Fixed Point Theory, 22(2021), No. 2, 587-602
DOI: 10.24193/fpt-ro.2021.2.39
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

AN EXTRAGRADIENT ALGORITHM FOR THE SPLIT EQUILIBRIUM PROBLEMS WITHOUT PRIOR KNOWLEDGE OF OPERATOR NORM

VAHID DADASHI* AND TRUONG MINH TUYEN**

*Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran
E-mail: vahid.dadashi@iausari.ac.ir

**Department of Mathematics and Informatics, Thai Nguyen University of Sciences,
Thai Nguyen, Vietnam
E-mail: tuyentm@tnus.edu.vn

Abstract. In this paper, using the hybrid projection method and an extragradient method of Hieu in [9], we present an extragradient algorithm for approximating a solution of the split equilibrium problem. The strong convergence theorem is proved in the framework of Hilbert spaces under some mild conditions. In particular, our algorithm does not depend on the norm of the transfer operator.
Key Words and Phrases: Hilbert space, split equilibrium problem, pseudomonotonicity, extragradient method.
2010 Mathematics Subject Classification: 68W10, 65K10, 65K15, 47H09, 47H10.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces endowed with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. We denote the strong convergence by $\to$. Let C and D be nonempty, closed and convex subsets of H_1 and H_2, respectively. Suppose that $f\colon C \times C \to \mathbb{R}$ is a bifunction. The equilibrium problem (EP) is to find $z \in C$ such that

$$f(z, x) \geq 0, \ \forall x \in C. \tag{1.1}$$

We denote the solution set of equilibrium problem (1.1) by $EP(f)$. Ky Fan [6, 7] introduced the equilibrium problem which includes optimization problem, variational inequality problem, fixed point problem, Nash equilibrium problem, saddle point problem and many other problems as a special case, (see [11, 16]).

Recently, Moudafi [15] (see also He [8]) has introduced the following split equilibrium problem (SEP) to find $z \in C$ such that

$$z \in EP(f) \cap T^{-1}(EP(g)), \tag{1.2}$$

where $T\colon H_1 \to H_2$ is a bounded linear operator and $g\colon D \times D \to \mathbb{R}$ be another bifunction. It is well known that SEP is a generalization of equilibrium problem by considering $g = 0$ and $D = H_2$. It also includes as a special case the split variational inequality problem, which is the generalization of split zero problems and split feasibility problems (see [1, 15, 14] and references therein).

There are some methods for obtaining a solution of EP such as the proximal method (see [2, 3, 13, 19] and references therein) and extragradient method (see [4, 5] for more details on extragradient algorithms). In the proximal method, the authors consider that the bifunctions are monotone and in the extragradient method, they consider the bifunctions are pseudomonotone. Tran et al. [18] suggested to use the introduced extragradient algorithm by Korpelevich [12]) for finding saddle points and other related problems.

Very recently, some authors introduced two parallel extragradient-proximal methods for solving split equilibrium problems [9, 10]. They assumed that one bifunction is monotone and the other one is pseudomonotone. By using extragradient method combined with proximal method, they obtained algorithms for solving these problems. However, in the results of [9, 10], the step-size is chosen depending on the norm of the transfer operator. This is a restriction in the algorithms of Hieu in [9] and Dinh et.al. in [10], because in the general case, it is not easy to define the norm of a bounded linear operator.

In this paper, motivated and inspired by the above literature, we assume that the two bifunctions are pseudomonotone and consider a new extragradient algorithm for solving split equilibrium problem. Moreover, we prove the strong convergence theorem without prior knowledge of operator norm. Section 4, we give an application of the main result for finding a solution of split variational inequality. Finally, in Section 5, we exhibit a numerical example to illustrate our result and observe the performance of our algorithm.

2. Preliminaries

We now provide some basic concepts, definitions and lemmas which will be used in the sequel. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Let C be a closed and convex subset of a Hilbert space H. The operator P_C is called a metric projection operator if it assigns to each $x \in H$ its nearest point $y \in C$ such that

$$\|x - y\| = \min\{\|x - z\| : z \in C\}.$$

The element y is called the metric projection of H onto C and denoted by $P_C x$. It exists and is unique at any point of the Hilbert space. It is known that the metric projection operator P_C is continuous.

Lemma 2.1. *Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Then, for all $x \in H$, the element $z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \ \forall y \in C.$$

The metric projection satisfies the following inequality:

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \ \forall x, y \in H. \tag{2.1}$$

Therefore the metric projection is a firmly nonexpansive operator in H.

It is easy to show that the following lemma holds for any Hilbert space H.

Lemma 2.2. *Let H be a real Hilbert space and let $\{x_n\}$ be a sequence in H. Then the following statements hold:*

i) *If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$; that is, the Hilbert space H has the Kadec-Klee property.*
ii) *If $x_n \rightharpoonup x$ as $n \to \infty$, then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.*

Definition 2.3. A bifunction $f\colon C \times C \to \mathbb{R}$ is said to be

- monotone on C if
$$f(x, y) + f(y, x) \le 0, \forall x, y \in C;$$
- pseudomonotone on C if
$$f(x, y) \ge 0 \Longrightarrow f(y, x) \le 0, \forall x, y \in C;$$
- Lipschitz-type continuous on C if there exist two positive constants c_1 and c_2 such that
$$f(x, y) + f(y, z) \ge f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \forall x, y, z \in C.$$

We assume that the bifunction f satisfies the following conditions:

(A1) f is pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$;
(A2) f is Lipschitz-type continuous on C with two constants c_1 and c_2;
(A3) $f(x, \cdot)$ is convex and subdifferentiable on C for every fixed $x \in C$;
(A4) f is weakly continuous on $C \times C$ in the sense that if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

It is easy to show that under assumptions (A1), (A3) and (A4), the solution set of $EP(f)$ is closed and convex (see, for instance [18]).

A mapping $A:\ C \to C$ is said to be

- monotone on C if
$$\langle Ax - Ay, x - y\rangle \ge 0, \forall x, y \in C;$$
- pseudomonotone on C if
$$\langle Ax, y - x\rangle \ge 0 \Longrightarrow \langle Ay, x - y\rangle \le 0, \forall x, y \in C;$$
- L_1-Lipschitz continuous on C if there exists a positive constant L_1 such that
$$\|Ax - Ay\| \le L_1\|x - y\|, \forall x, y \in C.$$

3. Main results

In this section, we present our main algorithm and prove the strong convergence theorem for finding a solution of split equilibrium problem of pseudomonotone and Lipschitztype continuous bifunctions in Hilbert space.

Let H_1 and H_2 be two real Hilbert spaces and C and D be nonempty closed and convex subsets of H_1 and H_2, respectively. Suppose that $f\colon C \times C \to \mathbb{R}$ and $g\colon D \times D \to \mathbb{R}$ be bifunctions satisfying (A1)–(A4) and $T\colon H_1 \to H_2$ be a bounded linear operator such that $EP(f) \cap T^{-1}(EP(g)) \ne \emptyset$. We introduce the following parallel extragradient algorithm for solving the split equilibrium problem.

Algorithm 3.1. Choose $x_1 \in H_1$. The control parameters μ_n, λ_n, r_n satisfy the following conditions

$$0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\},$$
$$0 < \underline{\mu} \le \mu_n \le \overline{\mu} < \min\left\{\frac{1}{2d_1}, \frac{1}{2d_2}\right\},$$
$$0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < \infty.$$

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = \arg\min\left\{\mu_n g(P_D(Tx_n), y) + \frac{1}{2}\|y - P_D(Tx_n)\|^2 : y \in D\right\}, \\ t_n = \arg\min\left\{\mu_n g(u_n, y) + \frac{1}{2}\|y - P_D(Tx_n)\|^2 : y \in D\right\}, \\ z_n = P_C\left(x_n + r_n T^*\left(t_n - Tx_n\right)\right), \\ v_n = \arg\min\left\{\lambda_n f(z_n, x) + \frac{1}{2}\|x - z_n\|^2 : x \in C\right\}, \\ y_n = \arg\min\left\{\lambda_n f(v_n, x) + \frac{1}{2}\|x - z_n\|^2 : x \in C\right\}, \\ C_n = \{z \in C : \|t_n - Tz\| \le \|Tx_n - Tz\|\}, \\ D_n = \{z \in C : \|y_n - z\| \le \|z_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1. \end{cases} \tag{3.1}$$

Remark 3.2. It is clear that the process of the computation for finding the sequence $\{x_n\}$ in Algorithm 3.1 does not depend on the norm of the bounded linear operator T. This overcomes the limitations in the algorithms of Hieu [9] and Dinh et.al. [10].

First, we need the following lemma to prove the convergence of Algorithm 3.1.

Lemma 3.3. *Let f and g satisfy the assumptions (A1)-(A3), such that $EP(f) \neq \emptyset$ and $EP(g) \neq \emptyset$. Then, we have:*

i) $\lambda_n\left(f(z_n, x) - f(z_n, v_n)\right) \ge \langle v_n - z_n, v_n - x\rangle, \forall x \in C;$

ii) $\|y_n - p\|^2 \le \|z_n - p\|^2 - (1 - 2\lambda_n c_1)\|z_n - v_n\|^2 - (1 - 2\lambda_n c_2)\|y_n - v_n\|^2, \forall p \in EP(f), \forall n \in \mathbb{N};$

iii) $\mu_n\left(g(P_D(Tx_n), y) - g(P_D(Tx_n), u_n)\right) \ge \langle u_n - P_D(Tx_n), u_n - y\rangle, \forall y \in D;$

iv) $\|t_n - y\|^2 \le \|P_D(Tx_n) - y\|^2 - (1 - 2\mu_n d_1)\|P_D(Tx_n) - u_n\|^2 - (1 - 2\mu_n d_2)\|t_n - u_n\|^2, \forall y \in EP(g), \forall n \in \mathbb{N}.$

Proof. i) We have

$$v_n = \arg\min\left\{\lambda_n f(z_n, x) + \frac{1}{2}\|x - z_n\|^2 : x \in C\right\}$$

if and only if

$$\lambda_n \partial_2 f(z_n, v_n) + v_n - z_n + N_C(v_n) \ni 0.$$

Thus, there exist $a_n \in \partial_2 f(z_n, v_n)$ and $b_n \in N_C(v_n)$ such that

$$\lambda_n a_n + v_n - z_n + b_n = 0.$$

From the definition of $N_C(v_n)$, we get that

$$\begin{aligned} 0 &\geq \langle b_n, x - v_n \rangle \\ &= \langle z_n - v_n - \lambda_n a_n, x - v_n \rangle \\ &= \langle v_n - z_n, v_n - x \rangle - \lambda_n \langle a_n, x - v_n \rangle, \end{aligned} \tag{3.2}$$

for all $x \in C$.
On the other hand, from the definition of $\partial_2 f(z_n, v_n)$, we have

$$f(z_n, x) - f(z_n, v_n) \geq \langle a_n, x - v_n \rangle \tag{3.3}$$

for all $x \in C$. So, from (3.2) and (3.3), we obtain

$$\lambda_n \left(f(z_n, x) - f(z_n, v_n) \right) \geq \langle v_n - z_n, v_n - x \rangle, \forall x \in C. \tag{3.4}$$

ii) From

$$y_n = \arg\min \left\{ \lambda_n f(v_n, x) + \frac{1}{2} \|x - z_n\|^2 \colon x \in C \right\}$$

and by an argument similar to the case i), we get

$$\lambda_n (f(v_n, x) - f(v_n, y_n)) \geq \langle y_n - z_n, y_n - x \rangle$$

for all $x \in C$.
Let $p \in EP(f)$. From $f(p, v_n) \geq 0$ and $f(v_n, p) + f(p, v_n) \leq 0$, we get $f(v_n, p) \leq 0$. Hence, we have

$$\lambda_n f(v_n, y_n) \leq \langle z_n - y_n, y_n - p \rangle. \tag{3.5}$$

It follows from the Lipschitz property of f that

$$f(v_n, y_n) \geq f(z_n, y_n) - f(z_n, v_n) - c_1 \|z_n - v_n\|^2 - c_2 \|v_n - y_n\|^2. \tag{3.6}$$

Now, in (3.4), replacing x by y_n, we get that

$$\lambda_n \left(f(z_n, y_n) - f(z_n, v_n) \right) \geq \langle v_n - z_n, v_n - y_n \rangle. \tag{3.7}$$

It follows from (3.5)–(3.7) that

$$\langle z_n - y_n, y_n - p \rangle \geq \langle v_n - z_n, v_n - y_n \rangle - \lambda_n (c_1 \|z_n - v_n\|^2 + c_2 \|v_n - y_n\|^2).$$

Combining the above inequality with the following equality

$$\langle z_n - y_n, y_n - p \rangle = \frac{1}{2} (\|z_n - p\|^2 - \|y_n - z_n\|^2 - \|y_n - p\|^2),$$

we obtain that

$$\begin{aligned} \|z_n - p\|^2 - \|y_n - z_n\|^2 - \|y_n - p\|^2 \geq 2 \langle v_n - z_n, v_n - y_n \rangle - 2\lambda_n (c_1 \|z_n - v_n\|^2 \\ + c_2 \|v_n - y_n\|^2). \end{aligned}$$

Thus

$$\begin{aligned}\|y_n-p\|^2 &\le \|z_n-p\|^2-\|y_n-z_n\|^2-2\langle v_n-z_n, v_n-y_n\rangle\\ &\quad+2\lambda_n(c_1\|z_n-v_n\|^2+c_2\|v_n-y_n\|^2)\\ &=\|z_n-p\|^2-\|(y_n-v_n)-(z_n-v_n)\|^2-2\langle v_n-z_n, v_n-y_n\rangle\\ &\quad+2\lambda_n(c_1\|z_n-v_n\|^2+c_2\|v_n-y_n\|^2)\\ &=\|z_n-p\|^2-(1-2\lambda_n c_1)\|z_n-v_n\|^2-(1-2\lambda_n c_2)\|v_n-y_n\|^2.\end{aligned}$$

iii) and iv) By arguments similar to the cases i) and ii), we get the proof for the estimates in iii) and iv).
This completes the proof. □

We begin our analysis of this algorithm with the following lemmas.

Lemma 3.4. *The sequence $\{x_n\}$ is well defined and bounded.*

Proof. First, we claim that C_n, D_n and Q_n are closed and convex subsets of H_1 for all $n \ge 0$. To see this, we rewrite, for each integer $n \ge 0$, the subsets C_n, D_n and Q_n in the following forms:

$$\begin{aligned}C_n &= \{z \in C:\ \langle Tx_n - t_n, Tz\rangle \le \frac{1}{2}(\|Tx_n\|^2-\|t_n\|^2)\}\\ &=\{z\in C:\ \langle T^*(Tx_n-t_n), z\rangle \le \frac{1}{2}(\|Tx_n\|^2-\|t_n\|^2)\},\\ D_n &= \{z\in C:\ \langle z_n-y_n, z\rangle \le \frac{1}{2}(\|z_n\|^2-\|y_n\|^2)\},\\ Q_n &= \{z\in C:\ \langle x_1-x_n, z\rangle \le \langle x_n, x_1-x_n\rangle\},\end{aligned}$$

respectively. Now it is easy to see that C_n, D_n and Q_n are closed and convex subsets of H_1 for all $n \ge 1$.

Next, we prove that $EP(f)\cap T^{-1}(EP(g))$ is contained in $C_n \cap D_n \cap Q_n$ for all $n \ge 1$. Let $p \in EP(f)\cap T^{-1}(EP(g))$. By Lemma 3.3, we have

$$\|y_n-p\|^2 \le \|z_n-p\|^2-(1-2\lambda_n c_1)\|z_n-v_n\|^2-(1-2\lambda_n c_2)\|y_n-v_n\|^2, \tag{3.8}$$

and

$$\begin{aligned}\|t_n-Tp\|^2 &\le \|P_D(Tx_n)-Tp\|^2-(1-2\mu_n d_1)\|P_D(Tx_n)-u_n\|^2\\ &\quad-(1-2\mu_n d_2)\|t_n-u_n\|^2,\end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$.

By assumption, we get $\|y_n-p\| \le \|z_n-p\|$ and since the metric projection is nonexpansive, we have

$$\|t_n-Tp\| \le \|P_D(Tx_n)-Tp\| = \|P_D(Tx_n)-P_D(Tp)\| \le \|Tx_n-Tp\|,$$

and hence $EP(f)\cap T^{-1}(EP(g)) \subset C_n \cap D_n$, for each $n \in \mathbb{N}$.

We prove that $EP(f)\cap T^{-1}(EP(g)) \subset Q_n$ by mathematical induction. We have $Q_1 = C$, so $EP(f)\cap T^{-1}(EP(g)) \subset Q_1$. Suppose that $EP(f)\cap T^{-1}(EP(g)) \subset Q_k$ for some $k \leq 1$. Then $EP(f) \cap T^{-1}(EP(g)) \subset C_k \cap D_k \cap Q_k$. From $x_{k+1} = P_{C_k \cap D_k \cap Q_k} x_1$, we have $x_{k+1} \in Q_k$ and it follows from Lemma 2.1 that

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0,$$

for all $z \in C_k \cap D_k \cap Q_k$. Since $EP(f) \cap T^{-1}(EP(g)) \subset C_k \cap D_k \cap Q_k$, we get

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0,$$

for all $z \in EP(f) \cap T^{-1}(EP(g))$. It follows from the definition of Q_{k+1} that $z \in Q_{k+1}$, that is, $EP(f) \cap T^{-1}(EP(g)) \subset Q_{k+1}$. So, $EP(f) \cap T^{-1}(EP(g)) \subset Q_n$ for all $n \geq 1$, and the sequence $\{x_n\}$ is well defined.

Since $EP(f) \cap T^{-1}(EP(g))$ is a nonempty, closed and convex subset of C, there exists a unique element $z_0 \in EP(f) \cap T^{-1}(EP(g))$ such that $z_0 = P_{EP(f) \cap T^{-1}(EP(g))} x_1$. From $x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1$, we have

$$\|x_{n+1} - x_1\| \leq \|x_1 - y\|,$$

for all $y \in C_n \cap D_n \cap Q_n$. Since $z_0 \in EP(f) \cap T^{-1}(EP(g)) \subset C_n \cap D_n \cap Q_n$, we get

$$\|x_{n+1} - x_1\| \leq \|x_1 - z_0\|, \tag{3.10}$$

for all $n \geq 1$. This implies that $\{x_n\}$ is bounded.
This completes the proof. □

Lemma 3.5. *The limit* $\lim_{n\to\infty} \|x_n - x_1\|$ *exists and is finite and* $\|x_n - x_{n+1}\| \to 0$ *as* $n \to \infty$.

Proof. Since $x_{n+1} \in Q_n$, we get

$$\begin{aligned} 0 &\leq \langle x_n - x_{n+1}, x_1 - x_n \rangle, \\ &\leq \langle x_n - x_1, x_1 - x_n \rangle + \langle x_1 - x_{n+1}, x_1 - x_n \rangle \\ &\leq -\|x_n - x_1\|^2 + \langle x_1 - x_{n+1}, x_1 - x_n \rangle, \end{aligned}$$

and hence, $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$. Combining this with the boundedness of $\{x_n\}$, we obtain that the limit $\lim_{n\to\infty} \|x_n - x_1\|$ exists and is finite.
Again, by $x_{n+1} \in Q_n$, we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_1) - (x_{n+1} - x_1)\|^2 \\ &\leq \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_1 \rangle + \|x_{n+1} - x_1\|^2 \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_n - x_{n+1}, x_1 - x_n \rangle \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 \to 0. \end{aligned}$$

This implies that

$$\|x_n - x_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

This completes the proof. □

Lemma 3.6. *We have* $\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|v_n - z_n\| = \lim_{n\to\infty} \|u_n - P_D(Tx_n)\| = 0.$

Proof. From $x_{n+1} \in C_n$ and Lemma 3.5, we have

$$\begin{aligned}\|t_n - Tx_n\| &\leq \|t_n - Tx_{n+1}\| + \|Tx_{n+1} - Tx_n\| \\ &\leq 2\|Tx_{n+1} - Tx_n\| \\ &\leq 2\|T\|\|x_{n+1} - x_n\| \to 0,\end{aligned}$$

and hence,

$$\|t_n - Tx_n\| \to 0. \tag{3.12}$$

It follows from $x_n \in C$, the definition of z_n and (3.12) that

$$\begin{aligned}\|z_n - x_n\| &= \|P_C\left(x_n + r_nT^*\left(t_n - Tx_n\right)\right) - P_C(x_n)\| \\ &\leq \|x_n + r_nT^*\left(t_n - Tx_n\right) - x_n\| \\ &\leq r_n\|T\|\|t_n - Tx_n\| \to 0. \end{aligned} \tag{3.13}$$

From $x_{n+1} \in D_n$, (3.11) and (3.13), we get

$$\begin{aligned}\|y_n - x_{n+1}\| &\leq \|z_n - x_{n+1}\| \\ &\leq \|z_n - x_n\| + \|x_n - x_{n+1}\| \to 0,\end{aligned}$$

and so,

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.14}$$

Then,

$$\|z_n - y_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| \to 0. \tag{3.15}$$

It follows from (3.9), (3.12) and the boundedness of the sequences $\{x_n\}$ and $\{t_n\}$ that

$$\begin{aligned}&(1 - 2\mu_n d_1)\|P_D(Tx_n) - u_n\|^2 + (1 - 2\mu_n d_2)\|t_n - u_n\|^2 \\ &\leq \|P_D(Tx_n) - Tp\|^2 - \|t_n - Tp\|^2 \\ &= \|P_D(Tx_n) - P_D(Tp)\|^2 - \|t_n - Tp\|^2 \\ &\leq \|Tx_n - Tp\|^2 - \|t_n - Tp\|^2 \\ &= \left(\|Tx_n - Tp\| - \|t_n - Tp\|\right)\left(\|Tx_n - Tp\| + \|t_n - Tp\|\right) \\ &\leq \|Tx_n - t_n\|\left(\|Tx_n - Tp\| + \|t_n - Tp\|\right) \to 0.\end{aligned}$$

and hence,

$$\|P_D(Tx_n) - u_n\| \to 0, \tag{3.16}$$

$$\|t_n - u_n\| \to 0. \tag{3.17}$$

Similarly, it follows from (3.8), (3.15) and boundedness of the sequences $\{z_n\}$ and $\{y_n\}$ that

$$\|z_n - v_n\| \to 0. \tag{3.18}$$

This completes the proof. □

The following theorem yields the strong convergence of the sequence generated by Algorithm 3.1.

Theorem 3.7. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $z_0 = P_{EP(f)\cap T^{-1}(EP(g))}x_1$.*

Proof. Because $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some p, as $k \to \infty$, consequently $\{Tx_{n_k}\}$ converges weakly to Tp. By (3.12), $\{t_{n_k}\}$ converges weakly to Tp. We show that $p \in EP(f) \cap T^{-1}(EP(g))$. We know that $x_n \in C$ and $t_n \in D$, for each $n \in \mathbb{N}$. Since C and D are closed and convex sets, so C and D are weakly closed, therefore, $p \in C$ and $Tp \in D$. It follows from (3.12), (3.16) and (3.17) that $\|Tx_{n_k} - P_D(Tx_{n_k})\| \to 0$, and hence $\{P_D(Tx_{n_k})\}$ converges weakly to $P_D(Tp) = Tp$. From (3.13) and (3.18), we get that $\{z_{n_k}\}$ and $\{v_{n_k}\}$ converges weakly to p. From (3.16), we also get that $\{u_{n_k}\}$ converges weakly to Tp. Algorithm 3.1 and assertion (i) in Lemma 3.3 imply that

$$\begin{aligned}\lambda_{n_k}\left(f(z_{n_k},x) - f(z_{n_k},v_{n_k})\right) &\geq \langle v_{n_k} - z_{n_k}, v_{n_k} - x\rangle \\ &\geq -\|v_{n_k} - z_n\|\|v_{n_k} - x\|, \forall x \in C,\end{aligned}$$

and

$$\begin{aligned}\mu_{n_k}\left(g(P_D(Tx_{n_k}),y) - g(P_D(Tx_{n_k}),u_{n_k})\right) &\geq \langle u_{n_k} - P_D(Tx_{n_k}), u_{n_k} - y\rangle \\ &\geq -\|u_{n_k} - P_D(Tx_{n_k})\|\|u_{n_k} - y\|, \forall y \in D.\end{aligned}$$

Hence, it follows that

$$f(z_{n_k},x) - f(z_{n_k},v_{n_k}) + \frac{1}{\lambda_{n_k}}\|v_{n_k} - z_{n_k}\|\|v_{n_k} - x\| \geq 0, \forall x \in C,$$

and

$$g(P_D(Tx_{n_k}),y) - g(P_D(Tx_{n_k}),u_{n_k}) + \frac{1}{\mu_{n_k}}\|u_{n_k} - P_D(Tx_{n_k})\|\|u_{n_k} - y\| \geq 0, \forall y \in D.$$

Letting $k \to \infty$ and using Lemma 3.6 and the weak continuity of f and g (condition (A4)), we obtain that

$$f(p,x) \geq 0, \forall x \in C, \text{ and } g(Tp,y) \geq 0, \forall y \in D.$$

This means that $p \in EP(f) \cap T^{-1}(EP(g))$.

Now, we show that the sequence $\{x_n\}$ converges strongly to p.

From $z_0 = P_{EP(f)\cap T^{-1}(EP(g))}x_1$, $p \in EP(f) \cap T^{-1}(EP(g))$ and (3.10), we have

$$\begin{aligned}\|x_1 - z_0\| \leq \|x_1 - p\| &\leq \liminf_{k\to\infty}\|x_1 - x_{n_k}\| \\ &\leq \limsup_{k\to\infty}\|x_1 - x_{n_k}\| \\ &\leq \|x_1 - z_0\|.\end{aligned}$$

Using the uniqueness of the nearest point z_0, we now see that $p = z_0$. We also have $\|x_{n_k} - x_1\| \to \|z_0 - x_1\|$. From $x_{n_k} - x_1 \rightharpoonup p - x_1 = z_0 - x_1$ and the Kadec-Klee property of H_1, we have $x_n \to z_0 = P_{EP(f)\cap T^{-1}(EP(g))}x_1$.
This completes the proof. □

Finally, in this section, we have the following corollary regarding the equilibrium problem in a real Hilbert space.

Corollary 3.8. *Let H_1 be a real Hilbert space and C be a nonempty closed and convex subset of H_1. Suppose that $f\colon C\times C\to\mathbb{R}$ be a bifunction such that assumptions (A1)–(A4) hold and $EP(f)\neq\emptyset$. Let $x_1\in H_1$ and $\{x_n\}$ be a sequence generated by the following extragradient algorithm:*

$$\begin{cases} u_n = \arg\min\left\{\lambda_n f(x_n,x)+\frac{1}{2}\|x-x_n\|^2\colon x\in C\right\},\\ t_n = \arg\min\left\{\lambda_n f(u_n,x)+\frac{1}{2}\|x-x_n\|^2\colon x\in C\right\},\\ C_n=\{z\in C\colon \|t_n-z\|\le\|x_n-z\|\},\\ Q_n=\{z\in C\colon \langle x_n-z,x_1-x_n\rangle\ge 0\},\\ x_{n+1}=P_{C_n\cap Q_n}x_1. \end{cases} \tag{3.19}$$

where, $0<\underline{\lambda}\le\lambda_n\le\overline{\lambda}<\min\left\{\frac{1}{2c_1},\frac{1}{2c_2}\right\}$. Then the sequence $\{x_n\}$ generated by (3.19) converges strongly to $z_0=P_{EP(f)}x_1$.

4. Application to the split variational inequality

In this section, we apply Theorem 3.7 for finding a solution of the split variational inequality. Let H_1 be a real Hilbert space, C be a nonempty convex subset of H_1 and $A\colon C\to C$ be a nonlinear operator. The variational inequality is to find $x^*\in C$ such that

$$\langle Ax^*, x-x^*\rangle\ge 0,\forall x\in C. \tag{4.1}$$

For each $x,y\in C$, we define $f(x,y)=\langle Ax,y-x\rangle$ Then the equilibrium problem (1.1) becomes the variational inequality problem (4.1). We denote the set of solutions of the problem (4.1) by $\mathrm{VI}(C,A)$. We assume that A satisfies the following conditions:

(B1) A is pseudomonotone on C;
(B2) A is weak to strong continuous on C, that is, $Ax_n\to Ax$ for each sequence $\{x_n\}\subset C$ converging weakly to x;
(B3) A is L_1-Lipschitz continuous on C for some positive constant $L_1>0$.

Let H_1 and H_2 be two real Hilbert spaces and C and D be nonempty closed and convex subsets of H_1 and H_2, respectively. Suppose that $A\colon C\to C$ and $B\colon D\to D$ be L_1 and L_2-Lipschitz continuous on C and D, respectively and $T\colon H_1\to H_2$ be a bounded linear operator. Using the idea of Hieu in [9] (see, Lemma 6, Theorem 3 and Theorem 4) and Theorem 3.7, we have the following theorem for solving the spilt variational inequality.

Theorem 4.1. *Let $A\colon C\to C$ and $B\colon D\to D$ be mappings such that assumptions (B1)–(B3) hold with some positive constant $L_1>0$ and $L_2>0$, respectively and $\Omega:=\mathrm{VI}(C,A)\cap T^{-1}(\mathrm{VI}(D,B))\neq\emptyset$. Suppose that control parameters μ_n,λ_n,r_n satisfy the following conditions*

$$0<\underline{\lambda}\le\lambda_n\le\overline{\lambda}<\frac{1}{L_1},\ 0<\underline{\mu}\le\mu_n\le\overline{\mu}<\frac{1}{L_2},\ 0<\liminf_{n\to\infty} r_n<\limsup_{n\to\infty} r_n<\infty.$$

For any $x_1 \in H_1$, Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = P_D\left(P_D(Tx_n) - \mu_n B\left(P_D(Tx_n)\right)\right), \\ t_n = P_D\left(P_D(Tx_n) - \mu_n B\left(u_n\right)\right), \\ z_n = P_C\left(x_n + r_n T^*\left(t_n - Tx_n\right)\right), \\ v_n = P_C\left(z_n - \lambda_n A z_n\right), \\ y_n = P_C\left(z_n - \lambda_n A v_n\right), \\ C_n = \{z \in C \colon \|t_n - Tz\| \leq \|Tx_n - Tz\|\}, \\ D_n = \{z \in C \colon \|y_n - z\| \leq \|z_n - z\|\}, \\ Q_n = \{z \in C \colon \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1. \end{cases} \tag{4.2}$$

Then $\{x_n\}$ converges strongly to $z_0 = P_\Omega x_1$.

5. Numerical Experiment

The algorithms were implemented in MATLAB 7.0 running on an HP Compaq 510, Core(TM) 2 Duo Processor T5870 with 2.00 GHz and 2GB RAM.

Example 5.1. Consider the problem of finding an element $x^\dagger \in \mathbb{R}^3$ such that

$$x^\dagger \in S = \operatorname{argmin}_{x\in\mathbb{C}} f(x) \cap T^{-1}(\operatorname{argmin}_{x\in\mathbb{D}} g(x)), \tag{5.1}$$

where $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $g: \mathbb{R}^3 \longrightarrow \mathbb{R}$ are defined by

$$f(x) := \langle A_1 x, x\rangle + \langle p_1, x\rangle + q_1,$$

$$g(x) := \langle A_2 x, x\rangle + \langle p_2, x\rangle + q_2,$$

with

$$A_2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$p_1 = \begin{pmatrix} -4 & -4 & 4 \end{pmatrix},\ p_2 = \begin{pmatrix} -4 & -4 & 0 \end{pmatrix},\ q_1,\ q_2 \text{ are any constants },$$

and $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a bounded linear operator which is defined by

$$Tx := \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

It is not difficult to see that f and g are two proper continuous convex functions on $\mathbb{R}^3$ and $\mathbb{R}^3$, respectively, and $x^\dagger \in S$ if and only if $\langle 2A_1 x^\dagger + p_1, x - x^\dagger\rangle \geq 0$ for all $x \in C$ and $\langle 2A_2(Tx^\dagger) + p_2, y - Tx^\dagger\rangle \geq 0$ for all $y \in D$.

Let $Ax = 2A_1 x + p_1$ and $Bx = 2A_2 x + p_2$ for all $x \in \mathbb{R}^3$. Then A and B satisfy all conditions (B1)–(B3) on $\mathbb{R}^3$. We now consider the following two cases.
a) If $C = D = \mathbb{R}^3$, then $x^\dagger \in S$ if and only if $2A_1 x^\dagger + p_1 = 0$ and $2A_2(Tx^\dagger) + p_2 = 0$. So, in this case, at the nth iterative step, we define the function TOL_n by

$$\mathrm{TOL}_n := \frac{1}{2}(\|2A_1 x_n + p_1\|^2 + \|2A_2(Tx_n) + p_2\|^2)$$

and use the stopping rule $\text{TOL}_n < \varepsilon$ for the iterative process, where ε is the predetermined error.

Now, applying iterative method (4.2) with $x_1 = (1, 2, 3)$, $r_n = 1/2$, $\lambda_n = 1/10$, and $\mu_n = 1/20$ for all $n \geq 1$, we obtain the following table of numerical results:

ε	n	TOL_n	x_n
10^{-2}	69	9.056894e−03	(−5.871033e−02,2.091475e+00, 3.305946e−02)
10^{-3}	83	8.833622e−04	(−6.272119e−02,2.089985e+00, 2.824746e−02)
10^{-4}	97	8.682724e−05	(−6.372180e−02,2.089768e+00, 2.650547e−02)
10^{-5}	117	9.714728e−06	(−6.417625e−02,2.089568e+00, 2.606984e−02)

Table 1. Table of numerical results for the case a)

Remark 5.2. It is not difficult to check that the set of solutions S in Example 5.1 is given by

$$S = \{(-5a + 5, 7a - 5, 2a - 2) : \ a \in \mathbb{R}\}$$

and $P_S^{\mathbb{R}^3}(x_1) = \left(-\frac{5}{78}, \frac{163}{78}, \frac{2}{78}\right) \approx (-0.06410256, 2.08974358, 0.02564102)$.

The behavior of TOL_n in the case where $\text{TOL}_n < 10^{-5}$ is described in the following figure:

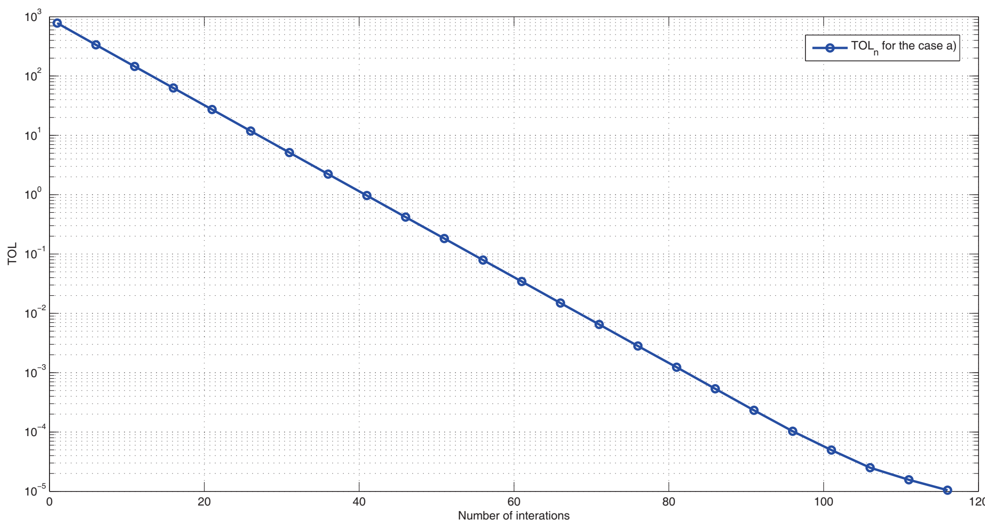

Figure 1. The behavior of TOL_n with $\text{TOL}_n < 10^{-5}$

b) Suppose that C and D are defined by

$$C = \{(x_1, x_2, x_3) : \ 1 \leq x_1 \leq 2, \ 1 \leq x_2 \leq 3, \ -2 \leq x_3 \leq -1\},$$
$$D = \{(x_1, x_2, x_3) : \ 0 \leq x_1 \leq 2, \ -4 \leq x_2 \leq -2, \ 1 \leq x_3 \leq 3\}.$$

In this case, it is easy to check that $x^\dagger = (1, 1, -1)$ is a unique solution of the problem. We define the function TOL_n by

$$\mathrm{TOL}_n := \|x_{n+1} - x_n\|$$

and use the stopping rule $\mathrm{TOL}_n < \varepsilon$ for the iterative process, where ε is the predetermined error.

Now, applying iterative method (4.2) with $x_1 = (2, 3, 4)$, $r_n = 1/2$, $\lambda_n = 1/10$, and $\mu_n = 1/20$ for all $n \geq 1$, we obtain the following table of numerical results:

ε	n	TOL_n	x_n
10^{-3}	33	8.045593e−04	(1.002258341, 1.001364071, −1.000001045)
10^{-4}	67	9.200339e−05	(1.000687143, 1.000086289, −1.000000044)
10^{-5}	90	2.121968e−06	(1.000420167, 1.000123676, −1.000001048)
10^{-6}	101	1.151519e−07	(1.000000000, 1.000280412, −1.000000000)

TABLE 2. Table of numerical results for the case b)

The behavior of TOL_n in the case where $\mathrm{TOL}_n < 10^{-6}$ is described in the following figure:

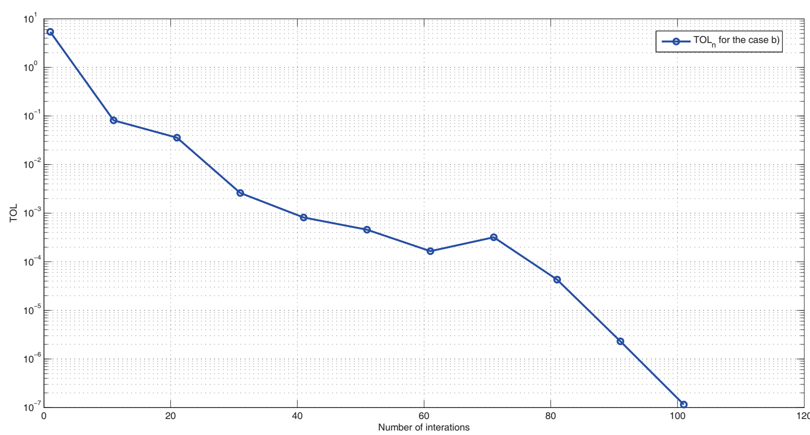

FIGURE 2. The behavior of TOL_n with $\mathrm{TOL}_n < 10^{-6}$

Remark 5.3. The generalized model of Problem (5.1) is the following split minimum point problem: Let C and D be two nonempty, closed and convex susets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $f_1 : \mathbb{R}^n \longrightarrow (-\infty, \infty]$ and $f_2 : \mathbb{R}^m \longrightarrow (-\infty, \infty]$ be two proper, lower semicontinuous and convex functions, and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a bounded linear operator. Find an element $x^* \in S$, with

$$S = \arg\min_{x \in C} f_1(x) \cap T^{-1}(\arg\min_{y \in D} f_2(y)) \neq \emptyset.$$

We know that this problem plays an important role in the optimization field. In particular, if

$$f(x) = \|x - P_C x\|^2/2 \text{ and } g(y) = \|y - P_D y\|^2/2$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then this problem becomes the split feasibility problem, that is, the problem of finding and element x^* such that $x^* \in C$ and $Tx^* \in D$.
In this case, we have

$$A = \nabla f = I - P_C \text{ and } B = \nabla g = I - P_D$$

satisfy the conditions (B1)–(B3). Thus, in Theorem 4.1, replacing

$$A = I - P_C \text{ and } B = I - P_D,$$

we obtain an extragradient method for solving the split feasibility problem.

Acknowledgement. The first author is supported by Sari Branch, Islamic Azad University. The second author is supported by Thai Nguyen University of Sciences. All the authors are very grateful to an anonymous referee for providing them with useful comments and helpful suggestions.

References

[1] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, **59**(2012), no. 2, 301-323.

[2] V. Dadashi, H. Khatibzadeh, *On the weak and strong convergence of the proximal point algorithm in reflexive Banach spaces*, Optimization, **66**(2017), no. 9, 1487-1494.

[3] V. Dadashi, M. Postolache, *Hybrid proximal point algorithm and applications to equilibrium problems and convex programming*, Journal of Optimization Theory and Applications, **174**(2017), 518-529.

[4] B.V. Dinh, L.D. Muu, *A projection algorithm for solving pseudomonotone equilibrium problems and its application to a class of bilevel equilibria*, Optimization, **64**(2015), 559-575.

[5] F. Facchinei, J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer, New York, 2003.

[6] K. Fan, *A Minimax Inequality and Applications. In Inequalities III*, Academic Press, New York, 1972.

[7] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann., **266**(1984), 519-537.

[8] Z. He, *The split equilibrium problem and its convergence algorithms*, J. Ineq. Appl., **2012**(2012), doi:10.1186/1029-242X-2012-16.

[9] D.V. Hieu, *Parallel extragradient-proximal methods for split equilibrium problems*, Mathematical Modelling and Analysis, **21**(2016), no. 4, 478-501.

[10] D.S. Kim, B.V. Dinh, *Parallel extragradient algorithms for multiple set split equilibrium problems in Hilbert spaces*, Numer. Algorithms, **77**(2018), 741-761.

[11] I.V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin, 2000.

[12] G.M. Korpelevich, *An extragradient method for finding saddle points and other problems*, Matecon, **12**(1976), 747-756.

[13] A. Moudafi, *Proximal point algorithm extended for equilibrium problems*, J. Nat. Geom., **15**(1999), 91-100.

[14] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inverse Problems, **26**(2010), 1-6.

[15] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl., **150**(2011), 275-283.

[16] L.D. Muu, *Stability property of a class of variational inequalities*, Optimization, **15**(1984), 347-353.

[17] N. Nadezhkina, W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM J. Optimization, **16**(2006), 1230-1241.

[18] T.D. Quoc, L.D. Muu, N.V. Hien, *Extragradient algorithms extended to equilibrium problems*, Optimization, **57**(2008), 749-776.

[19] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14**(1976), 877-898.

Received: June 26, 2019; Accepted: April 14, 2020.