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FIXED POINT RESULTS IN LOCALLY CONVEX SPACES WITH τ-KREIN-ŠMULIAN PROPERTY AND APPLICATIONS

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Abstract. In this paper, we present some new fixed point theorems in a locally convex space X with the so called τ-Krein-Šmulian property considering the concept of Φ^{τ}_{Λ}-measures of noncompactness, where τ is a weaker Hausdorff locally convex topology of X. Further, we apply our results to discuss the existence of solutions for a nonlinear functional integral equation in the Lebesgue space L^1.
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1. Introduction

The study of measures of noncompactness is one of the main tools of research in nonlinear functional analysis, in particular when we deal with existence of fixed points. In the previous decade, several papers gave some fixed point theorems in view of the measure of noncompactness. In [9], Darbo developed a fixed point theorem using the concept of k-set contraction, combining the Kuratowskii measures of noncompactness with the Schauder fixed point theorem. In [18], Sadovskii gave a generalization of Darbo theorem introducing the notion of condensing operators. In some early related works, the authors established several various forms of Schauder and Krasnoselskii fixed point result with respect to the De Blasi measure of weak noncompactness [4, 12, 14]. Recently, Banaś and Ben Amar [5], proved a new generalized fixed point theorem for the sum of two operators under some conditions for the operators as (τ-compactness, τ-sequential continuity, demi-τ-compactness), on convex subsets of Hausdorff topological vector spaces using τ-measures of noncompactenss. In a more recent paper [19], Wang and Zhou established an extension of

Krasnoselskii fixed point result in locally convex spaces with Krein-Šmulian property via the family of weak noncompactness introducing the concepts of ws-compactness and ww-compactness. Their results were applied to study the existence of nonlinear Volterra integral equation.

Following the ideas of the last mentioned paper, the object of this work is to provide some new variants of fixed point theorems in a Hausdorff locally convex topological vector space endowed with a family of seminorms $\{|\cdot|_p\}_{p\in\Lambda}$ and having the τ-Krein-Šmulian property (see Definition 2.6). Our results are formulated in terms of a family of axiomatic Φ_Λ^τ-measures of noncompactness (see Definition 2.1). As an application, we will study the solvability for the following nonlinear integral equation

$$x(t) = a(t) + \int_0^1 h(t,s)f(s,x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds, t \in [0,1], \tag{1.1}$$

in the Lebesgue space L^1, where a, h, f, u and φ are given functions satisfying certain conditions.

The outline of this paper is as follows. In section 2, we give some preliminaries, which are useful in the sequel. Section 3 is devoted to establish new results in fixed point theory for τ-sequentially continuous operators defined on a convex subset of a Hausdorff locally convex topological vector space. On basis of Theorem 3.2, we discuss several theorems for the sum $A+B$, where A is τ-sequentially continuous and B is a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction. In section 4, we discuss the applicability of theorems, proved in section 3, to study the existence of solutions of Eq. (1.1).

2. Preliminaries

At first, let us recall some notations and basic concepts. Let $(X, \{|\cdot|_p\}_{p\in\Lambda})$ be a Hausdorff locally convex vector space endowed with a family of seminorms $\{|\cdot|_p\}_{p\in\Lambda}$ generating its topology and let τ be a weaker Hausdorff locally convex topology on X. We denote by $\xrightarrow{\tau}$ the convergence in (X,τ) and by $\rightarrow$ the convergence in $(X, \{|\cdot|_p\}_{p\in\Lambda})$. We mean by τ-compact set, compact, set with respect to the topology τ. We also denote by $\mathcal{B}(X)$ the family of all nonempty bounded subsets of X (with respect to the topology generated by $\{|\cdot|_p\}_{p\in\Lambda}$).

Now, let us consider the following axiomatic definition of a family of measures of noncompactenss in a Hausdorff locally convex vector space.

Definition 2.1. A family of functions $\phi_{p_\tau} : \mathcal{B}(X) \rightarrow \mathbb{R}^+$, $(p \in \Lambda)$ is said to be a Φ_Λ^τ-measures of noncompactenss in X (Φ_Λ^τ-MNC, in short) if for each $p \in \Lambda$, it satisfies the following conditions:

(i) $\phi_{p_\tau}(\overline{conv}(M)) \leq \phi_{p_\tau}(M)$ for each $M \in \mathcal{B}(X)$, where $\overline{conv}(M)$ is the closure of the convex hull of M in (X,τ),

(ii) $M_1 \subseteq M_2 \Rightarrow \phi_{p_\tau}(M_1) \leq \phi_{p_\tau}(M_2)$, where $M_1, M_2 \in \mathcal{B}(X)$,

(iii) $\phi_{p_\tau}(\{x\} \cup M) = \phi_{p_\tau}(M)$ for any $x \in X$ and $M \in \mathcal{B}(X)$,

(iv) $\phi_{p_\tau}(M) = 0$ implies M is relatively τ-compact in X, and

(v) if $(M_n)_n$ is a sequence of closed sets of $\mathcal{B}(X)$ such that $M_{n+1} \subset M_n$, $n = 1, 2, ...$ and $\lim\limits_{n\to\infty} \phi_{p_\tau}(M_n) = 0$ for each $p \in \Lambda$, then $M_\infty = \bigcap_{n=1}^{\infty} M_n$ is nonempty relatively τ-compact subset of X.

The family Φ_Λ^τ-MNC is called:

(vi) Positively homogeneous, if for each $p \in \Lambda$, $\phi_{p_\tau}(\lambda M) = \lambda\phi_{p_\tau}(M)$, $\lambda > 0$, where $M \in \mathcal{B}(X)$.

(vii) Subadditive, if for each $p \in \Lambda$, $\phi_{p_\tau}(M_1 + M_2) \leq \phi_{p_\tau}(M_1) + \phi_{p_\tau}(M_2)$, where $M_1, M_2 \in \mathcal{B}(X)$.

Example 2.1. [19] The family of measures of weak noncompactenss in a locally convex space X, which is defined by:

$$\omega_p(M) = \inf\{r > 0 : \exists W \in \mathcal{W}(X) \,\text{such that}\, M \subseteq W + B_p(0, r)\},\ p \in \Lambda$$

is positively homogeneous and subadditive $\Phi_\Lambda^{\sigma(X,X^*)}$-MNC. Here, $B_p(0, r)$ is the closed ball centered at 0 with radius r, $\sigma(X, X^*)$ is the weak topology of X and $\mathcal{W}(X)$ is the set of all nonempty relatively weakly compact subsets of X. This formula is based on the notion of single measure of weak noncompactenss introduced by De Blasi [10].

Definition 2.2. Let M be a nonempty subset of X and let $\Phi_\Lambda^\tau := \{\phi_{p_\tau}, p \in \Lambda\}$ be a family of Φ_Λ^τ-MNC in X. An operator $A : M \to X$ is said Φ_Λ^τ-contraction if for any bounded subset S of M, $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$, there exists a constant $\beta_p \in [0, 1)$ such that $\phi_{p_\tau}(A(S)) \leq \beta_p\phi_{p_\tau}(S)$. The operator A is called Φ_Λ^τ-condensing if for any bounded subset S of M, $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$ such that $\phi_{p_\tau}(S) > 0$, $\phi_{p_\tau}(A(S)) < \phi_{p_\tau}(S)$.

Definition 2.3. [11] A topological (Hausdorff) space X is called angelic (or has countably determined compactness) if for every relatively countably compact subset M of X, the following holds:

(i) M is relatively compact.

(ii) For each $x \in \overline{M}$, there is a sequence in M which converges to x.

Note that all metrizable locally convex spaces endowed with their weak topology are angelic. (See Eberlein-Šmulian theorem [16].)

Remark 2.1. In angelic spaces the classes of compact, countably compact, and sequentially compact sets coincide (see [11, p. 31]).

Now, we define a class of operators needed in our considerations.

Definition 2.4. Let M be a nonempty subset of X. An operator $A : M \to X$ is said to be sequentially τ-closed on M if for each sequence $(x_n)_n \in M$ such that $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$, then $x \in M$ and $y = Ax$.

Definition 2.5. [5] Let M be a nonempty subset of X, $A : M \to X$ be an operator. We say that A is τ-sequentially continuous on M if for each sequence $(x_n)_n \subset M$ with $x_n \xrightarrow{\tau} x$ and $x \in M$, we have that $Ax_n \xrightarrow{\tau} Ax$.

Remark 2.2. (i) Clearly, every τ-sequentially continuous operator is sequentially τ-closed, but the converse is not true.

(ii) If X is angelic, then any τ-sequentially continuous map on a τ-compact set is τ-continuous.

Definition 2.6. We say that X has the τ-Krein-Šmulian property (τ-KS, in short) if the closed convex hull of a τ-compact set is τ-compact.

As an example, if X is a Banach space, then X has $\sigma(X, X^*)$-Krein-Šmulian property.

Definition 2.7. [8] Let X be a Hausdorff locally convex topological vector space. A mapping $A : X \to X$ is said to be a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction if for each $p \in \Lambda$, there exists $\alpha_p \in [0, 1)$ such that for all x_1, $x_2 \in X$

$$|Ax_1 - Ax_2|_p \leq \alpha_p |x_1 - x_2|_p.$$

Now, we provide an important result proved by Cain and Nashed in Hausdorff locally convex topological vector spaces [8].

Theorem 2.1. *Let X be a Hausdorff locally convex topological vector space. M is a sequentially complete subset of X and the mapping $A : M \to M$ is a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction. Then, A has a unique fixed point $x \in M$.*

3. Main results

Let $(X, \{|\cdot|_p\}_{p\in\Lambda})$ be a Hausdorff locally convex topological vector space, and τ is weaker Hausdorff locally convex vector topology of X.

Theorem 3.1. *Let (X, τ) be an angelic space. Let M be a nonempty, convex, and τ-compact subset of X. If $A : M \to M$ is a τ-sequentially continuous operator, then, A has a fixed point in M.*

Proof. The proof follows from Remark 2.2 and Schauder-Tychonoff fixed point theorem. □

Remark 3.1. Notice that Theorem 3.1 generalizes the Arino, Gautier and Penot fixed point theorem [3, Theorem 1].

Now, we state a generalization of Theorem 2.5 in [6].

Theorem 3.2. *Assume that (X, τ) is angelic and has the τ-KS property. Let M be a nonempty, closed convex subset of X and $A : M \to M$ be a τ-sequentially continuous operator. If $A(M)$ is relatively τ-compact, then A has a fixed point in M.*

Proof. Let $C = \overline{conv}(A(M))$. Clearly, $A(C) \subset C$. Since $A(M)$ is relatively τ-compact and X satisfy the τ-KS property, then C is relatively τ-compact. By Theorem 3.1, there exists $x \in C$ such that $Ax = x$. Which achieves the proof. □

As a consequence of the above Theorem we can formulate the following results.

Corollary 3.1. *Suppose that (X, τ) is an angelic space and has the τ-KS property. Let M be a nonempty, closed convex subset of X. Let $A : M \to M$ be a sequentially τ-closed operator such that $A(M)$ is relatively τ-compact. Then, A has a fixed point in M.*

Proof. Let $C = \overline{conv}(A(M))$. By the same arguments used in the proof of Theorem 3.2, C is relatively τ-compact and $A(C) \subset C$. Now, we prove that A is τ-sequentially continuous. For this purpose, let $(x_n)_n$ be a sequence in C, such that $x_n \xrightarrow{\tau} x$. Since $A(x_n) \subset A(C) \subset C$, there exists a subsequence (x_{n_k}) of $(x_n)_n$ such that $Ax_{n_k} \xrightarrow{\tau} y$. Taking into account that A is sequentially τ-closed, and $x_{n_k} \xrightarrow{\tau} x$, we get $y = Ax$. Next, we will show that

$$Ax_n \xrightarrow{\tau} Ax.$$

Suppose the contrary, then there exists a τ-neighborhood V^τ of Ax and a subsequence $(x_{n_i})_i$ of $(x_n)_n$ such that $Ax_{n_i} \notin V^\tau$ for all i. Moreover, $x_{n_i} \xrightarrow{\tau} x$, then arguing as before, there exists a subsequence $(x_{n_{i_k}})_k$ of $(x_{n_i})_i$ such that

$$Ax_{n_{i_k}} \xrightarrow{\tau} Ax,$$

which is absurd, since $Ax_{n_{i_k}} \notin V^\tau$ for all k. This prove that A is τ-sequentially continuous. Then, by Theorem 3.2, we deduce that A has a fixed point in M. $\square$

Now, we present a new variant of Darbo's fixed-point theorem [9] for τ-sequentially continuous maps involving a family of Φ_Λ^τ-MNC .

Corollary 3.2. *Assume that (X, τ) is angelic and has the τ-KS property. Let M be a nonempty, bounded, closed, and convex subset of X and $\Phi_\Lambda^\tau = \{\phi_{p_\tau},\, p \in \Lambda\}$ be a family of Φ_Λ^τ-MNC in X. Let $A : M \to M$ be a τ-sequentially continuous operator. If A is Φ_Λ^τ-contractive, then A has a fixed point in M.*

Proof. Let us define the sequence $(M_n)_n$ such that

$$M_1 = M \text{ and } M_{n+1} = \overline{conv}(A(M)).$$

Clearly, the sequence $(M_n)_n$ consists of nonempty closed, bounded, convex, and decreasing subsets of M. Let $p \in \Lambda$, using the property (i) of Definition 2.1 we obtain

$$\begin{aligned} \phi_{p_\tau}(M_2) &= \phi_{p_\tau}(\overline{conv}(A(M_1)) \\ &\leq \phi_{p_\tau}(A(M_1)). \end{aligned}$$

Further, since A is Φ_Λ^τ-contractive, there exists $\beta_p \in [0, 1)$ such that

$$\phi_{p_\tau}(M_2) \leq \beta_p \phi_{p_\tau}(M_1).$$

Proceeding by induction we get

$$\phi_{p_\tau}(M_n) \leq \beta_p^{n-1} \phi_{p_\tau}(M),$$

and therefore $\lim_{n\to\infty} \phi_{p_\tau}(M_n) = 0$. By using the property (v) of Φ_Λ^τ-MNC, we infer that $M_\infty = \cap_{n=1}^{\infty} M_n$ is nonempty, closed, convex, and relatively τ-compact subset of M. Moreover,

$$A(M_\infty) = A(\cap_{n=1}^{\infty} M_n) \subset \cap_{n=1}^{\infty} A(M_n) \subset M_\infty.$$

Accordingly, $A(M_\infty)$ is relatively τ-compact. Now, the use of Theorem 3.2 concludes the proof. $\square$

Note that Corollary generalizes Theorem 3.1 in [4] , and Theorem 2.2 in [17].

The next result is a generalization of Sadovskii's fixed point theorem for strong condensing map [18].

Corollary 3.3. *Assume that (X,τ) is angelic and has the τ-KS property. Let M be a nonempty, bounded,closed and convex subset of X and let $\Phi_\Lambda^\tau = \{\phi_{p_\tau},\, p \in \Lambda\}$ be a Φ_Λ^τ-MNC in X. Let $A : M \to M$ be a τ-sequentially continuous operator. If A is Φ_Λ^τ-condensing, then A has a fixed point in M.*

Proof. Let $x_0 \in M$, we define the set

$$\mathcal{S} = \{D \subset X;\ x_0 \in D \subset M,\ D \text{ is bounded, convex, and } A(D) \subset D\}.$$

Clearly, $M \in \mathcal{S}$, then $\mathcal{S}$ is nonempty. Denote by $C = \bigcap_{D\in\mathcal{S}} D$. Obviously $x_0 \in C$ and C is a bounded convex subset of X, and $A(C) \subset C$. It follows that

$$\overline{conv}\{A(C) \cup \{x_0\}\} \subset C. \tag{3.1}$$

Therefore,

$$A(\overline{conv}\{A(C) \cup \{x_0\}\}) \subset A(C) \subset \overline{conv}\{A(C) \cup \{x_0\}\}.$$

Hence,

$$\overline{conv}\{A(C) \cup \{x_0\}\} \in \mathcal{S}.$$

Consequently,

$$C \subset \overline{conv}\{A(C) \cup \{x_0\}\}. \tag{3.2}$$

When combining Eqs. (3.1) and (3.2), we get $\overline{conv}\{A(C) \cup \{x_0\}\} = C$. Using the properties of Φ_Λ^τ-MNC, we have

$$\phi_{p_\tau}(C) = \phi_{p_\tau}(\overline{conv}\{A(C) \cup \{x_0\}\}) \le \phi_{p_\tau}(A(C) \cup \{x_0\}) = \phi_{p_\tau}(A(C)).$$

By the Φ_Λ^τ-condensibility of A, we obtain $\phi_{p_\tau}(C) = 0$, then C is relatively τ-compact and consequently $A(C)$ is relatively τ-compact. So by Theorem 3.2, we deduce that there exists $x \in C$ such that $Ax = x$. □

Remark 3.2. As an easy consequence of Corollary 3.3, we may recapture Theorem 3.2 in [7], and Theorem 12 of [12].

Basing on Theorem 3.2, we prove the following fixed point theorem for the sum of a τ-sequentially continuous and a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction mapping.

Theorem 3.3. *Let X be a sequentially complete Hausdorff locally convex topological vector space. Assume that (X,τ) is angelic and has the τ-KS property. Suppose that M is a nonempty, bounded, closed, and convex subset of X. Let $\{\phi_{p_\tau},\, p \in \Lambda\}$ be a Φ_Λ^τ-MNC in X. Consider $A : M \to X$ and $B : X \to X$ two operators such that for all $p \in \Lambda$:*

(i) A is τ-sequentially continuous,

(ii) B is a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction and is sequentially τ-closed,

(iii) there exists $\lambda_p \in [0,1)$ such that $\phi_{p_\tau}(A(S) + B(S)) \le \lambda_p \phi_{p_\tau}(S)$ for all $S \subset M$, and

(iv) $(x = Bx + Ay, y \in M) \Longrightarrow x \in M$.

Then, there exists $x \in M$ such that $x = Ax + Bx$.

Proof. Let y be a fixed in M, the map which assigns to each $x \in X$ the value $Bx+Ay$ defines a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction from X into X for all $p \in \Lambda$. So, by Theorem 2.1, the equation $x = Bx + Ay$ has a unique solution $x \in X$. By assumption (iv), it follows that $x \in M$. Hence, $x = (I-B)^{-1}Ay \in M$ which, accordingly implies the inclusion

$$(I-B)^{-1}A(M) \subset M. \tag{3.3}$$

Now, define the sequence $(M_n)_n$ of subsets of M by:

$$M_1 = M \text{ and } M_{n+1} = \overline{conv}((I-B)^{-1}A(M_n)). \tag{3.4}$$

We claim that the sequence $(M_n)_n$ satisfies the conditions of property (v) of Φ_Λ^τ-MNC. Indeed, it is clear that the sequence $(M_n)_n$ consists of nonempty closed, convex and bounded subsets of M. By Eq. (3.3) one sees that it is also decreasing. Now, using Eq. (3.4) and the following equality

$$(I-B)^{-1}A = A + B(I-B)^{-1}A, \tag{3.5}$$

we obtain

$$(I-B)^{-1}A(M_n) \subseteq A(M_n) + B\,\overline{conv}((I-B)^{-1}A(M_n)) \subseteq A(M_n) + B(M_n). \tag{3.6}$$

Kipping in mind Eq. (3.6), and the properties (i) and (ii) of Definition 2.1, we obtain

$$\begin{array}{rcl} \phi_{p_\tau}(M_{n+1}) & = & \phi_{p_\tau}(\overline{conv}(I-B)^{-1}A(M_n)) \\ & \leq & \phi_{p_\tau}((I-B)^{-1}A(M_n)) \\ & \leq & \phi_{p_\tau}(AM_n + B(M_n)). \end{array}$$

Further, by assumption (iii), we deduce that

$$\phi_{p_\tau}(M_{n+1}) \leq \lambda_p \phi_{p_\tau}(M_n).$$

Proceeding by induction we get

$$\phi_{p_\tau}(M_n) \leq \lambda_p^{n-1}\phi_{p_\tau}(M),$$

and therefore $\lim\limits_{n\to\infty}\phi_{p_\tau}(M_n) = 0$. Now, applying the property (v) of Φ_Λ^τ-MNC we infer that $M_\infty = \cap_{n=1}^\infty M_n$ is nonempty, closed, convex , and relatively τ-compact subset of M. On the other hand one can easily verify that $(I-B)^{-1}A(M_n) \subset M_n$ for all n, thus we obtain $(I-B)^{-1}A(M_\infty) \subset M_\infty$. Consequently, $(I-B)^{-1}A(M_\infty)$ is relatively τ-compact. Next, let us show that $(I-B)^{-1}A : M_\infty \to M_\infty$ is τ-sequentially continuous . To this purpose, let $(x_n)_n$ be a sequence in M_∞ such that $x_n \xrightarrow{\tau} x$ in M_∞. Since

$$((I-B)^{-1}Ax_n)_n \subset (I-B)^{-1}AM_\infty,$$

there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that

$$(I-B)^{-1}Ax_{n_k} \xrightarrow{\tau} y \tag{3.7}$$

Going back to Eq. (3.5), using the τ-sequential continuity of the operator A and Eq. (3.7), it follows that

$$B(I-B)^{-1}Ax_{n_k} \xrightarrow{\tau} y - Ax. \tag{3.8}$$

Together with Eqs. (3.7), (3.8), and assumption (ii), we infer that $By = y - Ax$, hence $y = (I-B)^{-1}Ax$.

Now, we claim that

$$(I-B)^{-1}Ax_n \xrightarrow{\tau} (I-B)^{-1}Ax.$$

Suppose the contrary, then there exists a τ-neighborhood V^τ of $(I-B)^{-1}Ax$ and a subsequence $(x_{n_j})_j$ of $(x_n)_n$ such that $(I-B)^{-1}Ax_{n_j} \notin V^\tau$ for all j. Moreover, $x_{n_j} \xrightarrow{\tau} x$, then arguing as before, we can extract a subsequence $(x_{n_{j_k}})_k$ of $(x_{n_j})_j$ such that

$$(I-B)^{-1}Ax_{n_{j_k}} \xrightarrow{\tau} (I-B)^{-1}Ax,$$

which is absurd, since $(I-B)^{-1}Ax_{n_{j_k}} \notin V^\tau$ for all k.
Finally, $(I-B)^{-1}A$ is τ-sequentially continuous. Now, the use of Theorem 3.2 yields a point $x \in M$ such that $x = Ax + Bx$. □

Note that condition (iii) of Theorem 3.3, can be weakened. For this purpose let us recall the following definition.

Definition 3.1. [13] Let Q be the class of functions $\gamma : \mathbb{R}^+ \to [0,1)$ which satisfies the following condition: $\gamma(t_n) \to 1$ implies $t_n \to 0$.

Theorem 3.4. *Let X be a sequentially complete Hausdorff locally convex topological vector space. Assume that (X,τ) is angelic and has the τ-KS property. Suppose that M is a nonempty, bounded, closed, and convex subset of X. Consider $A : M \to X$ and $B : X \to X$ be two operators such that for all $p \in \Lambda$:*

(i) *A is τ-sequentially continuous,*

(ii) *B is $\{|\cdot|_p\}_{p\in\Lambda}$-contraction and is sequentially τ-closed,*

(iii) *$\phi_{p_\tau}(A(S)+B(S)) \le \gamma(\phi_{p_\tau}(S))\phi_{p_\tau}(S)$, for all $S \subset M$, $\gamma \in Q$, $p \in \Lambda$, and*

(iv) *$(x = Bx + Ay, y \in M) \Longrightarrow x \in M$.*

Then, there exists $x \in M$ such that $x = Ax + Bx$.

Proof. In view of the proof of Theorem 3.3, it is sufficient to establish that the sequence $(M_n)_n$ defined in (3.4), satisfies the conditions of property (v) of Definition 2.1. Using properties of Φ^τ_Λ-MNC and Eq. (3.6) we have

$$\begin{aligned}\phi_{p_\tau}(M_{n+1}) &= \phi_{p_\tau}(\overline{conv}((I-B)^{-1}A(M_n)))\\ &\le \phi_{p_\tau}((I-B)^{-1}A(M_n))\\ &\le \phi_{p_\tau}(A(M_n)+B(M_n)). \qquad (3.9)\end{aligned}$$

By assumptions (iii) and Eq. (3.9), we get

$$\begin{aligned}\phi_{p_\tau}(M_{n+1}) &\le \gamma(\phi_{p_\tau}(M_n))\phi_{p_\tau}(M_n) \qquad (3.10)\\ &\le \phi_{p_\tau}(M_n),\end{aligned}$$

this implies that $(\phi_{p_\tau}(M_n))_n$ is a positive decreasing sequence of real numbers. Thus, there is an $r \ge 0$ such that $\lim\limits_{n\to\infty} \phi_{p_\tau}(M_n) = r$. Now, we prove that $r = 0$. To this end, suppose that $r \ne 0$ and $\phi_{p_\tau}(M_n) \ne 0$ for $n \ge 1$. By Eq. (3.10), we obtain

$$\frac{\phi_{p_\tau}(M_{n+1})}{\phi_{p_\tau}(M_n)} \le \gamma(\phi_{p_\tau}(M_n)) < 1,$$

therefore, $\lim_{n\to\infty} \gamma(\phi_{p_\tau}(M_n)) = 1$. Since $\gamma \in Q$, we get $\lim_{n\to\infty} \phi_{p_\tau}(M_n) = 0$. Using property (v) of Φ^τ_Λ-MNC, we deduce that $M_\infty = \cap_{n=1}^\infty M_n$ is nonempty, closed, convex, and relatively τ-compact subset of M.
If there exists $k \geq 0$ such that $\phi_{p_\tau}(M_k) = 0$, this implies that M_k is relatively τ-compact. Arguing as in the proof of Theorem 3.3, we deduce that the operator $(I-B)^{-1}A : M_k \to M_k$ is τ-sequentially continuous, and $(I-B)^{-1}A(M_k)$ is relatively τ-compact. Then, Theorem 3.2 can be applied and this completes the proof. □

4. Application to nonlinear functional integral equation

In this section we are going to use our findings proved in the last section to present some existence results for a nonlinear integral equation in the Lebesgue space . At the beginning we provide some auxiliary notations and results which will be useful in our investigations. For more details the readers can refer to [1, 2, 4, 5, 15]. Let $I = [0, 1]$ be a compact interval in $\mathbb{R}$ and denote by $L^1 = L^1(I)$ the space of Lebesgue integrable real functions on the interval I with the norm

$$\|x\| = \int_0^1 |x(t)|dt.$$

Let $\xi = \xi(I)$ be the set of all real functions, Lebesgue measurable on I and denote by $m(M)$ the Lebesgue measure of a measurable subset M of $\mathbb{R}$. We define in ξ the metric ρ by the formula

$$\rho(x, y) = \inf\{a + m(\{s \in I : |x(s) - y(s)|) \geq a\}) : a > 0\},$$

so, ξ becomes a complete metric space. It is known that the convergence in measure coincides with the convergence generated by the metric ρ, but the convergence in measure of a sequence $(x_n)_n$ in L^1 does not imply the weak convergence of $(x_n)_n$ and conversely. While, we have the following results [4].

Lemma 4.1. *A sequence $(x_n)_n$ in L^1 converges strongly to $x \in L^1$ (i.e., converges in norm) if, and only if, $(x_n)_n$ converges in measure to x and is weakly compact.*

Lemma 4.2. *Let M be a bounded subset of the space L^1 consisting of all functions being a.e. nondecreasing (nonincreasing) on I. Then, M is compact in measure.*

The following Lemma states the case when a class of weakly sequentially continuous operators coincides with the class of continuous ones in L^1.

Lemma 4.3. *Let M be a bounded subset of the Lebesgue space L^1 which is compact in measure. If $A : M \to L^1$ is continuous, then it is also weakly sequentially continuous.*

Now, we give some basic results concerning the superposition operator [2].

Definition 4.1. Assume that the function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathédory conditions i.e., it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every function $x(t)$ being measurable on I, we may assign the function

$$F(x(t)) = f(t, x(t)), t \in I.$$

The operator F is such a way is called the superposition (Nemytskii) operator generated by the function f.

Theorem 4.1. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. Then, the superposition operator F generated by f transforms the space L^1 into itself if, and only if,*

$$|f(t,x)| \leq a(t) + b|x|,$$

for $t \in I$ and $x \in \mathbb{R}$, where $a(\cdot)$ is a function from the space L^1 and $b > 0$. Moreover, the operator F is continuous on the space L^1.

An important nonlinear integral equation is the Urysohn equation defined in Eq.(4.1) by

$$U(x(t)) = \int_I u(t,s,x(s))ds, \tag{4.1}$$

where $u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. A particular case of the Urysohn equation is the Hammerstein integral equation Eq.(4.2)

$$H(x(t)) = \int_I h(t,s)x(s)ds, \tag{4.2}$$

with $h : I \times I \to \mathbb{R}$. Now, we provide some conditions for continuity of integral operators U and H in the following results [15].

Theorem 4.2. *Let the functions $u(t,s,x)$ and $k(t,s,x)$ satisfy the Carathédory conditions. Let*

$$|u(t,s,x)| \leq k(t,s,x),\ t,s \in I, x \in \mathbb{R}$$

and suppose that the integral operator

$$K(x(t)) = \int_I k(t,s,x(s))ds$$

act from L^p to L^q, $q > 0$, and is continuous. Then, the integral operator U also acts from the space L^p to the space L^q and is continuous.

Theorem 4.3. *Let $h : I \times I \to \mathbb{R}$ be measurable with respect to both variables. Let the linear integral operator H with kernel $h(\cdot,\cdot)$ map L^p into L^q. Then, the integral operator H is continuous.*

Due to the works of Appell and De Pascale [1], we can express the De Blasi measure of weak noncompactness $\omega(.)$ in L^1 by the formula:

$$\omega(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |x(t)|\, dt : D \subset I, m(D) \leq \varepsilon \right] \right\} \right\}.$$

Now, we will discuss the solvability of Eq. (1.1) under the following assumptions :

($\mathcal{H}_1$) a is continuous and is decreasing on I.

($\mathcal{H}_2$) $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. There exists $l > 0$ such that

$$|f(t,x) - f(t,y)| \leq l\,|x-y|, \text{ for each } t \in I\,, x, y \in \mathbb{R},$$

and $f(t,0) \in L^1$.

$(\mathcal{H}_3)$

(*i*) $h : I \times I \to \mathbb{R}^+$ be measurable with respect to both variables and such that the integral operator H with the kernel $h(t,s)$ defined on L^1 by the formula

$$H(x(t)) = \int_0^1 h(t,s)x(s)ds. \tag{4.3}$$

(*ii*) The function $t \longmapsto h(t,s)$ is a.e. nondecreasing on I for almost all $t \in I$,

(*iii*) There exists a function $p \in L^1$ such that

$$h(t,s) \leq p(t),$$

$(\mathcal{H}_4)$

(*i*) $u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions, i.e., u is measurable with respect to (t,s) for any $x \in \mathbb{R}$ and is continuous in x for almost all $(t,s) \in I \times I$.

(*ii*) The function $t \longmapsto u(t,s,x)$ is a.e. nondecreasing on I for almost all $s \in I$ and for each $x \in \mathbb{R}$.

(*iii*) $|u(t,s,x)| \leq k(t,s)(q(t) + b_1 |x|)$ for $(t,s) \in I \times I$ and for $x \in \mathbb{R}$, where $q \in L^1$ and $b_1 \geq 0$.

(*iv*) $k : I \times I \to \mathbb{R}_+$ is measurable and such that the linear operator K generated by k maps L^1 into itself.

$(\mathcal{H}_5)$ $\varphi : [0,1] \to [0,1]$ is increasing, absolutely continuous such that there is a constant $b_2 > 0$ such that $\varphi'(t) \geq b_2$ for almost all $t \in I$.

$(\mathcal{H}_6)$ $\left(\dfrac{l}{b_2}\|p\| + b_1\|K\|\right) < 1$, where $\|K\|$ and $\|p\|$ denote the norm of the operator K and the function p respectively.

Theorem 4.4. *Assume that* $(\mathcal{H}_1) - (\mathcal{H}_6)$ *hold true. Then, Eq.* (1.1) *has at least one solution in* L^1.

Proof. We can rewrite Eq. (1.1) in the form $x(t) = Bx(t) + Ax(t)$, where

$$Bx(t) = a(t) + \int_0^1 h(t,s)f(t, x(\varphi(s)))ds, \text{and}$$
$$Ax(t) = \int_0^1 u(t,s,x(s))ds.$$

We will prove that the operators A and B satisfy conditions of Theorem 3.3.
Firstly, observe that the operator B can be written as

$$Bx(t) = a(t) + H(F(x(\varphi(t)))),$$

where F is the superposition operator generated by f, and H is the integral operator defined in Eq. (4.3). In view of assumption $(\mathcal{H}_2)$, we have

$$\begin{aligned} |f(t,x)| &\leq |f(t,x) - f(t,0)| + |f(t,0)| \\ &\leq |f(t,0)| + l|x|. \end{aligned} \tag{4.4}$$

From Eq. (4.4) and Theorem 4.1, it follows that F transforms L^1 into itself and is continuous. Moreover, by $(\mathcal{H}_3)(i)$ and Theorem 4.3, H is continuous on L^1. This fact together with assumption $(\mathcal{H}_1)$, yield that the operator B acts from L^1 to L^1 and is continuous.

Now to see that B is a contraction, let $x,\, y \in L^1$, then

$$\begin{aligned}
\|Bx - By\| &= \int_0^1 \left| \int_0^1 h(t,s)(f(t,x(\varphi(s))) - f(t,y(\varphi(s))))ds \right| dt \\
&\leq \int_0^1 \int_0^1 h(t,s) \left| f(t,x(\varphi(s))) - f(t,y(\varphi(s))) \right| dsdt \\
&\leq l \int_0^1 \int_0^1 p(t) \left| x(\varphi(s)) - y(\varphi(s)) \right| dsdt \\
&\leq l \int_0^1 \int_0^1 p(t) \left| x(\varphi(s)) - y(\varphi(s)) \right| \frac{\varphi'(t)}{b_2} dsdt \\
&\leq \frac{l}{b_2} \int_0^1 p(t) \int_0^1 \left| x(v) - y(v) \right| dvdt.
\end{aligned}$$

Thus,

$$\|Bx - By\| \leq \frac{l}{b_2} \|p\| \, \|x - y\| . \tag{4.5}$$

Using Eq. (4.5) and assumption $(\mathcal{H}_6)$ we deduce that B is a contraction. Further, by assumption $(\mathcal{H}_4)$ and Theorem 4.2 we infer that the operator A transforms L^1 into itself and is continuous.

Now, let N_r denotes the subset of L^1 consisting of all functions $x = x(t)$ being a.e. nondecreasing on I such that $\|x\| \leq r$, where r is a nonnegative constant that will be defined later. Firstly, we prove that the operator $A + B$ maps N_r into itself. By assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_3)(ii)$, $(\mathcal{H}_4)(ii)$, and $(\mathcal{H}_5)$ we obtain that the operator $A+B$ is a.e. nondecreasing.
Let $x \in N_r$, then

$$\|Ax + Bx\| = \int_0^1 |Ax(t) + Bx(t)| dt$$

$$\begin{aligned}
&= \int_0^1 \left| a(t) + \int_0^1 h(t,s) f(t,x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds \right| dt \\
&\leq \int_0^1 \left(|a(t)| + \int_0^1 h(t,s) \left| f(t,x(\varphi(s))) \right| ds + \int_0^1 |u(t,s,x(s))| \, ds \right) dt \\
&\leq \|a\| + \int_0^1 \left(\int_0^1 p(t) \left(|f(t,0)| + l \left| x\left(\varphi(s)\right) \right| \right) ds + \int_0^1 k(t,s) \left(q(t) + b_1 |x(s)| \right) ds \right) dt
\end{aligned}$$

$$\leq \|a\| + \int_0^1 \int_0^1 p(t) |f(t,0)| \, dsdt + l \int_0^1 \int_0^1 p(t) |x(\varphi(s))| \frac{\varphi'(t)}{b_2} dsdt$$
$$+ \int_0^1 \int_0^1 k(t,s) q(t) dsdt + b_1 \int_0^1 \int_0^1 k(t,s) |x(s)| \, dsdt$$
$$\leq \|a\| + \|p\| \|f(t,0)\| + \frac{l}{b_2} \|p\| \|x\| + \|K\| \|q\| + b_1 \|K\| \|x\|$$
$$= \alpha + \beta \|x\|,$$

where

$$\alpha = \|a\| + \|p\| \|f(t,0)\| + \|K\| \|q\|$$

and

$$\beta = \left(\frac{l}{b_2} \|p\| + b_1 \|K\| \right).$$

Let $r = \dfrac{\alpha}{1-\beta} > 0$, so for $x \in N_r$ we have

$$\| Ax + Bx \| \leq \alpha + \beta \|x\| = r. \tag{4.6}$$

Thus, the operator $A + B$ maps N_r into itself. Obviously, N_r is nonempty, bounded, and convex subset of L^1, and by Lemma 4.2 it is compact in measure. Next, we show that N_r is closed. For this purpose, let $(x_n)_n$ be a sequence in N_r converging to x, so

$$\|x_n - x\| \to 0$$

then Lemma 4.1 implies that the sequence $(x_n)_n$ converges in measure to x and there exists a subsequence $(x_{n_j})_j$ which converges a.e. to x on I. Let us prove that the function x is nondecreasing. To see that let t_1, $t_2 \in I$ such that $t_1 \leq t_2$ we have $x_{n_j}(t_1) \leq x_{n_j}(t_2)$ for every j,

$$\begin{aligned} x(t_1) - x(t_2) &= x(t_1) - x_{n_j}(t_1) + x_{n_j}(t_1) - x_{n_j}(t_2) + x_{n_j}(t_2) - x(t_2) \\ &\leq \left\| x(t_1) - x_{n_j}(t_1) \right\| + \left\| x_{n_j}(t_2) - x(t_2) \right\|. \end{aligned} \tag{4.7}$$

Since $(x_{n_j})_j$ is convergent, then by Eq. (4.7), and for each $\varepsilon > 0$ we get

$$x(t_1) - x(t_2) \leq \varepsilon,$$

which proves that $x(t_1) \leq x(t_2)$, hence N_r is closed. By Eq. (4.6), and assumption $(\mathcal{H}_1)$, we can see that the operators A and B converts N_r into itself continuously. Keeping in mind Lemma 4.3, we conclude that the operators A and B are sequentially weakly continuous. Now, We will verify assumption (iii) of Theorem 3.3. For this purpose let $S \subset N_r$ and $x \in S$. Further, let $\varepsilon > 0$ and take a measurable subset $D \subset [0,1]$ such that $m(D) \leq \varepsilon$ and $\varphi(D) \subset D$.

We obtain that

$$\int_D |Ax(t)+Bx(t)|dt \leq \int_D \left| a(t) + \int_0^1 h(t,s)f(t,x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds \right| dt$$

$$\leq \|a\|_{L^1(D)} + \int_D \left(\int_0^1 h(t,s)\left|f(t,x(\varphi(s)))\right| ds + \int_0^1 |u(t,s,x(s))|\, ds \right) dt$$

$$\leq \|a\|_{L^1(D)} + \int_D \int_0^1 p(t)\left(|f(t,0)| + l\,|x(\varphi(s))|\right) dsdt$$

$$+ \int_D \int_0^1 k(t,s)\left(q(t) + b_1\,|x(s)|\right) dsdt$$

$$\leq \|a\|_{L^1(D)} + \|p\|_{L^1(D)}\|f(t,0)\|_{L^1(D)} + l\|p\|_{L^1(D)} \int_D \int_0^1 |x(\varphi(s))|\, \frac{\varphi'(t)}{b_2} dsdt$$

$$+ \int_D \int_0^1 k(t,s)q(t)dsdt + b_1 \int_D \int_0^1 k(t,s)\,|x(s)|\, dsdt$$

$$\leq \|a\|_{L^1(D)} + \|p\|_{L^1(D)}\|f(t,0)\|_{L^1(D)} + \frac{l}{b_2}\|p\|_{L^1(D)} \int_{\varphi(D)} |x(v)|dv$$

$$+ \|K\|_{L^1(D)}\|q\|_{L^1(D)} + b_1\|K\|_{L^1(D)} \int_D |x(t)|dt.$$

This implies that,

$$\omega(A(S)+B(S)) \leq \left(\frac{l}{b_2}\|p\|_{L^1(D)} + b_1\|K\|_{L^1(D)} \right) \omega(S),$$

where $\omega(.)$ is the De Blasi measure of weak noncompactenss. Hence, all the hypotheses of Theorem 3.3 are satisfied. Then, there exists $x \in N_r$ such that $Ax+Bx=x$ which is a solution for Eq. (1.1). □

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