Fixed Point Theory, 22(2021), No. 2, 571-586
DOI: 10.24193/fpt-ro.2021.2.38
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

EXISTENCE AND STABILITY OF COUPLED FIXED POINT SETS FOR MULTI-VALUED MAPPINGS

BINAYAK S. CHOUDHURY*, T. GNANA BHASKAR**, N. METIYA*** AND S. KUNDU****

*Department of Mathematics, Indian Institute of Engineering Science and Technology,
Shibpur, Howrah - 711103, West Bengal, India
E-mail: binayak@math.iiests.ac.in

**Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901, USA
E-mail: gtenali@fit.edu

***Department of Mathematics, Sovarani Memorial College, Jagatballavpur,
Howrah-711408, West Bengal, India
E-mail: metiya.nikhilesh@gmail.com

****Department of Mathematics, Government Degree College, Salboni,
Paschim Mednipur - 721516, West Bengal, India
E-mail: sunirmalkundu2009@rediffmail.com

Abstract. The existence of a coupled fixed point for a multi-valued mapping satisfying a certain admissibility condition is established. Requirement that the mapping be a contraction is met implicitly through a function of several variables. Further, we study the stability of coupled fixed point sets for such mappings. Several illustrative examples are presented.

Key Words and Phrases: Metric space, admissible maps, implicit relation, coupled fixed point, stability.
2020 Mathematics Subject Classification: 47H10, 54H10, 54H25.

1. Introduction and mathematical preliminaries

The present paper is about the existence and stability of coupled fixed point sets associated with certain multivalued mappings defined on complete metric spaces. The idea of a coupled fixed point was introduced by Guo et al. [19] and received much attention after the appearance of the work of Bhaskar et al. [18]. Following the work of Neito et al. [28], Bhaskar et al. [18] studied the existence of coupled fixed points of mappings with *mixed monotone property* in a metric space endowed with a partial order. While an early result on fixed points in partially ordered metric spaces appeared in the work of Turinici [40], progress in this direction took place following the works of Ran et al. [34] and Neito et al. [28]. For some recent works from this area see [7, 14, 20, 27]. The main idea in this approach is that for the existence of fixed points the various types of contractive inequality conditions need only be satisfied by the elements of the metric space related through the partial order.

For some recent works that extended the study of coupled fixed points and the related properties of different types of mappings in various types of spaces, see [8, 9, 10, 15, 36]. The notion of coupled fixed point was extended to multivalued mappings in [41] and was followed up in [23, 29, 32, 33, 37]. In particular, Petrusel et al. studied the coupled fixed point problems for multi valued mappings in b-metric spaces (see [30, 31] with applications to systems of integral inclusions.

On the other hand, Samet et al. [38] showed that the above purpose of introducing the partial order can be also served by a set of conditions called admissibility conditions. These conditions, rather than introducing a new structure like partial order in metric spaces, are requirements on the operator under consideration. For some recent contributions in this direction [1, 11, 13, 16, 21, 22].

Several different types of contractive mappings are considered in fixed point theory. Contraction mappings defined implicitly through a function of several variables are used in [2, 3, 4, 39]. While the motivation for this approach may be to generalize and combine several existing results, it also makes it possible to apply this type of results to a wider class of problems.

Convergence of fixed point sets of a sequence of mappings, known as the stability of fixed points, has also been widely studied in various settings [5, 6, 24, 25, 26]. *The fixed point sets of a sequence of mappings are said to be stable if they converge to the set of fixed point of the limit mapping in the Hausdorff metric.* The study of fixed point sets of multivalued mappings is more involved than their singlevalued counterparts [24, 25, 26] and is of interest.

Motivated by the different approaches mentioned above, and with an intent to develop suitable results that may be of use in the study of Set Differential Equations (see [17]), in this paper we establish a fixed point result for coupled multivalued mappings on complete metric spaces. Further, we perform the stability analysis of the fixed point sets corresponding to a convergent sequence of coupled multivalued mappings. We begin with the following definition.

Definition 1.1. Let X be a nonempty set and $T : X \times X \to X$. A point $(x, y) \in X \times X$ is said to be a *coupled fixed point* of T if $x = T(x, y)$ and $y = T(y, x)$.

In the following we discuss some concepts and definitions that are used in the paper.

Let $(X,\ d)$ be a metric space. Then $X \times X$ is also a metric space under the metric ρ defined by $\rho((x, y),\ (u, v)) = \max\ \{d(x, u),\ d(y, v)\}$ for all (x, y) and $(u, v) \in X \times X$.

Let $N(X)$ denote the collection of all nonempty subsets of X, $CB(X)$ denote the collection of all nonempty closed and bounded subsets of X, and $C(X)$ denote the collection of all compact subsets of X. We use following notations and definitions:

$$D(x, B) = \ \inf\ \{d(x, y) : y \in B\},\ \text{where } x \in X \text{ and } B \in CB(X),$$

$$D(A, B) = \inf\ \{d(a, b) : a \in A,\ b \in B\},\ \text{where } A, B \in CB(X),$$

$$H(A, B) = \ \max\ \{\sup_{x \in A} D(x, B),\ \ \sup_{y \in B} D(y, A)\},\ \text{where } A, B \in CB(X).$$

H is known as the Hausdorff metric on $CB(X)$ [26]. Further, if (X,d) is complete then $(CB(X),H)$ is also complete and so is $(X\times X,\rho)$. Let H_ρ be the Hausdorff metric induced by ρ.

Lemma 1.1 ([12])**.** *Let $B\in C(X)$. Then for every $x\in X$ there exists $y\in B$ such that $d(x,y)=D(x,B)$.*

Definition 1.2. Let $T:X\times X\to N(X)$ be a multivalued mapping. A point $(x,y)\in X\times X$ is said to be a coupled fixed point of T if $x\in T(x,y)$ and $y\in T(y,x)$.

We denote the set of coupled fixed points of the mapping T by $F(T)$.

Let $\{T_n\colon X\times X\to CB(X)\}$ be a sequence of multivalued mappings that converges to a mapping $T\colon X\times X\to CB(X)$, that is, $T=\lim\limits_{n\to\infty}T_n$. That is,

$$M_n=\sup_{(x,y)\in X\times X}H(T_n(x,y),T(x,y))\to 0 \text{ as } n\to\infty.$$

Analogous to the notion of stability of fixed points in [24, 25, 26], we propose the following definition of stability of coupled fixed point sets.

Definition 1.3. Suppose that $\{F(T_n)\}$ is the sequence of coupled fixed point sets of the sequence of mappings $\{T_n\}$ and $F(T)$ is the coupled fixed point set of T. We say that the coupled fixed point sets of $\{T_n\}$ are stable if

$$\lim_{n\to\infty}H_\rho(F(T_n),\ F(T))=0.$$

As mentioned in the introduction, various admissibility criteria were introduced in the study of fixed points of mappings. In particular, we refer the reader to [1] and [38].

For our study, we introduce the cyclic (α,β)- admissibility for singlevalued and multivalued coupled mappings.

Definition 1.4. Let $T:X\times X\to X$ and $\alpha,\ \beta:X\to[0,\ \infty)$. We say that T is a cyclic (α,β)- admissible mapping if for $(x,y)\in X\times X$,
(i) $\alpha(x)\ge 1$ and $\beta(y)\ge 1\implies\beta(T(x,y))\ge 1$,
(ii) $\beta(x)\ge 1$ and $\alpha(y)\ge 1\implies\alpha(T(x,y))\ge 1$.

Definition 1.5. Let $T:X\times X\to N(X)$ be a multivalued mapping and $\alpha,\ \beta:X\to[0,\ \infty)$. We say that T is a cyclic (α,β)- admissible mapping if for $(x,y)\in X\times X$,
(i) $\alpha(x)\ge 1$ and $\beta(y)\ge 1\Longrightarrow\beta(u)\ge 1$ for all $u\in T(x,y)$,
(ii) $\beta(x)\ge 1$ and $\alpha(y)\ge 1\Longrightarrow\alpha(v)\ge 1$ for all $v\in T(x,y)$.

Example 1.1. Let $X=[0,\ 1]$ be equipped with usual metric, denoted as d. Let $T:X\times X\to C(X)$ be defined as $T(x,\ y)=[0,\ \frac{x+y}{16}]$. Let $\alpha,\ \beta:X\to[0,\ \infty)$ be defined as

$$\alpha(x)=\begin{cases} e^x, \text{ if } & 0\le x\le\frac{1}{2},\\ 0, & \text{otherwise}\end{cases}\quad\text{and}\quad\beta(x)=\begin{cases}\frac{e^x+e^{-x}}{2}, \text{ if } & 0\le x\le\frac{1}{2},\\ 0, & \text{otherwise.}\end{cases}$$

Suppose that $(x,y)\in X\times X$ and $\alpha(x)\ge 1$ and $\beta(y)\ge 1$. Then $x,\ y\in[0,\ \frac{1}{2}]$ and $T(x,y)=[0,\ \frac{x+y}{16}]\subseteq[0,\ \frac{1}{16}]\subseteq[0,\ \frac{1}{2}]$. It follows that $\beta(u)\ge 1$ for all $u\in T(x,y)$.

Similarly, if $(x, y) \in X \times X$ and $\beta(x) \geq 1$ and $\alpha(y) \geq 1$, it can be shown that $\alpha(v) \geq 1$ for all $v \in T(x, y)$. Therefore, T is a cyclic $(\alpha,\ \beta)$- admissible mapping.

Definition 1.6. We say that the metric space (X, d) has *regular property with respect to a mapping* $\alpha : X \to [0, \infty)$ if for every sequence $\{x_n\}$ converging to $x \in X$, $\alpha(x_n) \geq 1$ for all $n \Longrightarrow \alpha(x) \geq 1$.

Remark 1.1. For the metric space X and the mappings α, β as in Example 1.1, it can be easily verified that X is regular with respect to α and β. In fact, $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for every $x \in [0,\ \frac{1}{2}]$.

Next we describe two classes of functions in the following.
Let Φ denote the collection of all functions $\phi : [0, \infty) \longrightarrow [0, \infty)$ such that
(P_1): ϕ is nondecreasing;
(P_2): $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for $t \geq 0$ and $\phi(0) = 0$.

Let Γ denote the collection of all functions $G : [0, \infty)^7 \to \mathbb{R}$ such that
(P_1): G is continuous;
(P_2): G is nondecreasing in first coordinate and nonincreasing in sixth and seventh coordinates;
(P_3): there exists a function $\phi \in \Phi$ (defined above) such that

$$G(u,\ v,\ w,\ v,\ w,\ 0,\ 0) \leq 0 \Longrightarrow u \leq \phi(\max\ \{v,\ w\}).$$

Thus Γ is a collection of functions which is defined corresponding to a definite choice of ϕ from the class Φ.

2. Existence of coupled fixed points

Let (X, d) be a complete metric space and $\alpha,\ \beta : X \to [0,\ \infty)$. Denote

$$\Delta(x, y, u, v) = \alpha(x)\beta(y)\beta(u)\alpha(v) \text{ or } \beta(x)\alpha(y)\alpha(u)\beta(v).$$

We assume the following for the rest of the paper:
(A1) X has regular property with respect to α and β;
(A2) There exists $(x_0,\ y_0) \in X \times X$ such that $\alpha(x_0) \geq 1$ and $\beta(y_0) \geq 1$.
We establish the existence of coupled fixed points for mappings defined on the product space $X \times X$ under certain admissibility conditions.

Theorem 2.1. *Let $T : X \times X \to C(X)$ be a multivalued, cyclic (α, β)- admissible mapping. Suppose there exist $\phi \in \Phi$ and $G \in \Gamma$ corresponding to the function ϕ such that for $(x, y),\ (u, v) \in X \times X$ with $\Delta(x, y, u, v) \geq 1$,*

$$G\Big(H(T(x, y), T(u, v)),\ d(x, u),\ d(y, v),\ D(x, T(x, y)),\ D(y, T(y, x)),$$
$$D(u, T(x, y)),\ D(v, T(y, x))\Big) \leq 0.$$

Then T has a coupled fixed point.

Before proceeding to the proof of Theorem 2.1, we present a few special cases illustrating the applicability of the theorem.

Remark 2.1. In Theorem 2.1, taking $\alpha(x) = \beta(x) = 1$ for all $x \in X$, with $\phi(t) = kt$, where $0 \leq k < 1$, and choosing

(i) $G(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = t_1 - k \max\{t_2, t_3\}$,
(ii) $G(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = t_1 - k \max\{t_4, t_5\}$,
(iii) $G(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6, t_7\}$,
respectively, we have the following corollaries.

Corollary 2.1. *Let $T : X \times X \to C(X)$. If there exists $k \in [0,\ 1)$ for which*

$$H(T(x, y), T(u, v)) \leq k \ max\{d(x, u),\ d(y, v)\},\ for\ (x, y),\ (u, v) \in X \times X,$$

then T has a coupled fixed point.

Corollary 2.2. *Let $T : X \times X \to C(X)$. If there exists $k \in [0,\ 1)$ for which*

$$H(T(x, y), T(u, v)) \leq k\ max\{D(x, T(x, y)),\ D(y, T(y, x))\},\ for\ (x, y),\ (u, v) \in X \times X,$$

then T has a coupled fixed point.

Corollary 2.3. *Let $T : X \times X \to C(X)$. Suppose there exists $k \in [0,\ 1)$ such that for $(x,\ y),\ (u,\ v) \in X \times X$,*

$$H(T(x, y), T(u, v)) \leq k\ max\Big\{d(x, u),\ d(y, v),\ D(x, T(x, y)),\ D(y, T(y, x))$$
$$D(u, T(x, y)),\ D(v, T(y, x))\Big\}.$$

Then T has a coupled fixed point.

Further, we present the following illustrative example that verifies the requirements of Theorem 2.1.

Example 2.1. Using the metric space X, mappings α, β and the mapping T as in Example 1.1, we see that $X = [0,\ 1]$ is regular with respect to α and β (see Remark 1.1). Also, we know that T is a cyclic $(\alpha,\ \beta)$- admissible mapping. Now, let $\phi(t) = \frac{t}{2}$ and define $G : [0,\ \infty)^7 \to \mathbb{R}$ as

$$G(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = t_1 - \frac{1}{2} \max\{t_2, t_3, t_4, t_5\} - (e^{t_6 t_7} - 1).$$

Clearly, $\phi \in \Phi$ and $G \in \Gamma$. Let $(x,\ y), (u, v) \in X \times X$ such that $\Delta(x, y, u, v) \geq 1$. Now, $\Delta(x, y, u, v) \geq 1$ implies that $x, y, u, v \in [0,\ \frac{1}{2}]$. So, we now check the validity of the inequality in Theorem 2.1 for $x, y, u, v \in [0,\ \frac{1}{2}]$. Let $x, y, u, v \in [0,\ \frac{1}{2}]$. Now,

$$\begin{aligned} H(T(x, y),\ T(u, v)) = \frac{|\ x + y - u - v\ |}{16} &\leq \frac{|\ x - u\ |}{16} + \frac{|\ y - v\ |}{16} \\ &\leq \frac{1}{8}\max\{|\ x - u\ |,\ |\ y - v\ |\}. \end{aligned}$$

Then it follows that

$$H(T(x, y),\ T(u, v)) - \frac{1}{2} \max\{d(x, u),\ d(y, v),\ D(x, T(x, y)),\ D(y, T(y, x))\}$$
$$- [e^{D(u,T(x,y)).D(v,T(y,x))} - 1] \leq 0.$$

That is,

$$G\Big(H(T(x,y),T(u,v)),\ d(x,u),\ d(y,v),\ D(x,T(x,y)),\ D(y,T(y,x)),$$
$$D(u,T(x,y)),\ D(v,T(y,x))\Big) \leq 0,$$

is satisfied for all $(x,\ y),(u,v) \in X\times X$ with $\Delta(x,y,u,v) \geq 1$. Hence all the conditions of Theorem 2.1 are satisfied and $(0,\ 0)$ is a coupled fixed point of T.

We now present the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $(x_0,\ y_0) \in X \times X$ such that $\alpha(x_0) \geq 1$ and $\beta(y_0) \geq 1$. Since $T(x_0,\ y_0),\ T(y_0,\ x_0) \in C(X)$, by Lemma 1.1, there exist $x_1 \in T(x_0,\ y_0)$ and $y_1 \in T(y_0,\ x_0)$ such that $d(x_0,\ x_1) = D(x_0,\ T(x_0,\ y_0))$ and $d(y_0,\ y_1) = D(y_0,\ T(y_0,\ x_0))$. Since T is a cyclic (α,β)- admissible mapping, $\alpha(x_0) \geq 1$ and $\beta(y_0) \geq 1$, implies $\beta(x_1) \geq 1$ and $\alpha(y_1) \geq 1$. Similarly, as $T(x_1,\ y_1),\ T(y_1,\ x_1) \in C(X)$, there exist $x_2 \in T(x_1,\ y_1)$ and $y_2 \in T(y_1,\ x_1)$ such that $d(x_1,\ x_2) = D(x_1,\ T(x_1,\ y_1))$ and $d(y_1,\ y_2) = D(y_1,\ T(y_1,\ x_1))$. Further, $\beta(x_1) \geq 1$ and $\alpha(y_1) \geq 1$, implies $\alpha(x_2) \geq 1$ and $\beta(y_2) \geq 1$.

Proceeding in this manner, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$,

$$x_{n+1} \in T(x_n,\ y_n) \ \text{ and } \ y_{n+1} \in T(y_n,\ x_n), \tag{2.1}$$

$$\alpha(x_{2n}) \geq 1,\quad \beta(x_{2n+1}) \geq 1,\quad \beta(y_{2n}) \geq 1\ \text{ and } \alpha(y_{2n+1}) \geq 1, \tag{2.2}$$

$$d(x_n,\ x_{n+1}) = D(x_n,\ T(x_n,\ y_n)) \ \text{ and } \ d(y_n,\ y_{n+1}) = D(y_n,\ T(y_n,\ x_n)). \tag{2.3}$$

$$\text{Let } \ r_n = \max\ \{d(x_n,\ x_{n+1}),\ d(y_n,\ y_{n+1})\}, \quad \text{for all } n \geq 0. \tag{2.4}$$

As $\Delta(x_{2n}, y_{2n}, x_{2n+1}, y_{2n+1}) = \alpha(x_{2n})\beta(y_{2n})\beta(x_{2n+1})\alpha(y_{2n+1}) \geq 1$, using the assumption of the theorem, (2.1), (2.2), (2.3) and property (P_2) of G, we have

$$G\Big(H(T(x_{2n},y_{2n}),T(x_{2n+1},y_{2n+1})), d(x_{2n},x_{2n+1}), d(y_{2n},y_{2n+1}), D(x_{2n},T(x_{2n},y_{2n})),$$
$$D(y_{2n},T(y_{2n},x_{2n})), D(x_{2n+1},T(x_{2n},y_{2n})),\ D(y_{2n+1},T(y_{2n},x_{2n}))\Big) \leq 0.$$
$$G\Big(D(x_{2n+1},T(x_{2n+1},y_{2n+1})),\ d(x_{2n},x_{2n+1}),\ d(y_{2n},y_{2n+1}),\ d(x_{2n},x_{2n+1}),$$
$$d(y_{2n},y_{2n+1}),\ d(x_{2n+1},x_{2n+1}),\ d(y_{2n+1},y_{2n+1})\Big) \leq 0.$$

Thus,

$$G\Big(d(x_{2n+1},x_{2n+2}), d(x_{2n},x_{2n+1}), d(y_{2n},y_{2n+1}),$$
$$d(x_{2n},x_{2n+1}), d(y_{2n},y_{2n+1}), 0, 0\Big) \leq 0. \tag{2.5}$$

Now, taking $u = d(x_{2n+1},x_{2n+2}),\ v = d(x_{2n},x_{2n+1})$ and $w = d(y_{2n},y_{2n+1})$ and using property (P_3) of G, we have

$$d(x_{2n+1},x_{2n+2}) \leq \phi(\max\ \{d(x_{2n},x_{2n+1}),\ d(y_{2n},y_{2n+1})\}) = \phi(r_{2n}). \tag{2.6}$$

Again, $\Delta(x_{2n+1}, y_{2n+1}, x_{2n+2}, y_{2n+2}) = \beta(x_{2n+1})\alpha(y_{2n+1})\alpha(x_{2n+2})\beta(y_{2n+2}) \geq 1$. And, arguing as above, we have

$$\begin{aligned}
G\Big(&H(T(x_{2n+1},\ y_{2n+1}),\ T(x_{2n+2},\ y_{2n+2})),\ d(x_{2n+1},\ x_{2n+2}),\ d(y_{2n+1},\ y_{2n+2}),\\
&D(x_{2n+1},\ T(x_{2n+1},\ y_{2n+1})),\ D(y_{2n+1},\ T(y_{2n+1},\ x_{2n+1})),\\
&D(x_{2n+2},\ T(x_{2n+1},\ y_{2n+1})),\ D(y_{2n+2},\ T(y_{2n+1},\ x_{2n+1}))\Big) \leq 0\\
\Rightarrow G\Big(&D(x_{2n+2},\ T(x_{2n+2},\ y_{2n+2})),\ d(x_{2n+1},\ x_{2n+2}),\ d(y_{2n+1},\ y_{2n+2}),\\
&d(x_{2n+1},\ x_{2n+2}),\ d(y_{2n+1},\ y_{2n+2}),\ d(x_{2n+2},\ x_{2n+2}),\ d(y_{2n+2},\ y_{2n+2})\Big) \leq 0\\
\Rightarrow G\Big(&d(x_{2n+2}, x_{2n+3}),\ d(x_{2n+1}, x_{2n+2}),\ d(y_{2n+1}, y_{2n+2}),\ d(x_{2n+1}, x_{2n+2}),\\
&d(y_{2n+1}, y_{2n+2}),\ 0,\ 0\Big) \leq 0. \qquad (2.7)
\end{aligned}$$

Taking $u = d(x_{2n+2}, x_{2n+3}),\ v = d(x_{2n+1}, x_{2n+2})$ and $w = d(y_{2n+1}, y_{2n+2})$ and using the property (P_3) of G, we have

$$d(x_{2n+2}, x_{2n+3}) \leq \phi(\max\{d(x_{2n+1}, x_{2n+2}),\ d(y_{2n+1}, y_{2n+2})\}) = \phi(r_{2n+1}). \qquad (2.8)$$

Combining (2.6), (2.8), we have

$$d(x_{n+1},\ x_{n+2}) \leq \phi(\max\{d(x_n,\ x_{n+1}),\ d(y_n,\ y_{n+1})\}) = \phi(r_n). \qquad (2.9)$$

Now, $\Delta(y_{2n}, x_{2n}, y_{2n+1}, x_{2n+1}) = \beta(y_{2n})\alpha(x_{2n})\alpha(y_{2n+1})\beta(x_{2n+1}) \geq 1$. As before, we have

$$\begin{aligned}
G\Big(&H(T(y_{2n}, x_{2n}), T(y_{2n+1}, x_{2n+1})),\ d(y_{2n}, y_{2n+1}),\ d(x_{2n}, x_{2n+1}),\\
&D(y_{2n}, T(y_{2n}, x_{2n})), D(x_{2n}, T(x_{2n}, y_{2n})),\\
&D(y_{2n+1}, T(y_{2n}, x_{2n})),\ D(x_{2n+1}, T(x_{2n}, y_{2n}))\Big) \leq 0\\
\Rightarrow G\Big(&D(y_{2n+1}, T(y_{2n+1}, x_{2n+1})),\ d(y_{2n}, y_{2n+1}),\ d(x_{2n}, x_{2n+1}),\ d(y_{2n}, y_{2n+1}),\\
&d(x_{2n}, x_{2n+1}),\ d(y_{2n+1}, y_{2n+1}),\ d(x_{2n+1}, x_{2n+1})\Big) \leq 0.\\
\Rightarrow G\Big(&d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}),\\
&d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1}), d(x_{2n}, x_{2n+1}), 0, 0, \Big) \leq 0. \qquad (2.10)
\end{aligned}$$

Taking $u = d(y_{2n+1}, y_{2n+2}),\ v = d(y_{2n}, y_{2n+1})$ and $w = d(x_{2n}, x_{2n+1})$ and using property (P_3) of G, we have

$$d(y_{2n+1}, y_{2n+2}) \leq \phi(\max\{d(y_{2n}, y_{2n+1}),\ d(x_{2n}, x_{2n+1})\}) = \phi(r_{2n}). \qquad (2.11)$$

Again,

$$\Delta(y_{2n+1}, x_{2n+1}, y_{2n+2}, x_{2n+2}) = \alpha(y_{2n+1})\beta(x_{2n+1})\beta(y_{2n+2})\alpha(x_{2n+2}) \geq 1.$$

Also, we have

$$
\begin{aligned}
G\Big(&H(T(y_{2n+1},\ x_{2n+1}),\ T(y_{2n+2},\ x_{2n+2})),\ d(y_{2n+1},\ y_{2n+2}),\ d(x_{2n+1},\ x_{2n+2}),\\
&D(y_{2n+1},\ T(y_{2n+1},\ x_{2n+1})),\ D(x_{2n+1},\ T(x_{2n+1},\ y_{2n+1}))\\
&D(y_{2n+2},\ T(y_{2n+1},\ x_{2n+1})),\ D(x_{2n+2},\ T(x_{2n+1},\ y_{2n+1}))\Big) \leq 0.\\
\Rightarrow G\Big(&D(y_{2n+2},\ T(y_{2n+2},\ x_{2n+2})),\ d(y_{2n+1},\ y_{2n+2}),\ d(x_{2n+1},\ x_{2n+2}),\\
&d(y_{2n+1},\ y_{2n+2}), d(x_{2n+1},\ x_{2n+2}),\ d(y_{2n+2},\ y_{2n+2}),\ d(x_{2n+2},\ x_{2n+2})\Big) \leq 0.\\
\Rightarrow G\Big(&d(y_{2n+2}, y_{2n+3}),\ d(y_{2n+1}, y_{2n+2}),\ d(x_{2n+1}, x_{2n+2}),\ d(y_{2n+1}, y_{2n+2}),\\
&d(x_{2n+1}, x_{2n+2}),\ 0,\ 0\Big) \leq 0. \qquad (2.12)
\end{aligned}
$$

Taking $u = d(y_{2n+2}, y_{2n+3}),\ v = d(y_{2n+1}, y_{2n+2})$ and $w = d(x_{2n+1}, x_{2n+2})$ and using the property (P_3) of G, we have

$$d(y_{2n+2}, y_{2n+3}) \leq \phi(\max\ \{d(y_{2n+1}, y_{2n+2}),\ d(x_{2n+1}, x_{2n+2})\}) = \phi(r_{2n+1}). \qquad (2.13)$$

Combining (2.11) and (2.13), we have

$$d(y_{n+1},\ y_{n+2}) \leq \phi(\max\ \{d(y_n,\ y_{n+1}),\ d(x_n,\ x_{n+1})\}) = \phi(r_n). \qquad (2.14)$$

Combining (2.9) and (2.14), we obtain

$$r_{n+1} = \max\ \{d(x_{n+1},\ x_{n+2}),\ d(y_{n+1},\ y_{n+2})\} \leq \phi(r_n). \qquad (2.15)$$

By repeated application of (2.15) and using a property of ϕ, we have

$$r_{n+1} \leq \phi(r_n) \leq \phi^2(r_{n-1}) \leq \phi^3(r_{n-2}) \leq ... \leq \phi^{n+1}(r_0). \qquad (2.16)$$

Now we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. With the help of (2.15) and the property (P_2) of ϕ, we have

$$\sum_{n=1}^{\infty} d(x_n,\ x_{n+1}) \leq \sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{\infty} \phi^n(r_0) < \infty$$

and

$$\sum_{n=1}^{\infty} d(y_n,\ y_{n+1}) \leq \sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{\infty} \phi^n(r_0) < \infty,$$

which imply that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since (X, d) is complete, there exist $x,\ y \in X$ such that

$$\lim_{n\to\infty} x_n = x \ \text{ and } \ \lim_{n\to\infty} y_n = y. \qquad (2.17)$$

Using (2.2) and the assumption (A1), we have

$$\alpha(x) \geq 1, \quad \beta(x) \geq 1, \quad \alpha(y) \geq 1 \ \text{ and } \ \beta(y) \geq 1. \qquad (2.18)$$

By (2.2), (2.18), we have $\Delta(x_{2n}, y_{2n}, x, y) = \alpha(x_{2n})\beta(y_{2n})\beta(x)\alpha(y) \geq 1$. Using the assumption of the theorem, (2.1), (2.2), (2.3) and the property (P_2) of G, we have

$$\begin{aligned} &G\Big(H(T(x_{2n},\ y_{2n}),\ T(x,\ y)),\ d(x_{2n},\ x),\ d(y_{2n},\ y),\ D(x_{2n},\ T(x_{2n},\ y_{2n})),\\ &\qquad D(y_{2n},\ T(y_{2n},\ x_{2n})),\ D(x,\ T(x_{2n},\ y_{2n})),\ D(y,\ T(y_{2n},\ x_{2n}))\Big) \leq 0\\ &\Rightarrow G\Big(D(x_{2n+1},\ T(x,\ y)),\ d(x_{2n},\ x),\ d(y_{2n},\ y),\ d(x_{2n},\ x_{2n+1}),\\ &\qquad d(y_{2n},\ y_{2n+1}),\ d(x,\ x_{2n+1}),\ d(y,\ y_{2n+1})\Big) \leq 0. \end{aligned} \tag{2.19}$$

Taking limit as $n \to \infty$ in (2.19), using (2.17) and the continuity of G, we have

$$G\Big(D(x,\ T(x,\ y)),\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big) \leq 0. \tag{2.20}$$

Taking $u = D(x,\ T(x,\ y)),\ v = 0$ and $w = 0$ and using the property (P_3) of G, we have

$$D(x,\ T(x,\ y)) \leq \phi(\max\{0,\ 0\}) = 0. \tag{2.21}$$

Again from (2.2) and (2.18), we have

$$\Delta(y_{2n}, x_{2n}, y, x) = \beta(y_{2n})\alpha(x_{2n})\alpha(y)\beta(x) \geq 1.$$

Using the assumption of the theorem, (2.1), (2.2), (2.3) and the property (P_2) of G, we have

$$\begin{aligned} &G\Big(H(T(y_{2n},\ x_{2n}),\ T(y,\ x)),\ d(y_{2n},\ y),\ d(x_{2n},\ x),\ D(y_{2n},\ T(y_{2n},\ x_{2n})),\\ &\qquad D(x_{2n},\ T(x_{2n},\ y_{2n})),\ D(y,\ T(y_{2n},\ x_{2n})),\ D(x,\ T(x_{2n},\ y_{2n}))\Big) \leq 0\\ &\Rightarrow G\Big(D(y_{2n+1},\ T(y,\ x)),\ d(y_{2n},\ y),\ d(x_{2n},\ x),\ d(y_{2n},\ y_{2n+1}),\\ &\qquad d(x_{2n},\ x_{2n+1}),\ d(y,\ y_{2n+1}),\ d(x,\ x_{2n+1})\Big) \leq 0. \end{aligned} \tag{2.22}$$

Taking limit as $n \to \infty$ in (2.22), using (2.17) and the continuity of G, we have

$$G\Big(D(y,\ T(y,\ x)),\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big) \leq 0. \tag{2.23}$$

Taking $u = D(y,\ T(y, x)),\ v = 0$ and $w = 0$ and using the property (P_3) of G, we have

$$D(y,\ T(y,\ x)) \leq \phi(\max\{0,\ 0\}) = 0. \tag{2.24}$$

We have from (2.21) and (2.24) that $D(x,\ T(x,\ y)) = 0$ and $D(y,\ T(y,\ x)) = 0$, which imply that $x \in \overline{T(x,\ y)} = T(x,\ y)$ and $y \in \overline{T(y,\ x)} = T(y,\ x)$, where $\overline{T(x,\ y)}$ and $\overline{T(y,\ x)}$ denote the closures of $T(x,\ y)$ and $T(y,\ x)$ respectively. Therefore, $(x,\ y)$ is a coupled fixed point of T.

The following theorem is a special case of Theorem 2.1, obtained by treating $T : X \times X \to X$ as a multivalued mapping. That is, $T(x, y)$ is treated as a singleton set for every $(x, y) \in X \times X$.

Theorem 2.2. *Let $T : X \times X \to X$ be a cyclic $(\alpha,\ \beta)$- admissible mapping. Suppose that there exist $\phi \in \Phi$ and $G \in \Gamma$ corresponding to the function ϕ such that for $(x,\ y),\ (u,\ v) \in X \times X$ with $\Delta(x,y,u,v) \geq 1$,*

$$G\Big(d(T(x,y),T(u,v)),\ d(x,u),\ d(y,v),\ d(x,T(x,y)),\ d(y,T(y,x)),\ d(u,T(x,y)),\ d(v,T(y,x))\Big) \leq 0.$$

Then T has a coupled fixed point.

Proof. Since a singleton set in X is compact, we define a multivalued mapping $S : X \times X \to C(X)$ as $S(x,y) = \{T(x,y)\}$ for $(x,y) \in X \times X$.

Let $(x,\ y) \in X \times X$ such that $\alpha(x) \geq 1$ and $\beta(y) \geq 1$. Then by cyclic (α,β) - admissibility of T, we have $\beta(T(x,\ y)) \geq 1$, that is, $\beta(u) \geq 1$, where $u \in S(x,\ y) = \{T(x,\ y)\}$. Similarly, if $(x,\ y) \in X \times X$ such that $\beta(x) \geq 1$ and $\alpha(y) \geq 1$, then by cyclic (α,β) - admissibility of T, we have $\alpha(T(x,\ y)) \geq 1$, that is, $\alpha(v) \geq 1$, where $v \in S(x,\ y) = \{T(x,\ y)\}$. Therefore, S is a cyclic $(\alpha,\ \beta)$- admissible mapping.

Let $(x,y),(u,v) \in X \times X$ with $\Delta(x,y,u,v) \geq 1$. Then

$$G\Big(d(T(x,y),T(u,v)),\ d(x,u),\ d(y,v),\ d(x,T(x,y)),\ d(y,T(y,x)),\ d(u,T(x,y)),\ d(v,T(y,x))\Big) \leq 0.$$

$$\implies G\Big(H(S(x,y),S(u,v)),\ d(x,u),\ d(y,v),\ D(x,\ S(x,y)),\ D(y,\ S(y,x)),\ D(u,S(x,y)),\ D(v,S(y,x))\Big) \leq 0.$$

That is, S satisfies the assumptions of Theorem 2.1. So, all the conditions of Theorem 2.1 are satisfied and then there exists $(x,y) \in X \times X$ such that $x \in S(x,y) = \{T(x,y)\}$ and $y \in S(y,x) = \{T(y,x)\}$, that is, $x = T(x,y)$ and $y = T(y,x)$. Hence (x,y) is a coupled fixed point of T.

3. Stability of coupled fixed point sets

In this section, we investigate the stability of coupled fixed point sets of the set-valued contractions. We begin with the following Lemma.

Lemma 3.1. *Let (X,d) be a metric space and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that X is regular with respect to α,β. Let $\{T_n : X \times X \to C(X) : n \in N\}$ be a sequence of multivalued, cyclic $(\alpha,\beta)-$ admissible mappings that are uniformly convergent to a multivalued mapping $T : X \times X \to C(X)$. Further, let each T_n $(n \in \mathbb{N})$, satisfy the hypothesis of Theorem 2.1. Then T is cyclic $(\alpha,\beta)-$ admissible and satisfies the hypothesis of Theorem 2.1.*

Proof. First we prove that T is cyclic $(\alpha,\ \beta)$- admissible. Let $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for some $(x,y) \in X \times X$ and $u \in T(x,y)$. Since $T_n \to T$ uniformly, there exist a sequence $\{x_n\}$ in $\{T_n(x,\ y)\}$ such that $x_n \to u$ as $n \to \infty$. As each T_n $(n \in \mathbb{N})$ is cyclic $(\alpha,\ \beta)$ - admissible, we have $\beta(x_n) \geq 1$ for every $n \in \mathbb{N}$. Then by regular property of the space with respect to β, it follows that $\beta(u) \geq 1$. Again, let $\beta(x) \geq 1$ and $\alpha(y) \geq 1$ for some $(x,\ y) \in X \times X$ and $v \in T(x,\ y)$. Since $T_n \to T$ uniformly, there exist a sequence $\{y_n\}$ in $\{T_n(x,\ y)\}$ such that $y_n \to v$ as $n \to \infty$. As each T_n $(n \in \mathbb{N})$ is cyclic $(\alpha,\ \beta)$- admissible, we have $\alpha(y_n) \geq 1$ for every $n \in \mathbb{N}$. Then by regular property of the space with respect to α, it follows that $\alpha(v) \geq 1$. Hence T is cyclic $(\alpha,\ \beta)$-admissible.

Let $(x, y),\ (u, v) \in X \times X$ with $\Delta(x, y, u, v) \geq 1$. Now,

$$G\Big(H(T_n(x, y), T_n(u, v)),\ d(x, u),\ d(y, v),\ D(x, T_n(x, y)),\ D(y, T_n(y, x)),$$
$$D(u, T_n(x, y)),\ D(v, T_n(y, x))\Big) \leq 0.$$

Since G is continuous and T_n converges to T uniformly, taking limit as $n \to \infty$ in the above inequality, we have

$$G\Big(H(T(x, y), T(u, v)),\ d(x, u),\ d(y, v),\ D(x, T(x, y)),\ D(y, T(y, x)),$$
$$D(u, T(x, y)),\ D(v, T(y, x))\Big) \leq 0.$$

So T satisfies the hypothesis of Theorem 2.1.

Theorem 3.1. *Let (X, d) be a complete metric space, $T_l : X \times X \to C(X)$, $(l \in \{1,\ 2\})$ be two multivalued mappings and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions of Theorem 2.1 are satisfied by each T_l, $(l \in \{1,\ 2\})$. Then $F(T_l) \neq \emptyset$ for every $l \in \{1,\ 2\}$. Also suppose there exist $(x_0, y_0) \in F(T_1)$ and $(s_0, t_0) \in F(T_2)$ such that $\alpha(x_0) \geq 1$, $\beta(y_0) \geq 1$ and $\alpha(s_0) \geq 1$, $\beta(t_0) \geq 1$. Then*

$$H_\rho(F(T_1), F(T_2)) \leq \Theta(M),$$

where $\Theta(t) = \sum_{n=1}^{\infty} \phi^n(t)$ and $M = \sup_{(x,y)\in X\times X}\{H(T_1(x, y), T_2(x, y))\}$.

Proof. By Theorem 2.1, the set of coupled fixed points of T_l, $(l = 1,\ 2)$ are nonempty, that is, $F(T_l) \neq \emptyset$, for $l = 1,\ 2$. By the condition of the theorem, suppose $(x_0, y_0) \in F(T_1)$ with $\alpha(x_0) \geq 1$ and $\beta(y_0) \geq 1$. So $x_0 \in T_1(x_0, y_0)$ and $y_0 \in T_1(y_0, x_0)$. By Lemma 1.1, there exist $x_1 \in T_2(x_0, y_0)$ and $y_1 \in T_2(y_0, x_0)$ such that $d(x_0, x_1) = D(x_0, T_2(x_0, y_0))$ and $d(y_0, y_1) = D(y_0, T_2(y_0, x_0))$. Since T_2 is a cyclic (α, β)- admissible mapping, we have $\beta(x_1) \geq 1$ and $\alpha(y_1) \geq 1$. By Lemma 1.1, there exists $x_2 \in T_2(x_1, y_1)$ and $y_2 \in T_2(y_1, x_1)$ such that $d(x_1, x_2) = D(x_1, T_2(x_1, y_1))$ and $d(y_1, y_2) = D(y_1, T_2(y_1, x_1))$. Inductively, arguing as in the proof of Theorem 2.1, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$,

$$x_{n+1} \in T_2(x_n, y_n) \ \text{ and } \ y_{n+1} \in T_2(y_n, x_n),$$
$$\alpha(x_{2n}) \geq 1,\ \beta(x_{2n+1}) \geq 1,\ \beta(y_{2n}) \geq 1 \text{ and } \alpha(y_{2n+1}) \geq 1,$$
$$d(x_n, x_{n+1}) = D(x_n, T_2(x_n, y_n)) \text{ and } \ d(y_n, y_{n+1}) = D(y_n, T_2(y_n, x_n)).$$

Also,

$$d(x_{n+1}, x_{n+2}) \leq \phi(\max\{d(x_n, x_{n+1}),\ d(y_n, y_{n+1})\}) = \phi(r_n)$$

and

$$d(y_{n+1}, y_{n+2}) \leq \phi(\max\{d(x_n, x_{n+1}),\ d(y_n, y_{n+1})\}) = \phi(r_n),$$

where $r_n = \max\{d(x_n, x_{n+1}),\ d(y_n, y_{n+1})\}$.
Arguing as in the proof of Theorem 2.1, we can prove that:

(i) (2.16) is satisfied;

(ii) $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and so there exist $u,\ v \in X$ such that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$;

(iii) (u, v) is a coupled fixed point of T_2, that is, $u \in T_2(u, v)$ and $v \in T_2(v, u)$.

From the definition of M and the construction of the sequences $\{x_n\}$ and $\{y_n\}$, we have

$$d(x_0, x_1) = D(x_0, T_2(x_0, y_0)) \leq H(T_1(x_0, y_0), T_2(x_0, y_0)) \leq M \tag{3.1}$$

and

$$d(y_0, y_1) = D(y_0, T_2(y_0, x_0)) \leq H(T_1(y_0, x_0), T_2(y_0, x_0)) \leq M. \tag{3.2}$$

Using (2.16), (3.1), (3.2) and the properties of ϕ, we have

$$\begin{aligned} d(x_0,\ u) &\leq \sum_{i=0}^{n} d(x_i,\ x_{i+1}) + d(x_{n+1},\ u) \leq \sum_{i=0}^{n} \phi^i(r_0) + d(x_{n+1},\ u) \\ &= \sum_{i=0}^{n} \phi^i(\max\{d(x_0,\ x_1),\ d(y_0,\ y_1)\}) + d(x_{n+1},\ u) \\ &\leq \sum_{i=0}^{n} \phi^i(\max\{M,\ M\}) + d(x_{n+1},\ u) \leq \sum_{i=0}^{n} \phi^i(M) + d(x_{n+1},\ u). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(x_0,\ u) \leq \sum_{i=0}^{\infty} \phi^i(M) = \Theta(M). \tag{3.3}$$

Again, using (2.16), (3.1), (3.2) and the properties of ϕ, we have

$$\begin{aligned} d(y_0,\ v) &\leq \sum_{i=0}^{n} d(y_i,\ y_{i+1}) + d(y_{n+1},\ v) \leq \sum_{i=0}^{n} \phi^i(r_0) + d(y_{n+1},\ v) \\ &= \sum_{i=0}^{n} \phi^i(\max\{d(y_0,\ y_1),\ d(x_0,\ x_1)\}) + d(y_{n+1},\ v) \\ &\leq \sum_{i=0}^{n} \phi^i(\max\{M,\ M\}) + d(y_{n+1},\ v) \leq \sum_{i=0}^{n} \phi^i(M) + d(y_{n+1},\ v). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(y_0,\ v) \leq \sum_{i=0}^{\infty} \phi^i(M) = \Theta(M). \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\max\{d(x_0,\ u),\ d(y_0,\ v)\} \leq \Theta(M), \quad \text{that is,} \quad \rho((x_0,\ y_0),\ (u,\ v)) \leq \Theta(M).$$

We have that for $(x_0, y_0) \in F(T_1)$ there exists $(u, v) \in F(T_2)$ such that $\rho((x_0, y_0), (u, v)) \leq \Theta(M)$. Similarly, we can show that for $(s_0, t_0) \in F(T_2)$

there exists $(p,q) \in F(T_1)$ such that $\rho((s_0,t_0),(p,q)) \leq \Theta(M)$. Then it follows that $H_\rho(F(T_1), F(T_2))) \leq \Theta(M)$.

Theorem 3.2. *Let (X,d) be a complete metric space and $\alpha,\ \beta : X \to [0,\ \infty)$. Let $\{T_n : X \times X \to C(X) :\ n \in N\}$ be a sequence of multivalued mappings uniformly convergent to a multivalued mapping $T : X \times X \to C(X)$. Suppose that the assumptions of Theorem 2.1 are satisfied by each T_n $(n \in \mathbb{N})$. Then, $F(T_n) \neq \emptyset$ for every $n \in \mathbb{N}$ and $F(T) \neq \emptyset$. Also suppose there exist $(x_n, y_n) \in F(T_n)$ $(n \in \mathbb{N})$ and $(u,v) \in F(T)$ such that $\alpha(x_n) \geq 1,\ \beta(y_n) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha(u) \geq 1,\ \beta(v) \geq 1$, and $\Theta(t) \to 0$ as $t \to 0$, where $\Theta(t) = \sum_{n=1}^{\infty} \phi^n(t)$. Then the sets of coupled fixed points of T_n are stable.*

Proof. By Lemma 3.1 and Theorem 2.1, $F(T_n) \neq \emptyset$, for all $n \in \mathbb{N}$ and $F(T) \neq \emptyset$. Let

$$M_n = \sup_{(x,y) \in X \times X} H(T_n(x,y), T(x,y)).$$

By Theorem 3.1, we have

$$H_\rho(F(T_n), F(T)) \leq \Theta(M_n). \tag{3.5}$$

Since T_n converges to T uniformly, we have
$M_n = \sup_{(x,y) \in X \times X} H(T_n(x,y), T(x,y)) \to 0$ as $n \to \infty$. Taking limit as $n \to \infty$ in (3.5) and using the assumption on Θ, we have

$$\lim_{n \to \infty} H_\rho(F(T_n), F(T)) \leq 0,$$

which implies that $\lim_{n \to \infty} H_\rho(F(T_n), F(T)) = 0$, that is, coupled fixed point sets of the sequence $\{T_n\}$ are stable.

Example 3.1. Take the metric space (X,d) and the mappings α, β, G and ϕ as considered in Example 2.1. Let $T_n,\ T : X \times X \to C(X)$ be defined as

$$T_n(x,y) = \left[\frac{1+x+y}{64n}, \frac{1}{2}\right]$$

and $T(x,y) = [0,\ \frac{1}{2}]$ for $(x,y) \in X \times X$. Now

$$H(T_n(x,y),\ T_n(u,v)) = \frac{\mid x+y-u-v \mid}{64n}$$

for $(x,y),(u,v) \in X \times X$. Then as explained in Example 2.1, we can show that the assumptions of Theorem 2.1 are satisfied by each T_n $(n \in \mathbb{N})$. Here

$$F(T) = \left\{(x,y) : 0 \leq x \leq \frac{1}{2}; 0 \leq y \leq \frac{1}{2}\right\}$$

and

$$F(T_n) = \left\{(x,y) : \frac{64n}{(64n-1)(64n-1)-1} \leq x,\ y \leq \frac{1}{2}\right\},$$

where $n \in \mathbb{N}$. Also $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for any (x, y) belonging to $F(T_n)$ ($n \in \mathbb{N}$) and $F(T)$. Here $\Theta(t) \to 0$ as $t \to 0$, where

$$\Theta(t) = \sum_{n=1}^{\infty} \phi^n(t) = \sum_{n=1}^{\infty} \left(\frac{t}{2}\right)^n.$$

We see all the conditions of Theorem 3.2 are satisfied. Here

$$\lim_{n\to\infty} H_\rho(F(T_n), F(T)) = 0,$$

that is, coupled fixed point sets of the sequence $\{T_n\}$ are stable.

Remark 3.1. There is a similarity in the ideas regarding the concepts of data dependence of the fixed point sets of multi-valued weakly Picard operators discussed in [35] and the stability of multivalued coupled mappings and associated fixed point sets discussed in our work. In fact, the difference is in the construction of the sequences $\{x_n\}$ and $\{y_n\}$ in the proof of Theorem 3.1 which applies to coupled mappings, while in the case of the function considered in [35], a single sequence is constructed in a different manner.

Acknowledgment. The authors gratefully acknowledge the suggestions made by the learned referee.

References

[1] S. Alizadeh, F. Moradlou, P. Salimi, *Some fixed point results for $(\alpha - \beta)$-$(\psi - \varphi)$-contractive mappings*, Filomat, **28**(2014), no. 3, 635-647.

[2] I. Altun, D. Turkoglu, *Some fixed point theorems on fuzzy metric spaces with implicit relations*, Commun. Korean Math. Soc., **23**(2008), no. 1, 111-124.

[3] G.V.R. Babu, P.D. Sailaja, *Coupled fixed point theorems with new implicit relations and an application*, Journal of Operators, **2014**(2014), Article ID 389646, 16 pages.

[4] I. Beg, A.R. Butt, *Fixed point for mappings satisfying an implicit relation in ordered fuzzy metric spaces*, The Journal of Fuzzy Mathematics, **20**(2012), no. 3, 627-634.

[5] I. Bhaumik, B.S. Choudhury, *Uniform convergence and sequence of maps on a compact metric space with some chaotic properties*, Anal. Theory Appl., **26**(2010), 53-58.

[6] R.K. Bose, R.N. Mukherjee, *Stability of fixed point sets and common fixed points of families of mappings*, Indian J. Pure Appl. Math., **9**(1980), 1130-1138.

[7] S. Chandok, B.S. Choudhury, N. Metiya, *Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions*, J. Egypt. Math. Soc., **23**(2015), 95-101.

[8] B.S. Choudhury, K. Das, P. Das, *Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces*, Fuzzy Sets and Systems, **222**(2013), 84-97.

[9] B.S. Choudhury, A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal., **73**(2010), 2524-2531.

[10] B.S. Choudhury, P. Maity, *Coupled fixed point results in generalized metric spaces*, Math. Comput. Model., **54**(2011), 73-79.

[11] B.S. Choudhury, N. Metiya, C. Bandyopadhyay, *Fixed points of multivalued α-admissible mappings and stability of fixed point sets in metric spaces*, Rend. Circ. Mat. Palermo, **64**(2015), 43-55.

[12] B.S. Choudhury, N. Metiya, C. Bandyopadhyay, P. Maity, *Fixed points of multivalued mappings satisfying hybrid rational Pata-type inequalities*, J. Anal., 27 (2019), 813–828.

[13] B.S. Choudhury, N. Metiya, S. Kundu, *End point theorems of multivalued operators without continuity satisfying hybrid inequality under two different sets of conditions*, Rend. Circ. Mat. Palermo, II. Ser., **68**(2019), no. 1, 65-81.

[14] B.S. Choudhury, N. Metiya, M. Postolache, *A generalized weak contraction principle with applications to coupled coincidence point problems*, Fixed Point Theory Appl., **2013**(2013), 152.

[15] L.B. Ćirić, V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stoch. Anal. Appl., **27**(2009), no. 6, 1246-1259.

[16] A. Felhi, H. Aydi, D. Zhang, *Fixed points for α-admissible contractive mappings via simulation functions*, J. Nonlinear Sci. Appl., **9**(2016), 5544-5560.

[17] G. Galanis, T. Gnana Bhaskar, V. Lakshmikantham, *Set Differential Equations in Frechet Spaces*, J. Applied Anal., **14**(2008), no. 1, 103-113.

[18] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), 1379-1393.

[19] D. Guo, V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11**(1987), 623-632.

[20] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., **71**(2009), 3403-3410.

[21] N. Hussain, E. Karapinar, P. Salimi, F. Akbar, *α-admissible mappings and related fixed point theorems*, J. Inequal. Appl., **2013**(2013), 114.

[22] E. Karapinar, B. Samet, *Generalized $\alpha - \psi$ contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., **2012**(2012), Article ID 793486.

[23] C. Klanarong, S. Suantai, *Coupled coincidence point theorems for new types of mixed monotone multivalued mappings in partially ordered metric spaces*, Abstr. Appl. Anal., **2013**(2013), Article ID 604578, 7 pages.

[24] T.C. Lim, *Fixed point stability for set valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., **110**(1985), 436-441.

[25] J.T. Markin, *A fixed point stability theorem for nonexpansive set valued mappings*, J. Math. Anal. Appl., **54**(1976), 441-443.

[26] S.B. Nadler Jr., *Multivalued contraction mappings*, Pacific J. Math., **30**(1969), 475-488.

[27] H.K. Nashine, B. Samet, C. Vetro, *Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces*, Math. Comput. Modelling, **54**(2011), 712-720.

[28] J.J. Nieto, R. Rodrguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), 223-239.

[29] A. Petruşel, G. Petruşel, *Coupled Fixed Points and Coupled Coincidence Points Via Fixed Point Theory*, Mathematical Analysis and Applications, Book Editors: M. Ruzhansky, H. Dutta, R.P. Agarwal, Wiley, 2018, pp. 661-717.

[30] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Scalar and vectorial approaches for multi-valued fixed point and multi-valued coupled fixed point problems in b-metric spaces*, J. Nonlinear Convex Anal., **17**(2016), no. 10, 2049-2061.

[31] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Coupled fixed point theorems for symmetric multi-valued contractions in b-metric space with applications to systems of integral inclusions*, J. Nonlinear Convex Anal., **17**(2016), no. 7, 1265-1282.

[32] A. Petruşel, G. Petruşel, J.-C. Yao, *A study of a system of operator inclusions via a fixed point approach and applications to functional-differential inclusions*, Carpathian J. Math., **32**(2016), no. 3, 349-361.

[33] Z. Qiu, S. Hong, *Coupled fixed points for multivalued mappings in fuzzy metric spaces*, Fixed Point Theory Appl., **2013**(2013), 162.

[34] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435-1443.

[35] I.A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., **52**(2003), 1947-1959.

[36] B. Samet, *Coupled fixed point theorems for generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., **72**(2010), 4508-4517.

[37] B. Samet, C. Vetro, *Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces*, Nonlinear Anal., **74**(2011), no. 12, 4260-4268.

[38] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha - \psi$-contractive type mappings*, Nonlinear Anal., **75**(2012), 2154-2165.

[39] S. Sedghi, I. Altun, N. Shobe, *A fixed point theorem for multi-maps satisfying an implicit relation on metric spaces*, Appl. Anal. Discrete Math., **2**(2008), 189-196.
[40] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl., **117**(1986), 100-127.
[41] X. Zhang, *Fixed point theorems of multivalued monotone mappings in ordered metric spaces*, Appl. Math. Lett., **23**(2010), no. 3, 235-240.

Received: March 14, 2019; Accepted: January 7, 2020.