the topology of quasi-metric spaces, according to the definitions given in [5, 11, 22, 25, 28]. Other definition of quasi-metric space is given in [4, 20], where the symmetric property is relaxed, but the standard triangular inequality is maintained. As pointed out in [28], this definition is frequently used in general topology. Anyway, they are also included in our setting. Moreover, the metric structure introduced in this paper contains the S-metric spaces introduced by Sedghi *et al.* in [30]. Based on the mentioned structure, the aim of the current paper are two fold. First, we are interested in studying some properties of the topology generated by the generalized metric introduced here. Macías and Segovia in [22] proved that the topology of a quasi-metric space is metrizable. This fact remains valid whenever the symmetry condition is dropped, as we prove in this paper. An appropriate tool, allowing to verify compactness in these spaces, consists of a non-compactness measure, which is based on the generalized metrics defining their topologies. This concept allows us to obtain, in our context, old results by Kuratowski in [21] and Horvath in [18], for classical metric spaces. Also, we define condensing correspondences and derive some natural results from this definition. A number of authors such as [17, 27, 31] have derived results based on the existence of non-compactness measures, which are axiomatically defined. To this respect, we mention that in the current work a concrete non-compactness measure is introduced. The second aim of this work consisting of proving a number of fixed point theorems. Our results allow us to prove the existence of fixed points for a wide family of functions and correspondences, which are based on a type of Banach condition consisting of a recursive chain of inequalities that involve functions or correspondences, according to the case. These results are natural extensions of different types of contractions existing in the literature. For instance, the Banach contraction principle [6] for functions and the analogous result for correspondences by Nadler [26] are remarkable contractions in the context treated in this work. Also, Berinde [7], Chatterjea [9], Ćirić [10], Kannan [19], and Reich [29] contractions admit extensions for functions and correspondences in the setting that we present in this paper. Apart from Banach orbital type condition, we need certain continuity conditions to obtain fixed points for correspondences. Accordingly, we assume the correspondences satisfy similar or weaker properties than the upper semicontinuity. To this objective, we define metrically upper (and also lower) semicontinuity of a correspondence.

Including this section, we divide the entire paper into seven sections. In Section 2 we introduce a measure of length for paths in graphs, as a motivation to the proposal of this paper. In Section 3, we first extend quasi-metric spaces with suitable examples, which are extensions of quasi-metric and S-metric spaces. Also, direct properties are given in this section. Section 4 is devoted to define the topology of these spaces and their main topological characteristics. A non-compactness measure and condensing correspondences are defined in Section 5. Moreover, some results are presented. Indeed, conditions for the existence of a minimum in a function are established. In addition, existence of fixed points, both for single valued functions and correspondences are presented. In Section 6, the results corresponding to existence of fixed point for functions are stated. Examples for different contractions are shown in this section. Results for the existence of fixed point for correspondences and their corresponding examples are stated in Section 7.

2. Motivation

This section is dedicated to highlighting the importance of generalizing the quasi metric concept outlined in the introduction.

Let us consider a nonempty set X containing vertices of graphs and $w : X \times X \to \mathbb{R}_{\geq 0}$ be a function denoting conectivity and weight between vertices, i.e. $w(x, y) \geq 0$ and $w(x, y) = 0$, if and only if, $x = y$. For undirected graphs, w is a symmetric function. Let $x_1, \ldots, x_p \in X$. The total weight of the loop passing throught these vertices, which also could be considered as its length, is denoted by

$$u(x_1, \ldots, x_p) = w(x_1, x_2) + \cdots + w(x_{p-1}, x_p) + w(x_p, x_1).$$

Observe that, for each $i \in \{1, \ldots, p\}$ and $a \in X$, $u(x_i, \ldots, x_i, a) = 2w(x_i, a)$. Consequently, if $p = 3$ and w is a metric, we have

$$u(x_1, x_2, x_3) \leq u(x_1, x_1, a) + u(x_2, x_2, a) + u(x_3, x_3, a).$$

Notice that u satisfies the properties of a S-metric as introduced by Sedghi *et al.* in [30]. However, for $p > 3$ this concept need to be extended. On the other hand, if instead of a loop, we consider any path (not necessarily a loop), the length of this path is defined as

$$u(x_1, \ldots, x_p) = w(x_1, x_2) + \cdots + w(x_{p-1}, x_p).$$

In this case, for any $a \in X$, we have

$$u(x_1, \ldots, x_p) \leq 2(u(x_1, \ldots, x_1, a) + \cdots + u(x_p, \ldots, x_p, a)).$$

Different inequalities could result when w is not a metric or the graphs are directed (digraphs), in which case, the w could be nonsymmetric. On concepts related to graphs we recommend [13, 24].

We think the subject of this paper could be of interest for network theory. However, the current work is focused in the topological and metric properties of this concept of length and its relation with the fixed point theory.

3. Extension of quasi-metric spaces

In order to extend the concept of quasi-metric space, in this section we introduce the basic definitions and some examples.

Let X be a non-empty set, $p \geq 2$, $b \geq 1$ and $u : X^p \to [0, \infty)$ a function satisfying, for each $x_1, \ldots, x_p \in X$, the following condition: (u0) $u(x_1, \ldots, x_p) = 0$, if and only if, $x_1 = \cdots = x_p$. We say that

(u1) (X, u) is a $QM(p, b)$-metric space, if and only if

$$u(x_1, \ldots, x_p) \leq b \sum_{i=1}^{p} u(x_i, \ldots, x_i, a), \text{ for all } a \in X,$$

and

(u2) (X, u) is a $QM_1(p)$-metric space, if and only if (X, u) is a $QM(p, 1)$-metric space.

Observe that a $QM(2,b)$-metric space is a quasi-metric space, were the symmetry condition has been dropped. Moreover, a $QM_1(3)$-metric space is a S-metric space, as introduced by Sedghi *et al.* in [30].

Example 3.1. Let (X,d) be a metric space and $u : X^p \to \mathbb{R}$ be defined as

$$u(x_1,\dots,x_p) = d(x_1,x_2) + \cdots + d(x_{p-1},x_p) + d(x_p,x_1).$$

Let $x_1,\dots,x_p, a \in X$ and notice that, for each $i \in \{1,\dots,p\}$, we have

$$u(x_i,\dots,x_i,a) = 2d(x_i,a).$$

Since

$$\begin{aligned} u(x_1,\dots,x_p) &\leq 2\{d(x_1,a) + \cdots + d(x_{p-1},a) + d(x_p,a)\} \\ &= u(x_1,\dots,x_1,a) + \cdots + u(x_p,\dots,x_p,a), \end{aligned}$$

(X,u) is an $QM_1(p)$-metric space.

Example 3.2. Let (X,u) be a $QM_1(p)$-metric space and define $v : X^p \to \mathbb{R}$ as $v(x_1,\dots,x_p) = u(x_1,\dots,x_p)^\alpha$, where $\alpha > 1$. Since for each $u_1,\dots,u_p \geq 0$, $(u_1 + \cdots + u_p)^\alpha \leq p^{\alpha-1}(u_1^\alpha + \cdots + u_p^\alpha)$, we have (X,v) is a $QM(p,b)$-metric space, with $b = p^{\alpha-1}$.

Let (X,u) be a $QM(p,b)$ metric space. In what follows, $d_u : X^2 \to \mathbb{R}$ stands for the function defined as $d_u(x,y) = u(x,\dots,x,y)$.

Remarks 3.1. Let (X,u) be a $QM(p,b)$-metric space. Then, for all $x,y,a \in X$, we have

(i) $d_u(x,y) \leq b\{(p-1)d_u(x,a) + d_u(y,a)\}$ and
(ii) $d_u(x,y) \leq bd_u(y,x)$.
(iii) $d_u(x,y) = 0$, if and only if, $x = y$.

Consequently, if (X,u) is a $QM_1(p)$-metric space, then,

(iv) $d_u(x,y) \leq (p-1)d_u(x,a) + d_u(y,a)$ and
(v) $d_u(x,y) = d_u(y,x)$.

Notice that if (X,u) is a $QM_1(p)$-metric space, then (X,d_u) is a quasi-metric space. However, due to d_u is not symmetric, in general, (X,d_u) need not to be a quasi-metric space if (X,u) is a $QM(b,p)$-metric space with $p \geq 2$.

Given a $QM(p,b)$-metric space (X,u), $x \in X$ and a nonempty subset A of X, we denote

$$d_u(x,A) = \inf\{d_u(x,y) : y \in A\}.$$

Limit of sequences, Cauchy sequences and completeness in $QM(p,b)$-metric spaces are naturally defined. Indeed, let $\{x_n\}_{n\in\mathbb{N}}$ a sequence in E and $x^* \in E$. This sequence is said to converge to x^*, whenever for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_u(x^*,x_n) < \epsilon$, for all $n \geq N$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be Cauchy, whenever for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_u(x_n,x_{n+p}) < \epsilon$, for all $n \geq N$ and $p \in \mathbb{N}$. The space (X,u) is said to be complete, whenever every Cauchy sequence in X converges. For the time being, these definitions are only formal, because we have not yet defined a topology on X. In Section 4, a topology on X with uniform structure is given, which is consistent with the concepts of convergence and completeness defined here.

In most of the examples treated in the literature, for S-metric spaces, the function $d_u : X \times X \to \mathbb{R}$ turns out to be effectively a metric. This is the case, for instance, in our Example 3.1. Since a number of results are based exclusively on this function, it seems necessary to present at least one example of a genuine $QM_1(p)$-metric space, that is, a space where d_u is not a classical metric. Such an example is presented below.

Example 3.3. Let $X = \{1, \dots, n\}$, $I_p = \{1, \dots, p\}$ and, for each $x = (x_1, \dots, x_p) \in X^p$, $\sigma^x : I_p \to I_p$ be a bijective function satisfying $x_{\sigma^x(1)} \leq \cdots \leq x_{\sigma^x(p)}$. Let $u : X^p \to \mathbb{R}$ be defined as

$$u(x) = u(x_1, \dots, x_p) = \max\{x_{\sigma^x(2)} - x_{\sigma^x(1)}, \dots, x_{\sigma^x(p)} - x_{\sigma^x(p-1)}\}.$$

To avoid trivial cases, we assume $n \geq 2$. Since, for all $x, y \in X$, $d_u(x, y) = |x - y|$, we easily see that $u(x_1, \dots, x_p) \leq d_u(x_1, y) + \cdots + d_u(x_p, y)$, for all $y \in X$. Consequently, (X, u) is a $QM_1(p)$-metric space. However, (X, u) is not genuine. In order to obtain a genuine $QM_1(p)$-metric, we suppose $p \geq 3$ and choose $x^* = (1, \dots, 1, n) \in X^p$. Let $u^* : X^p \to \mathbb{R}$ be defined as $u^*(x) = u(x)$ if $x \neq x^*$ and $u^*(x^*) = u(x^*) + 1 (= n)$. We have (X, u^*) is a $QM_1(p)$-metric space. To see this, we only need to prove that

$$u^*(x^*) \leq d_{u^*}(1, y) + \cdots + d_{u^*}(1, y) + d_{u^*}(n, y), \quad \text{for all } y \in X.$$

Clearly, the equality hold for $y = 1$. For $y \geq 2$, we have

$$d_{u^*}(1, y) + \cdots + d_{u^*}(1, y) + d_{u^*}(n, y) = (p - 1)(y - 1) + n - y \geq n = u^*(x^*),$$

due to $p \geq 3$. Since $d_{u^*}(1, n - 1) + d_{u^*}(n - 1, n) < d_{u^*}(1, n)$, (X, u) is a genuine $QM_1(p)$-metric space.

4. Topological properties

Let (X, u) be a $QM(p, b)$-metric space. For all $a \in X$ and $\epsilon > 0$, the following notations are stated:

$$\begin{aligned}
&\Delta = \{(x, y) \in X \times X : x = y\},\\
&U_\epsilon = \{(x, y) \in X \times X : d_u(x, y) < \epsilon\},\\
&\mathcal{B}_u = \{U_\epsilon : \epsilon > 0\},\\
&\mathcal{U}_u = \{U \subset X : \exists V \in \mathcal{B}_u, V \subset U\} \text{ and}\\
&B_u(a, \epsilon) = \{x \in X : d_u(a, x) < \epsilon\}.
\end{aligned}$$

Theorem 4.1. *The family $\mathcal{B}_u$ is a fundamental system of entourages for the Hausdorff uniformity $\mathcal{U}_u$ on X, i.e., $\mathcal{B}_u$ is a filter base satisfying the following three conditions:*

(i) $\Delta = \bigcap_{\epsilon > 0} U_\epsilon$;
(ii) *for each $U \in \mathcal{B}_u$, there exists $V \in \mathcal{B}_u$, such that $V \subset U^{-1}$; and*
(iii) *for each $U \in \mathcal{B}_u$, there exists $W \in \mathcal{B}_u$, such that $W \circ W \subset U$.*

Proof. It is easy to verify that $\mathcal{B}_u$ is a filter base on $X \times X$. Conditions (i) follows directly from (u0) in the definition of u and condition (ii) is obtained by taking into account that, from (ii) in Remarks 3.1, for each $\epsilon > 0$, $U_{\epsilon/b}^{-1} \subset U_\epsilon$. Let us prove condition (iii). Let $U \in \mathcal{B}_u$ and $\epsilon > 0$ satisfy $U = U_\epsilon$.
From (i), $d_u(x,y) \leq b[(p-1)d_u(x,a)+d_u(y,a)]$, for all $x, y, a \in X$. Let $\alpha = \epsilon/2b^2(p-1)$ and $W = U_\alpha$. Hence, from (i) and (ii) in Remarks 3.1, we have $W \circ W \subset U$ and the proof is complete. □

In the sequel, we consider the space (X, u) provided with the topology induced by the uniformity $\mathcal{U}_u$, i.e., a subset A of X is open, if and only if, for all $a \in A$, there exists $\epsilon > 0$ such that $B_u(a, \epsilon) \subset A$. We denote by τ_u this topology. It is worth taking into account that, with this topology, not necessarily, d_u is a continuous function and either a ball $B_u(a, r)$ is an open set, (see [3]).

Notice that $\{U_{1/n}; n \in \mathbb{N} \setminus \{0\}\}$ is a countable fundamental system of entourages for $\mathcal{U}_u$. Consequently, by the Alexandroff-Urysohn theorem [2], the following corollary holds.

Corollary 4.1. *The topological space (X, τ_u) is metrizable. In particular, it satisfies the first countability axiom.*

Remark 4.1. Since, in general, d_u is not a metric, any metric generating τ_u need not coincide with d_u. According to Example 3.3, this fact is true even for $QM_1(p)$-metric spaces, for $p \geq 3$. However, in this case, (X, d_u) is a quasi-metric space and hence there exists a metric ρ generating the topology τ_u, such that $d_u^\beta \leq 2\rho \leq 2d_u^\beta$, where $0 < \beta \leq 1$ (see [1, 22, 28]).

Corollary 4.2. *Let (X, u) be a $QM(p, b)$-metric space, $x \in X$ and $f : X \to X$ a function. The following two conditions are equivalent:*

(i) *f is continuous at x, and*
(ii) *for all sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$.*

Corollary 4.3. *Let F be a nonempty subset X. The following three conditions are equivalent:*

(i) *F is closed;*
(ii) *for all $x \in X$, $d_u(x, F) = 0$ implies that $x \in F$, and*
(iii) *all sequence $\{x_n\}_{n\in\mathbb{N}\setminus\{0\}}$ in F, converging to $x \in X$, satisfies $x \in F$.*

Remarks 4.1. Suppose (X, u) is a $QM_1(p)$-metric space, A a nonempty subset of X and $f_A : X \to \mathbb{R}$ defined as $f_A(x) = d_u(x, A)$. Then, for all $x, y \in X$, we have $|f_A(x) - f_A(y)| \leq (p-1)d_u(x, y)$ and for all $a, b \in X$,

$$|d(x, y) - d(a, b)| \leq (p-1)\{d(x, a) + d(y, b)\}.$$

Consequently, in this type of spaces, the functions f_A and $d_u : X \times X \to \mathbb{R}$ are continuous. Moreover, for all $a \in X$ and $r > 0$, $B_u(a, r)$ is an open set. Indeed, for all $x \in B_u(a, r)$, $B_u(x, r') \subset B_u(a, r)$, where $r' = (r - d_u(x, a))/(p-1)$.

It is clear that the definition of completeness, according to the uniformity $\mathcal{U}$, is coherent with that given in Section 3.

A subset A of X is said to be bounded, whenever there exist $a \in X$ and $r > 0$ such that $A \subset B_u(a, r)$. We denote by $\mathcal{B}(X)$ the family of all bounded sets of X and by $\mathcal{C}(X)$ the family of all nonempty and closed subsets of X. In what follows, we denote $\mathcal{CB}(X) = \mathcal{C}(X) \cap \mathcal{B}(X)$ and $B_u(A, \epsilon) = \bigcup_{a \in A} B_u(a, \epsilon)$, for each $A \in \mathcal{B}(X)$ and $\epsilon > 0$.

The concepts of convergence, Cauchy sequence and completeness coincide with the corresponding ones, when they are defined in terms of the uniformity $\mathcal{U}_u$. In particular, a subset A of X is precompact (c.f. [8]), if and only if, for any $\epsilon > 0$, there exist $A_1, \ldots, A_r \subset X$ such that, for each $i \in \{1, \ldots, r\}$, $A_i \times A_i \subset U_\epsilon$ and $A \subset A_1 \cup \cdots \cup A_r$.

Let $T : X \to \mathcal{CB}(X)$ be a correspondence and B a subset of X. We denote

$$T^{-1}(B) = \{x \in X : Tx \cap B \neq \emptyset\}.$$

We say that T is lower (respectively, upper) semicontinuous, whenever for all open (respectively, closed) subset B of X, we have $T^{-1}(B)$ is an open (respectively, closed) subset of X. A correspondence is said to be continuous, whenever it is lower and upper semicontinuous.

Let $h_T : X \to [0, \infty)$ be the function defined as $h_T(x) = d_u(x, Tx)$. We say that T is metrically lower (respectively, upper) semicontinuous, whenever h_T is upper (respectively, lower) semicontinuous. If h_T is continuous, we say that T is metrically continuous. These concepts were previously introduced in [15] for classical metric spaces.

As we show below, in $QM_1(p)$-metric spaces, semicontinuity is stronger than metric semicontinuity.

Theorem 4.2. *Suppose (X, u) be a $QM_1(p)$-metric space and let $T : X \to \mathcal{CB}(X)$ be a correspondence.*

(i) *if T is lower semi continuous, then T is metrically lower semi continuous, and*

(ii) *if T is upper semi continuous, then T is metrically upper semi continuous.*

Proof. Let suppose T is lower semicontinuous. Let $\alpha > 0$ and

$$A = \{x \in E : d_u(x, Tx) < \alpha\}.$$

If A is an emptyset, A is open. Hence, in order to prove that, in general, A is an open set, we suppose $A \neq \emptyset$ and choose $a \in A$. Accordingly, there exist β and $y \in Ta$ such that $d_u(a, Ta) < d_u(a, y) < \beta < \alpha$. Consequently, there exists a neighbourhood $V'(a)$ of a such that $d_u(x, y) < \beta$, for all $x \in V'(a)$. Let

$$G = \{x \in X : d_u(x, y) < (\alpha - \beta)/(p - 1)\}.$$

Since G is open, $Ta \cap G \neq \emptyset$ and T is lower semicontinuous, there exists $V''(a)$, neighbourhood of a such that $Tx \cap G$, for all $x \in V''(a)$. Accordingly, for all $x \in V(a) = V'(a) \cap V''(a)$, $d_u(x, y) < \beta$ and there exists $b_x \in Tx$ such that $(p - 1)d_u(b_x, y) < \alpha - \beta$. Hence

$$d_u(x, Tx) \leq d_u(x, b_x) \leq (p - 1)d_u(b_x, y) + d_u(x, y) < \alpha, \quad \text{for all } x \in V(a),$$

which proves that T is metrically lower semicontinuous.

Next suppose T is upper semicontinuous. Let $\alpha \in \mathbb{R}$,

$$A = \{x \in E : d_u(x, Tx) > \alpha\}$$

and suppose $a \in A$. Choose $\beta \in \mathbb{R}$ such that $\beta > \alpha$ and $d_u(a, Ta) > \beta$. Let

$$G = \{y \in E : d_u(y, Ta) < (\beta - \alpha)/2(p-1)\}.$$

Since $Ta \subseteq G$, G is open and T is upper semicontinuous, there exists $U'(a)$ neighborhood of a such that for each $x \in U'(a)$, $Tx \subseteq G$. This implies that for each $x \in U'(a)$ and each $y \in Tx$, $d_u(y, Ta) < (\beta-\alpha)/2(p-1)$. Let $\eta > 0$ and choose $z(y) \in Ta$ such that $d_u(y, z(y)) < d_u(y, Ta) + \eta$. Hence

$$\begin{array}{rcl} \beta & < & d_u(a, Ta) \\ & \leq & (p-1)d_u(z(y), y) + d_u(y, a) \\ & < & (p-1)(d_u(y, Ta) + \eta) + d_u(a, y) \\ & < & (p-1)\eta + (\beta - \alpha)/2 + d_u(a, y). \end{array}$$

Since $\eta > 0$ is arbitrary, we have $d_u(a, y) \geq (\alpha+\beta)/2$ and hence, $d_u(a, Tx) \geq (\alpha+\beta)/2$. Let $U(a) = U'(a) \cap B(a, (\beta - \alpha)/(p-1))$ and note that for each $x \in U(a)$,

$$\beta < d_u(a, Tx) \leq (p-1)d_u(a, x) + d_u(x, Tx) < \beta - \alpha + d_u(x, Tx).$$

This proves that $U(a) \subseteq A$ and therefore, T is metrically upper semicontinuous, which concludes the proof. $\square$

Example 4.1. Let $u : \mathbb{R}^p \to \mathbb{R}$ be the function defined as

$$u(x_1, \dots, x_p) = (|x_1 - x_2| + \cdots + |x_{p-1} - x_p|)^2.$$

Hence, $(\mathbb{R}, u)$ is a $QM(p, p)$-metric space and $d_u(x, y) = (x - y)^2$, for all $x, y \in \mathbb{R}$. Let $S : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ and $T : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ be two correspondence defined as

$$Sx = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ [-1, 1] & \text{if} \quad x \neq 0 \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} [-1, 1] & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Observe that $S^{-1}(1/2, 1) = \{0\}$ and $T^{-1}[1/2, 1] = \mathbb{R} \setminus \{0\}$. Hence, S is not lower semicontinuous and T is not upper semicontinuous. But, on the other hand, $h_S(x) = x^2$ and

$$h_T(x) = \begin{cases} (|x| - 1)^2 & \text{if} \quad |x| > 1 \\ 0 & \text{if} \quad |x| \leq 1. \end{cases}$$

Thus, S and T are metrically continuous. Therefore, this example along with Theorem 4.2 show that, in $QM_1(p)$-metric spaces, metrical semicontinuity is strictly weaker than semicontinuity.

Remark 4.2. In Section 7, we show that upper metrically semicontinuity is related with the existence of fixed points for correspondences.

5. A non-compactness measure

Let (X, u) be a $QM(p, b)$-metric space. Even though an explicit metric generating the topology of a $QM(p, b)$-metric space is not defined in this paper, a non-compactness measure (NCM) can be defined by mean of its uniform structure. In [14] such a NCM is defined, which, according to our uniformity, is given by $\alpha : \mathcal{B}(X) \to [0, \infty)$ such that

$$\alpha(A) = \inf\{\epsilon > 0 : (\exists x_1, \ldots, x_r \in X)(A \subseteq B_u(x_1, \epsilon/2) \cup \cdots \cup B_u(x_r, \epsilon/2))\}.$$

From Theorem 3, Section §4.2, Chapter II in [8], $A \in \mathcal{B}(X)$ is precompact, if and only if, $\alpha(A) = 0$. It is easy to see that α satisfies the usual properties of a NCM. Indeed,

(a) $\alpha(A) \leq \alpha(B)$, whenever $A \subseteq B$, $\quad (A, B \in \mathcal{B}(X)$,

(b) $\alpha(A \cup B) \leq \max\{\alpha(A), \alpha(B)\}$, $\quad (A, B \in \mathcal{B}(X))$,

(c) $\alpha(\overline{A}) = \alpha(A)$, $\quad (A \in \mathcal{B}(X))$,

(d) $\alpha(B_u(A, \epsilon)) < \alpha(A) + \epsilon$, $\quad (A \in \mathcal{B}(X)$ and $\epsilon > 0)$.

Remark 5.1. Suppose (X, u) is a $QM(p, b)$-complete metric space and let $\{B_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets in $\mathcal{CB}(X)$ such that $\inf\{\alpha(B_\lambda) : \lambda \in \Lambda\} = 0$ and let

$$E = \bigcap_{\lambda\in\Lambda} B_\lambda.$$

Hence, from Theorem 5 in [14], an old result by Kuratowski in [21] and another by Horvath in [18], for classical metric spaces, are obtained in the context of this paper. Indeed, we have E is compact and nonempty.

The above remark enables us to obtain the following proposition.

Proposition 5.1. *Let (X, u) be a bounded $QM(p, b)$-complete metric space and $f : X \to \mathbb{R}$ be a lower semicontinuous function such that*

$$\inf_{x\in X} \alpha(\{y \in X : f(y) \leq f(x)\}) = 0.$$

Then, f is bounded from below and there exists $x^* \in X$ such that $f(x^*) \leq f(x)$, for all $x \in X$.

Proof. For each $x \in X$, let $B_x = \{y \in X : f(y) \leq f(x)\}$. Since f is lower semicontinuous, $\{B_x\}_{x\in X}$ is a family of subsets in $\mathcal{CB}(X)$ and, by assumption,

$$\inf\{\alpha(B_x) : x \in X\} = 0.$$

Since $\bigcap_{x\in X}\{y \in X : f(y) \leq f(x)\}$ is nonempty, there exists $x^* \in X$ such that $f(x^*) \leq f(x)$, for all $x \in X$, concluding the proof. $\square$

Let $T : D \subseteq X \to \mathcal{CB}(X)$ be a correspondence. We say that T is condensing, if $\alpha(T(A)) < \alpha(A)$, for all $A \in \mathcal{CB}(X)$ such that $\alpha(A) > 0$.

Example 5.1. Let (X, u) be a $QM(p, b)$-metric space and $f : X \to X$ a d_u-contraction, i.e., there exists $k \in [0, 1)$ such that $d_u(f(x), f(y)) \leq k d_u(x, y)$, for all $x, y \in X$. As in the case of classic metric spaces, f turns out to be condesing. Indeed, if for some $\eta > 0$ and $A \in \mathcal{CB}(X)$, we have $\alpha(A) = \eta$, then $\alpha(f(A)) \leq k\eta$ and thus $\alpha(f(A)) < \alpha(A)$.

Theorem 5.1. *Suppose (X, u) is a bounded $QM_1(p)$-metric space and let $T : X \to \mathcal{CB}(X)$ be a metrically upper semicontinuous correspondence such that, for each $x \in X$, there exists $y \in Tx \setminus \{x\}$ satisfying $d_u(y, Ty) < d_u(x, y)$. Additionally, suppose that one of the following two conditions hold:*

(i) *X is compact or*
(ii) *X is complete and T is condensing.*

Then, there exists $x^* \in X$ such that $x^* \in Tx^*$.

Proof. First, we assume condition (i) holds. Since $h_T : X \to \mathbb{R}$ is lower semicontinuous and X is compact, h_T has a minimum at some $x^* \in X$. Suppose $x^* \notin Tx^*$. By assumption, there exists $y^* \in Tx^* \setminus \{x^*\}$ such that

$$h_T(y^*) = d_u(y^*, Ty^*) < d_u(x^*, Tx^*) = h_T(x^*),$$

which contradicts that h_T attains a minimum at x^*. Hence, $x^* \in Tx^*$.

Next, if condition (ii) holds, it follows from Theorem 7 in [14] that there exists a precompact set $C \in \mathcal{CB}(X)$ such that $T(C) \subseteq C$. Since X is complete and C is closed, C is compact and from what was proved, there exists $x^* \in X$ such that $x^* \in Tx^*$, which concludes the proof. □

From Theorems 4.2 and 5.1, the following corollary holds.

Corollary 5.1. *Suppose (X, u) is a bounded $QM_1(p)$-metric space and let $T : X \to \mathcal{CB}(X)$ be an upper semicontinuous correspondence such that, for each $x \in X$, there exists $y \in Tx \setminus \{x\}$ satisfying $d_u(y, Ty) < d_u(x, y)$. Additionally, suppose that one of the following two conditions hold:*

(i) *X is compact or*
(ii) *X is complete and T is condensing.*

Then, there exists $x^* \in X$ such that $x^* \in Tx^*$.

A set-valued version of an old result due to Furi Vignoli [16] can be stated, in this setting, as follows.

Proposition 5.2. *Let D be a complete subset of a $QM(p, b)$-metric space X, $T : D \to \mathcal{CB}(X)$ be a condensing metrically upper semi-continuous correspondence and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in D such that $\{h_T(x_n)\}_{n\in\mathbb{N}}$ converges to zero. Then, $\{x_n\}_{n\in\mathbb{N}}$ is compact and any limit point x^* of $\{x_n\}_{n\in\mathbb{N}}$ satisfies $x^* \in Tx^*$.*

Proof. Let $\epsilon > 0$ and $M = \{x_n : n \in \mathbb{N}\}$ and $T(M) = \bigcup_{n\in\mathbb{N}} Tx_n$. There exists $N \in \mathbb{N}$ such that $M \subset \{x_0, \ldots, x_N\} \cup B_u(T(M), \epsilon)$ and hence, from property (d) of α, we have $\alpha(M) \leq \alpha(B_u(T(M), \epsilon)) \leq \alpha(T(M)) + \epsilon$.
From this, we obtain $\alpha(M) \leq \alpha(T(M))$ and since T is condensing, we have $\alpha(M) = 0$. Hence, $\{x_n\}_{n\in\mathbb{N}}$ is compact. Let $x^* \in D$ be a limit point of $\{x_n\}_{n\in\mathbb{N}}$ and $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence such that $\lim_{n\to\infty} d_u(x^*, x_{n_k}) = 0$. Since h_T is lower semicontinuous, we have

$$d_u(x^*, Tx^*) = h_T(x^*) \leq \liminf h_T(x_{n_k}) = \lim d_u(x^*, x_{n_k}) = 0.$$

Therefore, $x^* \in Tx^*$ and the proof is complete. □

6. Fixed point theorems

Lemma 6.1. *Let $x_0, x_1, \ldots, x_m \in X$. Then, for all $n \leq m-1$,*

$$d_u(x_0, x_m) \leq (p-1)\sum_{k=0}^{n} b^{2k+1} d_u(x_k, x_{k+1}) + b^{2n+1} d_u(x_m, x_{n+1}).$$

In particular,

$$d_u(x_0, x_m) \leq (p-1)\sum_{k=0}^{m-1} b^{2k+1} d_u(x_k, x_{k+1}).$$

Proof. It easily follows from the induction principle and conditions (i) and (ii) in Remarks 3.1. □

Given a function $f : X \to X$, we denote $f^1 = f$ and $f^{n+1} = f^n \circ f$, for all $n \in \mathbb{N}$.

Theorem 6.1. *Suppose (X, u) is complete and $f : X \to X$ is a continuous function such that for all $x \in X$, $d_u(f(x), f^2(x)) \leq k_f d_u(x, f(x))$, where $k_f b^2 \in [0, 1)$. Then, there exists a unique $x^* \in X$ such that $f(x^*) = x^*$ and, for all $x \in X$, the following two conditions hold:*

(i) $\lim_{n\to\infty} f^n(x) = x^*$, *and*
(ii) $d_u(x^*, f^n(x)) \leq b^2(p-1)k_f^n d_u(x, f(x))/(1 - k_f b^2)$, *for all $n \in \mathbb{N} \setminus \{0\}$.*

Proof. The uniqueness of x^* easily follows. Choose $x \in X$ and notice that

$$d_u(f^n(x), f^{n+1}(x)) \leq k_f d_u(f^{n-1}(x), f^n(x)) \leq \cdots \leq k_f^n d_u(x, f(x)).$$

By Lemma 6.1, we have

$$\begin{aligned} d_u(f^n(x), f^{n+m}(x)) &\leq (p-1)\sum_{k=0}^{m-1} b^{2k+1} d_u(f^{n+k}(x), f^{n+k+1}(x)) \\ &\leq (p-1)\sum_{k=0}^{m-1} b^{2k+1} k_f^{n+k} d_u(x, f(x)) \\ &= b(p-1)k_f^n \sum_{k=0}^{m-1} (k_f b^2)^k d_u(x, f(x)). \end{aligned}$$

Consequently, for all $m, n \in \mathbb{N} \setminus \{0\}$,

$$d_u(f^n(x), f^{n+m}(x)) \leq b(p-1)k_f^n d_u(x, f(x))/(1 - k_f b^2). \tag{6.1}$$

From this inequality, it is obtained that $\{f^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and since (X, u) is a complete $QM(p, b)$-metric space, there exists $x^* = x^*(x) \in X$ such that (i) holds. Observe that, for all $x, y \in X$,

$$d_u(x^*(x), x^*(y)) = \limsup d_u(f^n(x), f^n(y)) = 0.$$

Hence x^*, in (i), does not depend on x. Moreover, from (i) and the continuity of f at x^*, we have $f(x^*) = x^*$. On the other hand, $\lim_{m\to\infty} d_u(x^*, f^{m+n}(x)) = 0$ and

$$d_u(x^*, f^n(x)) \leq b(p-1)d_u(x^*, f^{m+n}(x)) + b d_u(f^n(x), f^{m+n}(x)).$$

Hence, by taking limit as m goes to ∞ in the above inequality, condition (ii) follows from (6.1), which concludes the proof. $\square$

Remark 6.1. The continuity of f in Theorem 4.1 is essential. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 + \frac{x}{2} & \text{if} \quad x < 2 \\ \frac{x}{2} - 1 & \text{if} \quad x \geq 2. \end{cases}$$

We have $|f(x) - f^2(x)| \leq (1/2)|x - f(x)|$, for all $x \in \mathbb{R}$, and f has no fixed point.

Corollary 6.1. *Suppose (X, u) is a complete $QM_1(p)$-metric space and $f : X \to X$ is a continuous function such that for all $x \in X$, $d_u(f(x), f^2(x)) \leq k_f d_u(x, f(x))$, where $k_f \in [0, 1)$. Then, there exists a unique $x^* \in X$ such that $f(x^*) = x^*$ and, for all $x \in X$, the following two conditions hold:*

(i) *$\lim_{n\to\infty} f^n(x) = x^*$, and*
(ii) *$d_u(f^n(x), x^*) \leq (p - 1)k_f^n d_u(x, f(x))/(1 - k_f)$, for all $n \in \mathbb{N} \setminus \{0\}$.*

Examples 6.1. The following examples illustrate Theorem 6.1.

(6.1.1) We say that f is a $QM(p, b)$-metric contraction, whenever

$$d_u(f(x), f(y)) \leq kd_u(x, y), \text{ for all } x, y \in X,$$

where $kb^2 \in [0, 1)$. Since f is continuous and $d_u(f(x), f^2(x)) \leq kd_u(x, f(x))$, for all $x \in X$, Theorem 6.1 applies with $k_f = k$.

(6.1.2) A continuous function f is said to be a Kannan $QM(p, b)$-metric contraction (see [19]), whenever

$$d_u(f(x), f(y)) \leq \alpha(d_u(x, f(x)) + d_u(y, f(y))), \quad \text{for all } x, y \in X,$$

where $0 \leq \alpha < 1/(1 + b^2)$. Hence, for all $x \in X$,

$$d_u(f(x), f^2(x)) \leq k_f d_u(x, f(x)),$$

where $k_f = \alpha/(1 - \alpha)$ and since, $kb^2 < 1$, it follows from Theorem 6.1 that f has a fixed point, x^*. Moreover, for all $x \in X$ and $n \in \mathbb{N} \setminus \{0\}$, we have

$$d_u(x^*, f^n(x)) \leq b^2\alpha(p - 1)\left(\frac{\alpha}{1 - \alpha}\right)^{n-1} \frac{d_u(x, f(x))}{1 - \alpha(1 + b^2)}.$$

Notice that, for $p = 2$ and $b = 1$, the original result by Kannan in [19] is recovered.

(6.1.3) A condition for a continuous function f to be a Chatterjea $QM(p, b)$-metric contraction (see [9]), with constant α, is naturally given by

$$d_u(f(x), f(y)) \leq \alpha[d_u(x, f(y)) + d_u(y, f(x))], \quad \text{for all } x, y \in X,$$

where $\alpha\{(p - 1)b + b^2\} < 1$. Hence, from Theorem 6.1, f has a fixed point, x^*, and for all $x \in X$ and $n \in \mathbb{N} \setminus \{0\}$, we have

$$d_u(x^*, f^n(x)) \leq \alpha b^3(p - 1)^2 \left(\frac{\alpha(p - 1)b}{1 - \alpha b^2}\right)^{n-1} \frac{d_u(x, f(x))}{1 - \alpha b^2(1 + (p - 1)b)}.$$

If $p = 2$ and $b = 1$, as in Chatterjea [9], Theorem 6.1 applies for $\alpha < 1/2$.

(6.1.4) A $QM(p,b)$-Reich contraction is defined as a continuous function f satisfying

$$d_u(f(x), f(y)) \leq \alpha d_u(x, y) + \beta d_u(x, f(x)) + \gamma d_u(y, f(y)),$$

for all $x, y \in X$, where α, β and γ are constants such that $0 \leq (\alpha+\beta)b^2 < 1-\gamma$. Accordingly, Theorem 6.1 implies that such a f has a fixed point, x^*, and for all $x \in X$ and $n \in \mathbb{N} \setminus \{0\}$, we have

$$d_u(x^*, f^n(x)) \leq b^2(p-1)\left(\frac{\alpha+\beta}{1-\gamma}\right)^n \frac{(1-\gamma)d_u(x, f(x))}{1-\gamma-(\alpha+\beta)b^2}.$$

Conclusion of Theorem 3 in Reich [29] is obtained by applying Theorem 6.1 with $p = 2$ and $b = 1$.

(6.1.5) We extend the concept of weak contraction by Berinde [7], as follows. We say $f : X \to X$ is a $QM(p,b)$-contraction, if f is continuous and there exists $\delta \in [0, 1/b^2)$ and $L \geq 0$, such that

$$d_u(f(x), f(y)) \leq \delta d_u(x, y) + L d_u(y, f(x)).$$

Since for each $x \in X$, $d_u(f(x), f^2(x)) \leq \delta d_u(x, f(x))$, Theorem 6.1 implies that f has a fixed point, x^*, and for all $x \in X$ and $n \in \mathbb{N} \setminus \{0\}$, we have

$$d_u(x^*, f^n(x)) \leq \frac{b^2(p-1)\delta^n d_u(x, f(x))}{(1-\delta b^2)}, \quad \text{for all } n \in \mathbb{N} \setminus \{0\}.$$

(6.1.6) A version of quasi contraction (Ćirić) mapping with constant α, is defined by stating f is continuous and satisfies

$$\begin{aligned} d_u(f(x), f(y)) \leq \alpha \max\{&d_u(x, y), d_u(x, f(x)), \\ &d_u(y, f(y)), d_u(x, f(y)), d_u(y, f(x))\}, \end{aligned}$$

for all $x, y \in X$, such that $\alpha b(p-1+b) < 1$. Indeed, we have

$$d_u(f(x), f^2(x)) \leq \alpha \max\{d_u(x, f(x)), d_u(f(x), f^2(x)), d_u(x, f^2(x))\}.$$

But $\alpha < 1/2$ and hence

$$\begin{aligned} d_u(f(x), f^2(x)) &\leq \alpha \max\{d_u(x, f(x)), d_u(x, f^2(x))\} \\ &\leq \alpha\{b(p-1)d_u(x, f(x)) + b d_u(f^2(x), f(x))\}. \end{aligned}$$

That is

$$d_u(f(x), f^2(x)) \leq \left(\frac{\alpha b(p-1)}{1-\alpha b^2}\right) d_u(x f(x)).$$

By Theorem 6.1, f has a fixed point, x^*, and for all $x \in X$ and $n \in \mathbb{N} \setminus \{0\}$, we have

$$d_u(x^*, f^n(x)) \leq \alpha b^3(p-1)^2 \left(\frac{\alpha b(p-1)}{1-\alpha b^2}\right)^{n-1} \frac{d_u(x, f(x))}{1-\alpha b(p-1+b)}.$$

Condition $\alpha b(p-1+b) < 1$, in Example (6.1.6), may be improved in order to obtain existence of fixed point for quasi contractions. To this end, an independent result of Theorem 6.1 is stated below, which is an extension of the main result of Ćirić in [10].

The concept of α-quasi contraction defined in Ćirić [10] may be extended in our setting as a function $f : X \to X$ satisfying

$$d_u(f(x), f(y)) \leq \alpha \max\{d_u(x, y), d_u(x, f(x)), d_u(y, f(y)), d_u(x, f(y)), d_u(y, f(x))\},$$

for all $x, y \in X$, such that $\alpha b^2 < 1$. Indeed, we have

Theorem 6.2. *Let (X, u) be a complete $QM(p, b)$ metric space and $f : X \to X$ be a quasi contraction, as defined in Example (6.1.6), with $\alpha b^2 < 1$. Then, there exists $x^* \in X$ such that the following two condition hold:*

(i) $\lim_{n\to\infty} f^n(x) = x^*$, *for all* $x \in X$, *and*
(ii) $d_u(x^*, f^n(x)) \leq (\alpha b)^n (p-1) b^2 d_u(x, f(x))/(1 - \alpha b^2)$, *for all* $n \in \mathbb{N} \setminus \{0\}$.

Moreover, x^* is a fixed point for f, if one of the following three conditions hold:

(iii) f *is continuous at* x^*,
(iv) $d_u(x^*, \cdot)$ *is continuous, or*
(v) $d_u(\cdot, x^*)$ *is continuous.*

Proof. For each $n \in \mathbb{N} \setminus \{0\}$ and $x \in X$, let $O(x, n) = \{f^k(x) : 0 \leq k \leq n\}$, where $f^0(x) = x$, and $O(x, \infty) = \bigcup_{n\geq 1} O(x, n)$. Given a nonempty subset A of X, we denote by $\delta[A] = \sup\{d_u(x, y) : x, y \in A\}$ the diameter of A. Hence, for each $i, j \in \{1, \dots, n\}$ and $x \in X$, we have

$$d_u(f^i(x), f^j(x)) \leq \alpha \delta[O(x, n)].$$

In particular, since $\alpha < 1$, there exists $k_n \leq n$ such that

$$M_n(x) := \max\{d_u(x, f^{k_n}(x)), d_u(f^{k_n}(x), x)\} = \delta[O(x, n)]. \tag{6.2}$$

Moreover,

$$\begin{aligned} d_u(x, f^{k_n}(x)) &\leq (p-1) b d_u(x, f(x)) + b d_u(f^{k_n}(x), f(x)) \\ &\leq (p-1) b d_u(x, f(x)) + \alpha b M_n(x). \end{aligned}$$

Hence, $M_n(x) \leq (p-1) b^2 d_u(x, f(x))/(1 - \alpha b^2)$ and consequently,

$$\delta[O(x, \infty)] \leq \frac{(p-1) b^2}{1 - \alpha b^2} d_u(x, f(x)). \tag{6.3}$$

For each $n, r \in \mathbb{N}$ and $x \in X$, we have

$$d_u(f^n(x), f^{n+r}(x)) \leq \alpha \delta[O(f^{n-1}(x), r+1)]$$

and, as in (6.2), there exists $k \leq r$ such that $\delta[O(f^{n-1}(x), r+1)] = M'_n(x)$, where

$$M'_n(x) = \max\{d_u(f^{n-1}(x), f^{k+n}(x)), d_u(f^{k+n}(x), f^{n-1}(x))\}.$$

As before, $d_u(f^{n-1}(x), f^{k+n}(x)) \leq \alpha \delta[O(f^{n-2}(x), r+2)]$ and hence

$$M'_n(x) = \delta[O(f^{n-1}(x), r+1)] \leq \alpha b \delta[O(f^{n-2}(x), r+2)].$$

Thus, inductively, we have

$$d_u(f^n(x), f^{n+r}(x)) \leq \alpha \delta[O(f^{n-1}(x), r+1)] \leq \cdots \leq \alpha^n b^{n-1} \delta[O(x, r+n)]$$

and from (6.3), we obtain

$$d_u(f^n(x), f^{n+r}(x)) \leq (\alpha b)^n \frac{(p-1) b}{1 - \alpha b^2} d_u(x, f(x)). \tag{6.4}$$

Thus, for each $x \in X$, $\{f^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence and since (X,u) is complete, there exists $x^* \in X$ such that condition (i) holds. From (6.4),

$$d_u(x^*, f^n(x)) \le b(p-1)d_u(x^*, f^{n+r}(x)) + (\alpha b)^n \frac{(p-1)b^2}{1-\alpha b^2} d_u(x, f(x)),$$

and by taking limit as r goes to ∞, condition (ii) is obtained.

From (i), x^* is a fixed point for f, when f is continuous at x^*. Next, we suppose $d_u(\cdot, x^*)$ or $d_u(x^*, \cdot)$ are continuous. Notice that

$$\begin{aligned} d_u(x^*, f(x^*)) &\le (p-1)bd_u(x^*, f^n(x^*)) + bd_u(f(x^*), f^n(x^*)) \\ &\le (p-1)bd_u(x^*, f^n(x^*)) + \alpha b M_n, \end{aligned}$$

where $M_n = \max A_n$ and

$$\begin{aligned} A_n = \{&d_u(x^*, f^{n-1}(x^*)), d_u(x^*, f(x^*)), d_u(f^{n-1}(x^*), f^n(x^*)), \\ &d_u(x^*, f^n(x^*)), d_u(f^{n-1}(x^*), f(x^*))\}. \end{aligned}$$

It is easy to see that $\limsup M_n \le bd_u(x^*, f(x^*))$. Consequently,

$$d_u(x^*, f(x^*)) \le \alpha b^2 d_u(x^*, f(x^*))$$

and since $\alpha b^2 < 1$, we have $d_u(x^*, f(x^*)) = 0$, which concludes the proof. □

Corollary 6.2. *Suppose (X,u) is a $QM_1(p)$ metric space and $f: X \to X$ is a quasi contraction, as defined in Example (6.1.6), with $\alpha < 1$. Then, there exists a fixed point $x^* \in X$ for f, such that the following two condition hold:*

(i) $\lim_{n\to\infty} f^n(x) = x^*$, *for all* $x \in X$, *and*
(ii) $d_u(x^*, f^n(x)) \le \alpha^n (p-1) d_u(x, f(x))/(1-\alpha)$, *for all* $n \in \mathbb{N} \setminus \{0\}$.

Proof. It directly follows from Theorem 6.2 and Remarks 4.1. □

Remark 6.2. Even though most of the conclusions of Examples 6.1 may be also obtained from Theorem 6.2, both Theorems 6.1 and 6.2 have their own importance and they are independent, in the sense that, each of them has an independent proof. In particular, it is not possible to obtain the conclusions of Example (6.1.5) by means of Theorem 6.2 or Corollary 6.2. However, Corollary 6.2 implies that the continuity in Examples (6.1.2)-(6.1.4) could be dropped when (X,u) is a $QM_1(p)$-metric space.

7. Fixed point theorems for correspondences

Theorem 7.1. *Let (X,u) be a complete $QM(p,b)$-metric space, $T: X \to \mathcal{CB}(X)$ be a correspondence and $k_T \in [0, 1/b^2)$. Assume that for all $x \in X$,*

$$\inf_{y\in Tx} d_u(y, Ty) \le k_T d_u(x, Tx).$$

Then, for all $x_0 \in X$ and $\rho \in (k_T, 1/b^2)$, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$, and the following three conditions hold:

(i) $d_u(x_n, Tx_n) \le d_u(x_n, x_{n+1}) \le \rho^n d_u(x_0, Tx_0)$,
(ii) $\lim_{n\to\infty} d_u(x^*, Tx_n) = 0$ *and*
(iii) $d_u(x^*, Tx_n) \le b^2(p-1)\rho^{n+1} d_u(x_0, Tx_0)/(1-\rho b^2)$, *for all* $n \in \mathbb{N}$.

Moreover, $x^* \in Tx^*$, whenever one of the two following conditions hold:

(iv) *T is metrically upper continuous, or*

(v) *T is upper semicontinuous.*

Proof. Fix $x_0 \in X$ and let ρ satisfy $k_T b^2 < \rho b^2 < 1$. If $d_u(x_0, Tx_0) = 0$, we define $x_n = x_0$, for all $n \geq 1$. Otherwise, from assumption, there exist $x_1 \in Tx_0$ such that $d_u(x_1, Tx_1) < \rho d_u(x_0, Tx_0)$. If $d_u(x_1, Tx_1) = 0$, we define $x_n = x_1$, for all $n \geq 2$. Otherwise, there exist $x_2 \in Tx_1$ such that $d_u(x_2, Tx_2) < \rho d(x_1, Tx_1) < \rho^2 d(x_0, Tx_0)$. It follows by induction that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that, for all $n \in \mathbb{N}$, $d_u(x_n, Tx_n) \leq d_u(x_n, x_{n+1}) \leq \rho^n d_u(x_0, Tx_0)$ and $x_{n+1} \in Tx_n$. Hence, condition (i) holds.

By Lemma 6.1, for all $n \in \mathbb{N}$ and $m \geq n+1$, we have

$$\begin{aligned}
d_u(x_n, x_{n+m}) &\leq (p-1)\sum_{k=0}^{m-1} b^{2k+1} d_u(x_{n+k}, x_{n+k+1}) \\
&\leq (p-1)\sum_{k=0}^{m-1} b^{2k+1}\rho^{n+k} d_u(x_0, Tx_0) \\
&= b(p-1)\rho^n \sum_{k=0}^{m-1} (\rho b^2)^k d_u(x_0, Tx_0) \\
&\leq b(p-1)\rho^n d_u(x_0, Tx_0)/(1-\rho b^2).
\end{aligned}$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and, accordingly, there exists $x^* \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. In particular, condition (ii) holds. Condition (iii) is obtained from the inequality Notice that

$$\begin{array}{rcl}
d_u(x^*, x_{n+1}) & \leq & b(p-1)d_u(x^*, x_{n+m}) + b d_u(x_{n+1}, x_{n+m}) \\
 & & b(p-1)d_u(x^*, x_{n+m}) + b^2(p-1)\rho^{n+1} d_u(x_0, Tx_0)/(1-\rho b^2).
\end{array}$$

Hence, condition (iii) is obtained from the above inequality by taking limit as m goes to ∞.

Under condition (iv), we have

$$d_u(x^*, Tx^*) \leq \liminf d_u(x_n, Tx_n) \leq \liminf d_u(x_n, x_{n+1}) = 0,$$

and since Tx^* is closed, it is obtained that $x^* \in Tx^*$. Next, we suppose that $x^* \notin Tx^*$ and condition (v) holds. Let G an open set such that $x^* \notin G$ and $Tx^* \subset G$. Since T is upper semicontinuous, there exists a neighborhood V of x^* such that $T(x) \subset G$, for all $x \in V$. By the convergence of $\{x_n\}_{n\in\mathbb{N}}$ to x^*, there exists $N \in \mathbb{N}$ such that $x_n \in V$, for all $n \geq N$, and hence $x_{n+1} \in Tx_n \subset G$. This fact implies that $x^* \in G$, which is a contradiction. Therefore, $x^* \in Tx^*$ and the proof is complete. □

Corollary 7.1. *Let (X, u) be a complete $QM_1(p)$-metric space, $T : X \to \mathcal{CB}(X)$ be an upper semicontinuous correspondence and $k_T \in [0, 1)$. Assume that, for all $x \in X$, $\inf_{y\in Tx} d_u(y, Ty) \leq k_T d_u(x, Tx)$. Then, for all $x_0 \in X$ and $\rho \in (k_T, 1/b^2)$, there exist $x^* \in X$ such that $x^* \in Tx^*$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$, and the following three conditions hold:*

(i) $d_u(x_n, Tx_n) \leq d_u(x_n, x_{n+1}) \leq \rho^n d_u(x_0, Tx_0)$,

(ii) $\lim_{n\to\infty} d_u(x^*, Tx_n) = 0$ *and*
(iii) $d_u(x^*, Tx_n) \leq (p-1)\rho^{n+1} d_u(x_0, Tx_0)/(1-\rho)$, *for all* $n \in \mathbb{N}$.

Proof. It follows from Theorems 4.2 and 7.1. □

Suppose (X, u) is a $QM(p, b)$-metric space and let $\mathcal{H}_u : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ be defined as

$$\mathcal{H}_u(A, B) = \max\left\{\sup_{x\in A} d_u(x, B), \sup_{x\in B} d_u(x, A)\right\}.$$

Remark 7.1. For all $A, B, C \in \mathcal{CB}(X)$, $\mathcal{H}_u$ satisfies the following three conditions:

(i) $\mathcal{H}_u(A, B) = 0$ if $A = B$,
(ii) $\mathcal{H}_u(A, B) = \mathcal{H}_u(B, A)$,
(iii) $\mathcal{H}_u(A, B) \leq b\{(p-1)\mathcal{H}_u(A, C) + \mathcal{H}_u(B, C)\}$, whenever d_u is symmetric, and
(iv) $\mathcal{H}_u(A, B) \leq b\{(p-1)\mathcal{H}_u(A, C) + b\mathcal{H}_u(B, C)\}$, in general.

Moreover, if (X, u) is a non genuine $QM_1(p)$-metric space, $\mathcal{H}_u$ is the classic Hausdorff metric generated by the metric d_u.

Theorem 7.2. *Let* (X, u) *be a complete* $QM(p, b)$*-metric space,* $T : X \to \mathcal{CB}(X)$ *be a correspondence and* $k_T \in [0, 1/b^2)$*. Assume that for all* $x \in X$,

$$\inf_{y\in Tx} d_u(y, Ty) \leq k_T d_u(x, Tx).$$

Then, for all $x_0 \in X$ *and* $\rho \in (k_T, 1/b^2)$*, there exist* $x^* \in X$ *and a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converging to* x^* *such that* $x_{n+1} \in Tx_n$*, for all* $n \in \mathbb{N}$*, and the following three conditions hold:*

(i) $d_u(x_n, Tx_n) \leq d_u(x_n, x_{n+1}) \leq \rho^n d_u(x_0, Tx_0)$,
(ii) $\lim_{n\to\infty} d_u(x^*, Tx_n) = 0$ *and*
(iii) $d_u(x^*, Tx_n) \leq b^2(p-1)\rho^{n+1} d_u(x_0, Tx_0)/(1-\rho b^2)$, *for all* $n \in \mathbb{N}$.

Moreover, $x^* \in Tx^*$*, whenever one of the following two conditions hold:*

(iv) $\lim_{n\to\infty} \mathcal{H}_u(Tx_n, Tx^*) = 0$, *or*
(v) T *is upper semicontinuous.*

Proof. From Theorem 7.1, for all $x_0 \in X$ and $\rho \in (k_T, 1/b^2)$, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x^* \in X$ such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$, and conditions (i)-(iii) hold. Moreover, $x^* \in Tx^*$ if T is upper semicontinuous. Hence, by supposing that $\lim_{n\to\infty} \mathcal{H}_u(Tx_n, Tx^*) = 0$, it only remains to prove that $x^* \in Tx^*$. We have,

$$d_u(x^*, Tx^*) \leq b(p-1)d_u(x^*, x_{n+1}) + b^2\mathcal{H}_u(Tx_n, Tx^*)$$

and by taking limit as n goes to ∞, we obtain $d_u(x^*, Tx^*) = 0$ due to condition (v). Since Tx^* is closed, we have $x^* \in X$ and the proof is complete. □

Remark 7.2. In Theorem 7.2, if (X, u) is a metric space and T has compact images, then, condition (iv) is obtained from condition (v).

Examples 7.1. Let (X, u) be a complete $QM(p, b)$-metric space and $T : X \to \mathcal{CB}(X)$ be a correspondence.

(7.1.1) We say T is a multivalued $QM(p,b)$-metric contraction, whenever $\mathcal{H}_u(Tx,Ty) \leq kd_u(x,y)$, for all $x,y \in X$, where $k \in [0,1/b^2)$. Notice that, for all $x \in X$ and $y \in Tx$, $d_u(y,Ty) \leq \mathcal{H}_u(Tx,Ty) \leq kd_u(x,y)$. Hence, $\inf_{y\in Tx} d_u(y,Ty) \leq kd_u(x,Tx)$. Theorem 7.2 implies that, for all $x_0 \in X$ and $\rho \in (k,1/b^2)$, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$, and the following inequality holds:

$$d_u(x^*,Tx_n) \leq \left(\frac{b^2(p-1)\rho^{n+1}}{1-\rho b^2}\right) d_u(x_0,Tx_0), \quad \text{for all } n \in \mathbb{N}.$$

Since $\mathcal{H}_u(Tx^*,Tx_n) \leq kd_u(x^*,x_n)$, we have $\lim_{n\to\infty} \mathcal{H}_u(Tx^*,Tx_n) = 0$. Hence, Theorem 7.2 implies that $x^* \in Tx^*$.

(7.1.2) In our setting, T is a Kannan multivalued $QM(p,b)$-metric contraction, whenever

$$\mathcal{H}_u(Tx,Ty) \leq \alpha(d_u(x,Tx) + d_u(y,Ty)), \quad \text{for all } x,y \in X,$$

where $0 \leq \alpha(b^2+1) < 1$. For all $x \in X$ and $y \in Tx$, we have

$$d_u(y,Ty) \leq \mathcal{H}_u(Tx,Ty) \leq \alpha\{d_u(x,Tx) + d_u(y,Ty)\}$$

and hence $\inf_{y\in Tx} d_u(y,Ty) \leq k_T d_u(x,Tx)$, where $k_T = \alpha/(1-\alpha) < 1/b^2$. From Theorem 7.2, for all $x_0 \in X$ and $\alpha^* \in (\alpha, 1/(b^2+1)$, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$, and the following inequality holds holds:

$$d_u(x^*,Tx_n) \leq \alpha^* b^2(p-1)\left\{\frac{\alpha^*}{1-\alpha^*}\right\}^n \frac{d_u(x_0,Tx_0)}{1-\alpha^*(b^2+1)},$$

for all $n \in \mathbb{N}$. Moreover

$$\begin{aligned} \mathcal{H}_u(Tx_n,Tx^*) &\leq \alpha\{d_u(x_n,Tx_n) + d_u(x^*,Tx^*)\} \\ &\leq \alpha\{d_u(x_n,Tx_n) + b(p-1)d_u(x^*,Tx_n) \\ &\quad + b^2\mathcal{H}_u(Tx_n,Tx^*)\} \end{aligned}$$

and consequently,

$$\mathcal{H}_u(Tx_n,Tx^*) = \frac{\alpha\{d_u(x_n,x_{n+1}) + b(p-1)d_u(x^*,Tx_n)\}}{1-\alpha b^2}.$$

Hence, $\lim_{n\to\infty} \mathcal{H}_u(Tx^*,Tx_n) = 0$ and, from Theorem 7.2, $x^* \in Tx^*$.

(7.1.3) The correspondence T is a Chatterjea $QM(p,b)$-metric contraction, with constant α satisfying $\alpha b^2(b(p-1)+1) < 1$, whenever

$$\mathcal{H}_u(Tx,Ty) \leq \alpha\{d_u(x,Ty) + d_u(y,Tx)\}, \quad \text{for all } x,y \in X.$$

For all $x \in X$ and $y \in Tx$, we have

$$\mathcal{H}_u(Tx,Ty) \leq \alpha d_u(x,Ty) \leq \alpha b(p-1)d_u(x,Tx) + \alpha b^2\mathcal{H}_u(Tx,Ty)$$

and hence

$$d_u(y,Ty) \leq \mathcal{H}_u(Tx,Ty) \leq \left(\frac{\alpha b(p-1)}{1-\alpha b^2}\right) d_u(x,Tx).$$

Hence, Theorem 7.2 implies that, for all $x_0 \in X$ and

$$\alpha^* \in (\alpha, 1/b^2(b(p-1)+1)),$$

there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$, converging to x^*, such that $x_{n+1} \in Tx_n$ and the following inequality holds:

$$d_u(x^*, Tx_n) \leq \alpha^* b^3(p-1)^2 \left\{\frac{\alpha^* b(p-1)}{1-\alpha^* b^2}\right\}^n \frac{d_u(x_0, Tx_0)}{1-\alpha^* b^2(b(p-1)+1)},$$

for all $n \in \mathbb{N}$.
Notice that

$$\begin{aligned}\mathcal{H}_u(Tx_n, Tx^*) &\leq \alpha\{d_u(x_n, Tx^*) + d_u(x^*, Tx_n)\}\\ &\leq \alpha\{b(p-1)d_u(x_n, x_{n+1}) + b^2\mathcal{H}_u(Tx_n, Tx^*)\\ &\quad + d_u(x^*, Tx_n)\}.\end{aligned}$$

Hence

$$\mathcal{H}_u(Tx_n, Tx^*) = \frac{\alpha\{b(p-1)d_u(x_n, x_{n+1}) + d_u(x^*, Tx_n)\}}{1-\alpha b^2}$$

and thus $\lim_{n\to\infty} \mathcal{H}_u(Tx_n, Tx^*) = 0$. Therefore, by Theorem 7.2, $x^* \in Tx^*$.

(7.1.4) We say the correspondence T is a multivalued $QM(p,b)$-Reich contraction whenever

$$\mathcal{H}_u(Tx, Ty) \leq \alpha d_u(x,y) + \beta d_u(x, Tx) + \gamma d_u(y, Ty), \quad \text{for all } x, y \in X,$$

where α, β and γ are constants such that $0 \leq (\alpha+\beta)b^2 + \gamma < 1$. Notice that, for all $x \in X$ and $y \in Tx$,

$$d_u(y, Ty) \leq \frac{\alpha d_u(x,y) + \beta d_u(x, Tx)}{1-\gamma}.$$

Hence,

$$\inf_{y\in Tx} d_u(y, Ty) \leq \frac{\alpha+\beta}{1-\gamma} d_u(x, Tx).$$

From Theorem 7.2, for all $x_0 \in X$ and ρ such that

$$\frac{\alpha+\beta}{1-\gamma} < \rho < \frac{1}{b^2},$$

there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that $x_{n+1} \in Tx_n$ and the following inequality holds:

$$d_u(x^*, Tx_n) \leq \left(\frac{b^2(p-1)\rho^{n+1}}{1-\rho b^2}\right) d_u(x_0, Tx_0),$$

for all $n \in \mathbb{N}$.

(7.1.5) The concept of weak contraction by Berinde [7] is extended for correspondences as follows. We say T is a $QM(p,b)$-weak contraction, if there exists $\delta \in [0, 1/b^2)$ and $L \geq 0$, such that

$$\mathcal{H}_u(Tx, Ty) \leq \delta d_u(x,y) + L d_u(y, Tx).$$

Since for all $x \in X$ and $y \in Tx$, $d_u(y, Ty) \leq \delta d_u(x, Tx)$, Theorem 7.2 implies that, for all $x_0 \in X$ and $\delta^* \in (\delta, 1)$, there exist $x^* \in X$ and a sequence

$\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that $x_{n+1} \in Tx_n$ and the following inequality holds:

$$d_u(x^*, Tx_n) \leq \left(\frac{b^2(p-1)\delta^{*n+1}}{1-\delta^* b^2}\right) d_u(x_0, Tx_0), \qquad \text{for all } n \in \mathbb{N}.$$

(7.1.6) The correspondence T is say to be a multivalued quasi contraction (Ćirić), with constant α, whenever

$$\mathcal{H}_u(Tx, Ty) \leq \alpha \max\{d_u(x,y), d_u(x,Tx), d_u(y,Ty), d_u(x,Ty), d_u(y,Tx)\},$$

for all $x, y \in X$, such that $\alpha(b^3(p-1)+b^2) < 1$. In this case, for all $x \in X$ and $y \in Tx$, we have

$$d_u(y, Ty) \leq \mathcal{H}_u(Tx, Ty) \leq \alpha \max\{d_u(x,y), d_u(x,Ty)\}$$

and hence

$$\inf_{y\in Tx} d_u(y, Ty) \leq \left(\frac{\alpha b(p-1)}{1-\alpha b^2}\right) d_u(x, Tx).$$

Since $0 \leq \alpha b(p-1)/(1-\alpha b^2) < 1/b^2$, Theorem 7.2 implies that for all $x_0 \in X$ and $\alpha^* \in (\alpha, 1/\{b^3(p-1)+b^2\})$ there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n+1} \in Tx_n$ and the following inequality holds:

$$d_u(x^*, Tx_n) \leq \alpha^* b^3 (p-1)^2 \left\{\frac{\alpha^* b(p-1)}{1-\alpha^* b^2}\right\}^n \frac{d_u(x_0, Tx_0)}{1-\alpha^* b^2(b(p-1)+1)},$$

for all $n \in \mathbb{N}$.

Remark 7.3. It is well known that in Example (7.1.6), when (X, u) is a classic metric space, one condition for the existence of a fixed point and the other conclusions in this example is $0 \leq \alpha < 1/2$. These results are recovered in Example (7.1.6), by making $p = 2$ and $b = 1$. It is worth mentioning that it is an open problem the existence of fixed points for this type of contractions when $1/2 \leq \alpha < 1$. Even for classic metric spaces.

Acknowledgment. This work was partially supported by Chilean Council for Scientific and Technological Research, under grant FONDECYT 1160868.

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Received: October 23, 2019; Accepted: February 7, 2020.