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MANN ITERATIVE ALGORITHM IN CONVEX METRIC SPACES ENDOWED WITH A DIRECTED GRAPH

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Abstract. The aim of this paper is to introduce Mann iterative algorithm by using the convex structure in the metric space endowed with a directed graph. First of all, the concept of the convex metric space endowed with a directed graph is given. Moreover, Mann iteration scheme and the corresponding convergence theorems for the $G-$monotone contractive mappings and the $G-$monotone nonexpansive mappings in convex metric spaces endowed with a directed graph are established respectively. In addition, an example is shown to illustrate that the Mann iterative sequence does not necessarily converge to the fixed point of the $G-$monotone nonexpansive mapping.
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1. Introduction

In the early 20th century, the Polish mathematician Banach [2] gave the famous Banach contraction principle and proved the theorem by Picard iteration. Due to its importance, numerous kinds of extensions and generalizations of this theorem have appeared over the years. In 2004, Ran and Reurings [7] extended the Banach contraction principle in the context of partially ordered set and gave a meaningful application to linear and nonlinear matrix equations. Moreover, Nieto and Rodríguez-López [6] extended the main fixed point theorem in [7] and used it to solve some problems of differential equations. In [4], Jachymski introduced the graphs into the general metric spaces and used them to replace the previous partially ordered structures. Furthermore, Jachymski extended the Banach contraction principle to the metric space endowed with a directed graph. Since then, many of new fixed point theorems are presented in the metric space endowed with a directed graph. For example, Alfuraidan and Khamsi [1] discussed the existence of fixed points for set-valued G-monotone quasi-contractive mappings and Reich contractive mappings in the metric space endowed with a directed graph.

In 1970, Takahashi [9] introduced the concepts of the convex structure and the convex metric space, he also obtained some fixed point theorems for nonexpansive mappings in the convex metric space. Moreover, Goebel and Kirk [3] studied some iterative processes for nonexpansive mappings in the hyperbolic metric space. In this work, we introduce Mann iterative algorithm by using the convex structure in the metric space endowed with a directed graph. First of all, the concept of the convex metric space endowed with a directed graph is given. Furthermore, Mann iteration and the convergence theorems for the $G-$monotone contractive mappings and the $G-$monotone nonexpansive mappings in convex metric spaces endowed with a directed graph are presented respectively. An example is shown to state that the Mann iterative sequence does not necessarily converge to the fixed point of the $G-$monotone nonexpansive mapping.

2. Basic concept and notations

As usual, in this paper the symbols $\mathbb{R}$ and $\mathbb{N}$ will denote the sets of all real numbers and all positive integers, respectively.

Definition 2.1. (Takahashi [9]) Let (X, d) be a metric space. If a mapping W from $X \times X \times [0,1]$ to X satisfies

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in [0,1]$, then the space (X, d, W) is said to be a convex metric space.

Obviously, a Banach space is a convex metric space. However, a Fréchet space is not necessary a convex metric space.

In 1983, Goebel and Kirk [3] obtained the following important inequality:

Theorem 2.2. (Goebel and Kirk [3]) Suppose (X, d) is a metric space of hyperbolic type, let $\{\alpha_n\} \subset [0,1)$, and suppose $\{x_n\}$ and $\{y_n\}$ are sequences in X, which satisfy for all $n \in \mathbb{N}$,

(1) $x_{n+1} \in \mathrm{seg}[x_n, y_n]$ with $d(x_n, x_{n+1}) = \alpha_n d(x_n, y_n)$,
(2) $d(y_{n+1}, y_n) \leq k d(x_{n+1}, x_n)$ among $0 < k < 1$.

Then for all $i, n \in \mathbb{N}$,

$$k\left[1+\sum_{s=i}^{i+n-1} \alpha_s\right] d(x_i, y_i) \leq d(x_i, y_{i+n}) + \prod_{s=i}^{i+n-1} (1-\alpha_s)^{-1}\left[k d(x_i, y_i) - d(x_{i+n}, y_{i+n})\right].$$

Next, we show some basic definitions and notations of graphs.

The graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the binary relation on $V(G)$. Elements of $E(G)$ are called edges. If each edge of a graph is given a direction, then the graph G is called a directed graph; otherwise, it is called an undirected graph.

By G^{-1} we denote the conversion of a directed graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Given a directed graph G, one may generate a graph $\widetilde{G}$ where we ignore the directions and replace the resulting multiple edges by single edges. We define

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}),$$

then $\widetilde{G}$ is a symmetric directed graph. The directed graph G is called reflexive, if the set $E(G)$ contains all loops, i.e., $(x, x) \in G$ for each $x \in V(G)$. Moreover, a directed graph G is called transitive whenever

$$(x, y) \in E(G) \text{ and } (y, z) \in E(G) \textit{ imply that } (x, z) \in E(G)$$

for all $x, y, z \in V(G)$.

Throughout this paper, we always assume that the directed graph G with edge weights by assigning the distance between two vertices to each edge is symmetric, reflexive and transitive.

Definition 2.3. (Monther, Mostafa and Khamsi [5]) Let (G, d) be a weighted directed graph. A sequence $\{x_n\} \in V(G)$ is said to be

(i) $G-$increasing if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$,
(ii) $G-$decreasing if $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$,
(iii) $G-$monotone if $\{x_n\}$ is either $G-$increasing or $G-$decreasing.

Definition 2.4. (Monther, Mostafa and Khamsi [5]) Let (G, d) be a weighted directed graph. G is said to be $G-$complete if any Cauchy $G-$monotone sequence $\{x_n\}$ is convergent to a point in $V(G)$.

Definition 2.5. (Alfuraidan and Khamsi [1]) Let (G, d) be a weighted directed graph. $T\colon V(G) \to V(G)$ is said to be a $G-$monotone nonexpansive mapping if

(i) T is $G-$monotone, i.e., $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$,
(ii) T is nonexpansive, i.e., $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in V(G)$ with $(x, y) \in E(G)$.

Definition 2.6. (Jachymski [4]) Let (G, d) be a weighted directed graph. $T\colon V(G) \to V(G)$ is said to be a $G-$monotone contraction mapping if

(i) T is $G-$monotone, i.e., $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$,
(ii) T is contractive, i.e., $d(Tx, Ty) \leq kd(x, y)$, $0 < k < 1$, for any $x, y \in V(G)$ with $(x, y) \in E(G)$.

The point $x \in V(G)$ is called a fixed point of T if $Tx = x$.

3. Main results

In the general linear space X, Mann iterative process is usually written as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n,\ \alpha_n \in (0,1),\ n \in \mathbb{N}.$$

However, there is no addition operation in the above sense for a graph. Next, we will extend Mann iterative algorithm to the metric space endowed with a directed graph by using the convex structure.

Definition 3.1. Let (G,d) be a metric space endowed with a directed graph. Assume that a mapping W from $V(G)\times V(G)\times[0,1]$ to $V(G)$ satisfies

$$d(u, W(x,y;\alpha)) \le \alpha d(u,x) + (1-\alpha)d(u,y)$$

for all $x,y,u \in V(G)$ and $\alpha \in [0,1]$. Then the space (G,d,W) is said to be a convex metric space endowed with a directed graph.

Proposition 3.2. For any $x,y \in V(G)$ and $\alpha \in [0,1]$, we have

$$d(x,y) = d(x, W(x,y;\alpha)) + d(W(x,y;\alpha), y).$$

Proof. Since (G,d,W) is a convex metric space endowed with a directed graph, we obtain

$$\begin{aligned} d(x,y) &\le d(x, W(x,y;\alpha)) + d(W(x,y;\alpha), y) \\ &\le \alpha d(x,x) + (1-\alpha)d(x,y) + \alpha d(x,y) + (1-\alpha)d(y,y) \\ &\le d(x,y). \end{aligned}$$

From the above inequalities, we can get the following equalities

$$\begin{gathered} d(x,y) = d(x, W(x,y;\alpha)) + d(W(x,y;\alpha), y), \\ d(x, W(x,y;\alpha)) = (1-\alpha)d(x,y), \\ d(W(x,y;\alpha), y) = \alpha d(x,y) \end{gathered}$$

for any $x,y \in V(G)$ and $\alpha \in [0,1]$. □

Remark 3.3. Let $(x,y) \in E(G)$. If there exist $u \in V(G)$ and $\alpha \in [0,1]$ such that

$$d(x,u) = (1-\alpha)d(x,y) \ \textit{and}\ d(u,y) = \alpha d(x,y),$$

then we denote $u \in [x,y]$. Obviously, $W(x,y;\alpha) \in [x,y]$.

Definition 3.4. A convex metric space endowed with a directed graph (G,d,W) is said to satisfy property (C) if for any $(x,y) \in E(G)$ and $u \in [x,y]$, we have $(x,u) \in E(G)$ and $(u,y) \in E(G)$.

Example 3.5. Let

$$E(G) = \{(x,y) : x \le y,\ x,\ y \in \mathbb{R}\}, V(G) = \mathbb{R},$$

and $d(x,y) = |x-y|$. We choose $u \in [x,y]$, it is not difficult to see that

$$[x,y] = \{z : z = \alpha x + (1-\alpha)y,\ \alpha \in [0,1]\}.$$

Obviously, $(x,u) \in E(G)$ and $(u,y) \in E(G)$.

Now we construct the Mann iterative sequence in a convex metric space endowed with a directed graph.

Lemma 3.6. Let (G, d, W) be a convex metric space endowed with a directed graph satisfying Property (C). Let a mapping $T : V(G) \to V(G)$ be $G-$monotone. Choose $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. Then the Mann iterative sequence $\{x_n\}$ is defined by

$$x_{n+1} = W(x_n, Tx_n; \alpha_n),\ n \in \mathbb{N}, \alpha_n \in (0, 1). \tag{3.1}$$

Then $(x_i, x_j) \in E(G)$, $(x_l, Tx_k) \in E(G)$ for any $i, j, l, k \in \mathbb{N}$.

Proof. From Proposition 3.2, it follows that

$$d(x_1, x_2) = d(x_1, W(x_1, Tx_1; \alpha_1)) = (1 - \alpha_1)d(x_1, Tx_1),$$

$$d(x_2, Tx_1) = d(W(x_1, Tx_1; \alpha_1), Tx_1) = \alpha_1 d(x_1, Tx_1)$$

which imply $x_2 \in [x_1, Tx_1]$. By property (C), we obtain

$$(x_1, x_2) \in E(G) \text{ and } (x_2, Tx_1) \in E(G).$$

We can conclude that $(Tx_1, Tx_2) \in E(G)$ since T is $G-$monotone. Noticing that G is transitive, we also get $(x_2, Tx_2) \in E(G)$. By induction, for any $p \in \mathbb{N}$, we deduce that $x_{p+1} \in [x_p, Tx_p]$, and

$$(x_p, x_{p+1}) \in E(G),\ (x_{p+1}, Tx_p) \in E(G),$$

$$(Tx_p, Tx_{p+1}) \in E(G),\ (x_{p+1}, Tx_{p+1}) \in E(G).$$

Therefore, we complete the proof by the transitivity of G. $\square$

Theorem 3.7. Let (G, d, W) be a G-complete convex metric space endowed with a directed graph satisfying Property (C). Suppose $T : V(G) \to V(G)$ is a $G-$monotone nonexpansive mapping. We choose $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. If the sequence $\{x_n\}$ is defined by (3.1) for $\alpha_n \in (0, 1)$ such that $\sum_{n=1}^{\infty}(1 - \alpha_n) < +\infty$, then $\{x_n\}$ converges to some point $x \in V(G)$.

Proof. By Lemma 3.6, we deduce that $(x_n, x_{n+1}) \in E(G)$ which implies the sequence $\{x_n\}$ is $G-$monotone. From Proposition 3.2, it follows that

$$d(x_n, x_{n+1}) = d(x_n, W(x_n, Tx_n; \alpha_n)) = (1 - \alpha_n)d(x_n, Tx_n).$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) \\ &\le d(W(x_{n-1}, Tx_{n-1}; \alpha_{n-1}), Tx_{n-1}) + d(x_{n-1}, x_n) \\ &= \alpha_{n-1} d(x_{n-1}, Tx_{n-1}) + (1 - \alpha_{n-1})d(x_{n-1}, Tx_{n-1}) \\ &= d(x_{n-1}, Tx_{n-1}) \le ... \le d(x_1, Tx_1), \end{aligned}$$

which shows that the sequence $\{d(x_n, Tx_n)\}$ is nonincreasing.

For all $n, p \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_n, x_{n+p}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &\le [(1-\alpha_n) + (1-\alpha_{n+1}) + ... + (1-\alpha_{n+p-1})]\, d(x_1, Tx_1) \\ &= \sum_{s=n}^{\infty} (1-\alpha_s) d(x_1, Tx_1). \end{aligned}$$

Since $\sum_{1}^{\infty}(1-\alpha_n) < +\infty$, we have $\sum_{s=n}^{\infty}(1-\alpha_s) \to 0$ as $n \to \infty$. Hence, we conclude that $\{d(x_n, x_{n+p})\}$ converges uniformly to 0 with respect to p when $n \to \infty$, which implies that $\{x_n\}$ is a Cauchy $G-$monotone sequence. Since (G, d, W) is $G-$complete, then $\{x_n\}$ is convergent to some point $x \in V(G)$. □

Remark 3.8. Theorem 3.7 claims that the Mann iteration $\{x_n\}$ is convergent. Unfortunately, we have not verified that $\{x_n\}$ converges to a fixed point of the $G-$monotone nonexpansive mapping T. Actually, we can see from the following example that $\{x_n\}$ does not necessarily converge to the fixed point of T under the conditions of Theorem 3.7.

Before giving an example, we recall a useful lemma.

Lemma 3.9. (Stein and Shakarchi [8]) For a sequence $\{a_n\} \subseteq \mathbb{R}$, if there exists $n_0 \in \mathbb{N}$ such that $a_n > 0$ or $a_n \in [-1, 0]$ for all $n \ge n_0$, then $\prod_{n=1}^{\infty}(1 + a_n)$ and $\sum_{n=1}^{\infty} a_n$ are simultaneously convergent or divergent.

Example 3.10. Let $V(G) = [0, 1]$. For any $x \in V(G)$, set $Tx = \frac{x}{2}$. Define

$$E(G) = \{(x, y) : x, y \in V(G)\}.$$

We pick the initial value $x_1 = 1$, then $Tx_1 = \frac{1}{2}$. Obviously, $(x_1, Tx_1) \in E(G)$. Assume that the sequence $\{x_n\}$ is defined by (1) and $\alpha_n = 1 - \frac{1}{2^n}$. We also define $d(x, z) = |x - z|$ which is the Euclidean metric. Then $\{x_n\}$ converges to some point $x \in V(G)$ which is not a fixed point of T.

Proof. First of all, we will show that the sequence $\{x_n\}$ converges to some point $x \in V(G)$ which coincides with the conclusion of Theorem 3.7. Since $d(x_1, Tx_1) = \dfrac{1}{2}$, we obtain

$$\begin{aligned} d(x_1, x_2) &= d(x_1, W(x_1, Tx_1; \alpha_1)) \\ &= (1-\alpha_1) d(x_1, Tx_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} d(Tx_1, x_2) &= d(Tx_1, W(x_1, Tx_1; \alpha_1)) \\ &= \alpha_1 d(x_1, Tx_1) = \left(1 - \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

Therefore,

$$x_2 = x_1 - d(x_1, x_2) = 1 - \frac{1}{2^2} = \frac{3}{4},$$
$$Tx_2 = \frac{x_2}{2} = \frac{1}{2}\left(1 - \frac{1}{2^2}\right) = \frac{3}{8},$$
$$d(x_2, Tx_2) = \frac{1}{2}\left(1 - \frac{1}{2^2}\right) = \frac{3}{8}.$$

By calculating, we get

$$d(x_2, x_3) = (1-\alpha_2)d(x_2, Tx_2) = \frac{1}{2^2} \cdot \frac{1}{2}\left(1 - \frac{1}{2^2}\right) = \frac{1}{2^3}\left(1 - \frac{1}{2^2}\right),$$

$$d(Tx_2, x_3) = \alpha_2 d(x_2, Tx_2) = \left(1 - \frac{1}{2^2}\right)\frac{1}{2}\left(1 - \frac{1}{2^2}\right).$$

Hence,

$$x_3 = x_2 - d(x_2, x_3) = \left(1 - \frac{1}{2^2}\right) - \frac{1}{2^3}\left(1 - \frac{1}{2^2}\right) = \left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{2^2}\right),$$

$$Tx_3 = \frac{1}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^3}\right),$$

$$d(x_3, Tx_3) = \frac{1}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^3}\right).$$

By induction, we conclude that

$$\begin{aligned} d(x_{n-1}, x_n) &= (1-\alpha_{n-1})d(x_{n-1}, Tx_{n-1}) \\ &= (1-\alpha_{n-1})\frac{x_{n-1}}{2} = \frac{1}{2^n}\prod_{i=1}^{n-2}\left(1 - \frac{1}{2^{i+1}}\right), \end{aligned}$$

$$d(Tx_{n-1}, x_n) = \alpha_{n-1}\frac{x_{n-1}}{2} = \frac{1}{2}\left(1 - \frac{1}{2^n}\right)\prod_{i=1}^{n-2}\left(1 - \frac{1}{2^{i+1}}\right).$$

Thus,

$$x_n = x_{n-1} - d(x_{n-1}, x_n) = \prod_{i=1}^{n-1}\left(1 - \frac{1}{2^{i+1}}\right) = \prod_{i=2}^{n}\left(1 - \frac{1}{2^i}\right), \tag{3.2}$$

$$Tx_n = \frac{x_n}{2} = \frac{1}{2}\prod_{i=2}^{n}\left(1 - \frac{1}{2^i}\right). \tag{3.3}$$

For large enough n, we have

$$\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) < \prod_{i=2}^{n}\left(1 - \frac{1}{2^i}\right) < 1.$$

Noticing that $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, it follows from Lemma 3.9 that $\lim_{n\to\infty} \prod_{i=2}^{n} \left(1 - \frac{1}{2^i}\right)$ exists. Assume that

$$\lim_{n\to\infty} \prod_{i=2}^{n} \left(1 - \frac{1}{2^i}\right) = x.$$

Combining with (2) and (3), we get $x_n \to x$ and $Tx_n \to \frac{x}{2}$ when $n \to \infty$. Since

$$\lim_{n\to\infty} \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{1}{2}$$

which implies that $x \neq 0$, then x is not a fixed point of T. □

Now, we will consider the convergence of Mann iteration for $G-$monotone contraction mappings.

Theorem 3.11. Let (G, d, W) be a convex metric space endowed with a directed graph satisfying Property (C). Suppose T: $V(G) \to V(G)$ is a $G-$monotone contraction mapping. Choose $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. If the sequence $\{x_n\}$ is defined by (1) for $\alpha_n \in (0, 1)$, then for all $i, n \in \mathbb{N}$,

$$k\left[1 + \sum_{s=i}^{i+n-1} (1 - \alpha_s)\right] d(x_i, Tx_i) \leq d(x_i, Tx_{i+n}) + \prod_{s=i}^{i+n-1} \alpha_s^{-1} \left[kd(x_i, Tx_i) - d(x_{i+n}, Tx_{i+n})\right].$$

Proof. By Proposition 3.2, Definition 2.6 and Definition 3.1, it is not difficult to see that the sequences $\{x_n\}$ and $\{Tx_n\}$ satisfy

(1) $x_{n+1} = W(x_n, Tx_n; \alpha_n)$ with $d(x_n, x_{n+1}) = (1 - \alpha_n) d(x_n, Tx_n)$ and $d(Tx_{n+1}, x_{n+1}) \leq \alpha_n d(Tx_{n+1}, x_n) + (1 - \alpha_n) d(Tx_{n+1}, Tx_n)$,

(2) $d(Tx_{n+1}, Tx_n) \leq kd(x_{n+1}, x_n)$ among $0 < k < 1$,

for all $n \in \mathbb{N}$.

The remaining proof is similar to Theorem 2.2 (see [3]), so we omit it here. □

Theorem 3.12. Let (G, d, W) be a G-complete convex metric space endowed with a directed graph satisfying Property (C). Suppose T: $V(G) \to V(G)$ is a $G-$monotone contraction mapping. Choose $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. If the sequence $\{x_n\}$ is defined by (1) for $\alpha_n \in [a, b] (0 < a < b < 1)$, then $\{x_n\}$ converges to some point $x \in V(G)$ which is a fixed point of T.

Proof. Similar to the procedure of Theorem 3.7, we conclude that the sequence $\{d(x_n, Tx_n)\}$ is nonincreasing. Then the limit $\lim_{n\to\infty} d(x_n, Tx_n)$ exists. For any $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, Tx_n) + d(Tx_n, Tx_m) + d(Tx_m, x_m) \\ &\leq d(x_n, Tx_n) + kd(x_n, x_m) + d(Tx_m, x_m), \end{aligned}$$

which implies

$$(1 - k) d(x_n, x_m) \leq d(x_n, Tx_n) + d(Tx_m, x_m). \tag{3.4}$$

From Theorem 3.11, it follows that

$$k\left[1+\sum_{s=i}^{i+p-1}(1-\alpha_s)\right]d(x_i,Tx_i) \le d(x_i,Tx_{i+p}) + \prod_{s=i}^{i+p-1}\alpha_s^{-1}\left[kd(x_i,Tx_i) - d(x_{i+p},Tx_{i+p})\right], \tag{3.5}$$

for all $i,p \in \mathbb{N}$. Noticing that

$$\lim_{i\to\infty}[kd(x_i,Tx_i) - d(x_{i+p},Tx_{i+p})] \le 0,$$

we will consider the following two cases.
(i) If $\lim\limits_{i\to\infty}[kd(x_i,Tx_i) - d(x_{i+p},Tx_{i+p})] = 0$, it follows from $k<1$ that

$$\lim_{i\to\infty} d(x_i,Tx_i) = 0.$$

(ii) If $\lim\limits_{i\to\infty}[kd(x_i,Tx_i) - d(x_{i+p},Tx_{i+p})] < 0$, noting that

$$\begin{aligned} d(x_i,Tx_{i+p}) &\le d(x_i,Tx_i) + d(Tx_i,Tx_{i+p}) \\ &\le d(x_i,Tx_i) + kd(x_i,x_{i+p}), \end{aligned}$$

$$b^{-p} \le \prod_{s=i}^{i+p-1}\alpha_s^{-1} \le a^{-p}$$

and combining with (3.5), for large enough i, we get

$$\left\{k\left[1+\sum_{s=i}^{i+p-1}(1-\alpha_s)\right]-1\right\} d(x_i,Tx_i) \le kd(x_i,x_{i+p}),\ \ p\in\mathbb{N}. \tag{3.6}$$

Since

$$\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty,$$

we have

$$k\left[1+\sum_{s=i}^{i+p-1}(1-\alpha_s)\right]-1 > 0$$

for large enough p. From (3.4) and (3.6), it follows that

$$\begin{aligned} d(x_i,Tx_i) &\le \frac{k}{k\left[1+\sum\limits_{s=i}^{i+p-1}(1-\alpha_s)\right]-1} d(x_i,x_{i+p}) \\ &\le \frac{k}{k\left[1+\sum\limits_{s=i}^{i+p-1}(1-\alpha_s)\right]-1}\,\frac{1}{(1-k)}\left[d(x_i,Tx_i) + d(Tx_{i+p},x_{i+p})\right]. \end{aligned}$$

By letting $i,p\to\infty$, we infer that

$$\lim_{i\to\infty} d(x_i,Tx_i) = 0.$$

Combining with (3.4), we obtain $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence, $\{x_n\}$ is a Cauchy $G-$monotone sequence. Since (G, d, W) is $G-$complete, then $\{x_n\}$ converges to some point $x \in V(G)$. Since

$$\begin{aligned}\overline{\lim}_{n\to\infty} d(x_n, Tx) &\le \overline{\lim}_{n\to\infty} d(x_n, Tx_n) + \overline{\lim}_{n\to\infty} d(Tx_n, Tx)\\ &\le \overline{\lim}_{n\to\infty} d(x_n, Tx_n) + k\overline{\lim}_{n\to\infty} d(x_n, x) = 0,\end{aligned}$$

we deduce that

$$\lim_{n\to\infty} d(x_n, Tx) = 0.$$

By the uniqueness of the limit, we obtain $x = Tx$ which shows that x is a fixed point of T. $\square$

Example 3.13. Let $V(G) = [0, 1]$ and define

$$E(G) = \{(x, y) : x, y \in V(G)\}.$$

We also define $d(x, z) = |x - z|$ for all $x, z \in V(G)$. For any $x \in V(G)$, set $Tx = \frac{x}{2}$. We pick the initial value $x_1 = 1$, then $(x_1, Tx_1) \in E(G)$. Assume that the sequence $\{x_n\}$ is defined by (3.1) and $\alpha_n = \varepsilon$ for each $n \in \mathbb{N}$, where $\varepsilon \in (0, 1)$. Then $\{x_n\}$ is convergent to 0 which is a fixed point of T.

Proof. Similar to the procedure of Example 3.10, we get

$$d(x_{n-1}, x_n) = (1 - \alpha_{n-1}) d(x_{n-1}, Tx_{n-1}) = (1 - \alpha_{n-1}) \frac{x_{n-1}}{2}$$

and

$$\begin{aligned}x_n &= x_{n-1} - d(x_{n-1}, x_n) = x_{n-1} - (1 - \alpha_{n-1}) \frac{x_{n-1}}{2} = \frac{x_{n-1}}{2}(1 + \alpha_{n-1})\\ &= x_1 \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} (1 + \alpha_i) = (\frac{1+\varepsilon}{2})^{n-1}.\end{aligned}$$

Then

$$Tx_n = \frac{x_n}{2} = \frac{1}{2}(\frac{1+\varepsilon}{2})^{n-1}. \tag{3.7}$$

Therefore, letting $n \to \infty$ in (3.7), we obtain

$$x_n \to 0 \textit{ and } Tx_n \to 0.$$

Hence, 0 is a fixed point of T. $\square$

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