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NOTES ON KRASNOSELSKII-TYPE FIXED-POINT THEOREMS AND THEIR APPLICATION TO FRACTIONAL HYBRID DIFFERENTIAL PROBLEMS

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Abstract. In this paper we prove a new version of Kransoselskii's fixed-point theorem under a (ψ, θ, φ)-weak contraction condition. The theoretical result is applied to prove the existence of a solution of the following fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators orders of $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^{\alpha}[x(t) - f(t, x(t))] = g(t, x(t), I^{\beta}(x(t))), \text{ a.e. } t \in J, \ \beta > 0, \\ x(t_0) = x_0, \end{cases}$$

where D^{α} is the Riemann-Liouville fractional derivative order of α, I^{β} is Riemann-Liouville fractional integral operator order of $\beta > 0$, $J = [t_0, t_0 + a]$, for some fixed $t_0 \in \mathbb{R}$, $a > 0$ and the functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy certain conditions. An example is also furnished to illustrate the hypotheses and the abstract result of this paper.
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1. Introduction

Fixed-point theory has experienced quick improvement over the most recent quite a few years. The development has been firmly advanced by the vast number of utilization in the existence theory of functional, fractional, differential, partial differential, and integral equations. Two fundamental theorems concerning fixed points are those of Schauder and of Banach. Schauder's fixed point theorem, involving a compactness condition, may be stated as "*if S is a closed convex and bounded subset of a Banach space X, then every completely continuous operator $A : S \to S$ has at least one fixed point*". Note that an operator A on a Banach space X is called completely continuous if it is continuous and $A(D)$ is totally bounded for any bounded subset D of X. Banach's fixed point theorem, involving a metric assumption on the mapping, states that "*if X is complete metric space and if A is a contraction on X, then it has a* *unique fixed point, i.e., there is a unique point $x^* \in X$ such that $Ax^* = x^*$. Moreover, the sequence $A^n x$ converges to x^* for every $x \in X$,".* The idea of the hybrid fixed point theorems, that is, a blend of the nonlinear contraction principle and Schauder's fixed-point theorem goes back to 1964, with Krasnoselskii [15], who still maintains an interest in the subject. He gave intriguing applications to differential equations by finding the existence of solutions under some hybrid conditions. Burton [4] extended Krasnoselskii's result for a wide class of operators in 1998. In 2013, Dhage [6] and Dhage and Lakshmikantham [12] proposed an important Krasnoselskii-type fixed-point theorems and applied them the following first-order hybrid differential equation with linear perturbations of first type:

$$\begin{cases} \frac{d}{dx}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \text{ a.e. } t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $J = [t_0, t_0 + a]$, for some fixed $t_0 \in \mathbb{R}$, $a > 0$ and $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$. In the same year, Dhage and Jadhav [11] studied the existence of solution for hybrid differential equation with linear perturbations of second type:

$$\begin{cases} \frac{d}{dx}\Big[x(t) - f(t,x(t))\Big] = g(t,x(t)), \ t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$. They established the existence and uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison result. In [17], Lu et al. proved an existence theorem for fractional hybrid differential equations under a φ-Lipschitz contraction condition and applying this theorem they develop the theory of fractional hybrid differential equations with linear perturbations of second type involving Riemann-Liouville differential operators order of $0 < q < 1$:

$$\begin{cases} D^q\Big[x(t) - f(t,x(t))\Big] = g(t,x(t)), \text{ a.e. } t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $f, g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

In recent years, a number of excellent results concerning the existence of solutions and their approximation solutions for nonlinear initial value problems of first and second orders hybrid functional, differential, integro-differential equations have been obtained by Dhage et al. in [10]-[9]. In [10], the existence and approximation result for the following first order nonlinear initial value problem of hybrid functional integro-differential equations is proven

$$\begin{cases} \frac{d}{dx}\Big[x(t) - f(t,x(t))\Big] = \int_0^t g(s,x_s)ds, \ t \in J, \\ x_0 = \phi, \end{cases} \tag{1.4}$$

where $\phi, f, g \in C(J \times \mathbb{R}, \mathbb{R})$. In [7], it is proven some basic hybrid fixed point theorems for the sum and product of two operators defined in a Banach algebra. And applications of the newly developed abstract hybrid fixed point theorems to nonlinear hybrid linearly perturbed and quadratic integral equations are given. In [8], it is studied the following nonlinear initial value problem of second order hybrid functional differential equations

$$\begin{cases} \frac{d}{dx}\Big[x'(t) - f(t, x_t)\Big] = g(t, x_t), \ \ t \in J, \\ x_0 = \phi, \ \ x'(0) = \eta \end{cases} \tag{1.5}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$. The existence and approximation result for these differential equations is proven. In [9], it is investigated a system of two non-homogeneous boundary value problems of coupled hybrid integro-differential equations of fractional order.

In line with the above works, our purpose in this paper is to developed a new version of Kransoselskii's fixed-point theorem under a weak contraction condition and utilizing this theorem to show the existence of a solution of the following fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators orders of $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^{\alpha}[x(t) - f(t, x(t))] = g(t, x(t), I^{\beta}(x(t))), \ \ \text{a.e.} \ \ t \in J, \ \ \beta > 0, \\ x(t_0) = x_0, \end{cases} \tag{1.6}$$

where $J = [t_0, t_0 + a]$, for some fixed $t_0 \in \mathbb{R}$ and $a > 0$ and $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in \mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The paper is organized as follows. In Section 2, we review some definitions and rigorous results of fixed point theory, fractional calculus and partially ordered Banach space. In Section 3, we prove a new version of Kransoselskii's fixed-point theorem under a weak contraction condition. Utilizing this theorem, in Section 4, we prove the existence of a solution of a fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators orders of $0 < \alpha < 1$ and $\beta > 0$. In Section 5, we provide an illustrative example to highlight the realized improvements.

2. Preliminaries

2.1. **Integrals and derivatives of fractional order.** In this subsection, we recall some basic notions, concepts and definitions of integrals and derivatives of fractional order. The following are discussions of some of the concepts we will need. Let $C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \to g(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$,
(ii) the map $x \to g(t, x, y)$ is continuous for each $x \in \mathbb{R}$,
(iii) the map $y \to g(t, x, y)$ is continuous for each $y \in \mathbb{R}$.

The class $\mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R} \times \mathbb{R}$, which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

Definition 2.1 ([14])**.** The form of the Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} f(s) ds.$$

Definition 2.2 ([14])**.** The Riemann-Liouville derivative of fractional order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-s)^{m-\alpha-1} f(s) ds,$$

where $m = [\alpha] + 1$.

The following lemma will be used in the sequel.

Lemma 2.3 ([14, 18])**.** *Let $0 < \alpha < 1$ and $f \in L^1(0,1)$. Then*

(1) *the equality $D^\alpha I^\alpha f(x) = f(x)$ holds;*
(2) *the equality*

$$I^\alpha D^\alpha f(x) = f(x) - \frac{[D^{\alpha-1} f(x)]_{x=0}}{\Gamma(\alpha)} x^{\alpha-1}$$

holds for almost everywhere on J.

2.2. **Partially ordered Banach space.** Throughout this paper $X := (X, \|\cdot\|)$ stands for a real Banach space. Let $P := P_X$ be a nonempty subset of X.

Definition 2.4. P is called a *closed convex cone* (or shortly, *cone*) with vertex 0 if

(1) P is closed, non-empty and $P \neq \{0\}$;
(2) $\alpha x + \beta y \in P$ for all $x, y \in P$ and non-negative real numbers α, β;
(3) $P \cap (-P) = \{0\}$;
(4) a cone P is called to be positive if $P \circ P \subseteq P$, where $\circ$ is a multiplication composition in X.

We define an order relation $\preceq$ in X as follows. Let $x, y \in X$. Then $x \preceq y$ if and only if $y - x \in P$. The notation $y \prec x$ indicates that $y \preceq x$ and $x \neq y$, while $y \ll x$ will show $x - y \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. From now on, it is assumed that $\mathrm{int}P \neq \emptyset$. The triple $(X, \|\cdot\|, \preceq)$ is called a *partially ordered Banach space*. A cone P is said to be *normal* if the norm $\|\cdot\|$ is semi-monotone increasing on P, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in X$ with $x \preceq y$. It is known that if the cone P is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [13]. Next we define an *upper comparable property* and use in the proof of the main theorem.

Definition 2.5. We say that a partially ordered Banach space $(X, \|\cdot\|, \preceq)$ has an upper comparable property if for every $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

3. Main results

The aim of this section is to prove the following a new version of Kransoselskii's fixed-point theorem under a (ψ, θ, φ)-weak contraction. We need the following definition.

Definition 3.1. The function $\psi : [0, \infty) \to [0, \infty)$ is called a generalized altering distance function if the following properties are satisfied:

(1) ψ is a lower semi-continuous and non-decreasing;
(2) $\psi(t) = 0$ if and only if $t = 0$.

Our main theorem is the following.

Theorem 3.2. *Let $(X, \|\cdot\|, \preceq)$ be a partially ordered Banach space with an upper comparable property. Assume that S be a nonempty, closed, convex, and bounded subset of X. Let $A : X \to X$ and $B : S \to X$ be two operators verifying the following hypotheses:*

(a) *there exist a generalized altering distance function $\psi : [0, \infty) \to [0, \infty)$, an upper semi-continuous function $\theta : [0, \infty) \to [0, \infty)$, and a lower semi-continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that*
$$\psi(\|Ax - Ay\|) \leq \theta(\|x - y\|) - \varphi(\|x - y\|), \quad \text{for} \quad x \succeq y,$$
where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$;
(b) *there exists $x_0 \in X$ such that $x_0 \preceq Ax_0$;*
(c) *A is continuous, non-decreasing (w.r.t. $\preceq$) and $(I - A)^{-1}$ exists and it is continuous;*
(d) *B is continuous, BS is contained in a compact subset of positive cone P;*
(e) *$x = Ax + By \Rightarrow x \in S$ for all $y \in S$.*

Then the operator $A + B$ has a fixed point in S.

Proof. Let us fix arbitrarily $y \in S$. Define $fx := Ax + By$. We show that f has a fixed point in S. By hypothesis (b), there exists $x_0 \in X$ such that $x_0 \preceq Ax_0$. Since $By \in P$ for all $y \in S$ one has $0 \preceq By$. It implies $x_0 \preceq Ax_0 \preceq Ax_0 + By = fx_0 := x_1$. Because the operator A is non-decreasing we have $Ax_0 \preceq Ax_1 = A(Ax_0 + By)$. It implies $x_1 = Ax_0 + By \preceq Ax_1 + By = A(Ax_0 + By) + By = f^2x_0 := x_2$. Proceeding by induction, we obtain $x_{n+1} := fx_n = Ax_n + By$ such that $x_n \preceq x_{n+1}$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some $n \geq 0$ then f has a fixed point. Assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. By using hypothesis (a), we get
$$\begin{aligned}\psi(\|x_n - x_{n+1}\|) &= \psi(\|fx_{n-1} - fx_n\|) \\ &= \psi(\|Ax_{n-1} - Ax_n\|) \leq \theta(\|x_{n-1} - x_n\|) - \varphi(\|x_{n-1} - x_n\|).\end{aligned} \tag{3.1}$$
On the other hand we have
$$\psi(\|x_{n-1} - x_n\|) - \theta(\|x_{n-1} - x_n\|) + \varphi(\|x_{n-1} - x_n\|) > 0$$
since $\|x_{n-1} - x_n\| > 0$. It implies
$$\frac{\psi(\|x_n - x_{n+1}\|)}{\psi(\|x_{n-1} - x_n\|)} \leq \frac{\theta(\|x_{n-1} - x_n\|) - \varphi(\|x_{n-1} - x_n\|)}{\psi(\|x_{n-1} - x_n\|)} < 1.$$
Thus
$$\psi(\|x_n - x_{n+1}\|) < \psi(\|x_{n-1} - x_n\|). \tag{3.2}$$
Since ψ is a generalized altering distance function we get
$$\|x_n - x_{n+1}\| < \|x_{n-1} - x_n\|. \tag{3.3}$$
Hence the sequence $\big(\|x_n - x_{n+1}\|\big)_n$ is decreasing and bounded below. We show that
$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.4}$$

Suppose $\lim_{n\to\infty}\|x_n - x_{n+1}\| = r \neq 0$. For $k \geq 1$, define

$$a_k = \sup\Big\{\theta(\|x_{n-1}-x_n\|) - \varphi(\|x_{n-1}-x_n\|) : \ \ n \geq k\Big\};$$
$$b_k = \inf\Big\{\psi(\|x_n - x_{n+1}\|) : \ \ n \geq k\Big\}.$$

It is clear

$$a_n \geq \theta(\|x_{n-1}-x_n\|) - \varphi(\|x_{n-1}-x_n\|)$$
$$b_n \leq \psi(\|x_n - x_{n+1}\|).$$

These and from (3.1) it follows that

$$b_n \leq \psi(\|x_n - x_{n+1}\|) \leq \theta(\|x_{n-1}-x_n\|) - \varphi(\|x_{n-1}-x_n\|) \leq a_n. \tag{3.5}$$

By using the definitions of ψ, θ and φ we get

$$\begin{aligned}\lim_{n\to\infty} a_n &= \limsup_{n\to\infty} \theta(\|x_{n-1}-x_n\|) - \varphi(\|x_{n-1}-x_n\|) \leq \theta(r) - \varphi(r),\\ \lim_{n\to\infty} b_n &= \liminf_{n\to\infty} \psi(\|x_n - x_{n+1}\|) \geq \psi(r).\end{aligned} \tag{3.6}$$

As a consequence of (3.5) and (3.6) we obtain

$$\psi(r) \leq \theta(r) - \varphi(r)$$

which is a contradiction to hypothesis (a). Hence $r = 0$. Next we show that the sequence (x_n) is a Cauchy sequence in X. Assume the contrary, that is, (x_n) is not a Cauchy sequence. Then we can find an $\varepsilon > 0$ and sub-sequences (x_{n_i}) and (x_{m_i}) of (x_n) with $n_i > m_i > i$ such that

$$\|x_{m_i} - x_{n_i}\| \geq \varepsilon. \tag{3.7}$$

Let n_i be the smallest index with $n_i > m_i > i$ and satisfying (3.7). This means

$$\|x_{m_i} - x_{n_i-1}\| < \varepsilon. \tag{3.8}$$

By using the triangular inequality we get

$$\|x_{m_i} - x_{n_i}\| \leq \|x_{m_i} - x_{n_i-1}\| + \|x_{n_i-1} - x_{n_i}\|. \tag{3.9}$$

Combining (3.7)-(3.9) we get

$$\varepsilon \leq \|x_{m_i} - x_{n_i}\| < \|x_{n_i-1} - x_{n_i}\| + \varepsilon.$$

Taking the limit as $i \to \infty$ we have

$$\lim_{i\to\infty} \|x_{m_i} - x_{n_i}\| = \varepsilon. \tag{3.10}$$

On the other hand, the triangular inequality implies

$$\|x_{m_i} - x_{n_i-1}\| \leq \|x_{m_i} - x_{n_i}\| + \|x_{n_i} - x_{n_i-1}\|. \tag{3.11}$$

Letting $i \to \infty$ in (3.9) and (3.11) we obtain

$$\varepsilon \leq \lim_{i\to\infty} \|x_{m_i} - x_{n_i-1}\| \leq \varepsilon.$$

Hence

$$\lim_{i\to\infty} \|x_{m_i} - x_{n_i-1}\| = \varepsilon. \tag{3.12}$$

Similarly, it can be shown that

$$\lim_{i\to\infty} \|x_{m_i+1} - x_{n_i}\| = \varepsilon. \tag{3.13}$$

From hypothesis (a) it follows that

$$\begin{aligned}\psi(\|x_{m_i+1} - x_{n_i}\|) &= \psi(\|fx_{m_i} - fx_{n_i-1}\|) = \psi(\|Ax_{m_i} - Ax_{n_i-1}\|)\\ &\le \theta(\|x_{m_i} - x_{n_i-1}\|) - \varphi(\|x_{m_i} - x_{n_i-1}\|)\end{aligned} \tag{3.14}$$

since $x_{m_i} \preceq x_{n_i-1}$. The same manner as in (3.5) and (3.6) it can be shown

$$\begin{aligned}&\limsup_{i\to\infty} \theta(\|x_{m_i} - x_{n_i-1}\|) - \varphi(\|x_{m_i} - x_{n_i-1}\|) \le \theta(\varepsilon) - \varphi(\varepsilon),\\ &\liminf_{i\to\infty} \psi(\|x_{m_i+1} - x_{n_i}\|) \ge \psi(\varepsilon),\end{aligned} \tag{3.15}$$

and $\psi(\varepsilon) \le \theta(\varepsilon) - \varphi(\varepsilon)$ which is a contradiction due to $\varepsilon > 0$. Therefore (x_n) is a Cauchy sequence. Since X is a Banach space there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

According to hypotheses (c) and (d) the operator f is continuous, thus

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fx^*.$$

Hence $x^* \in X$ is a fixed point of f. Next we show that f has a unique fixed point. Suppose f has another fixed point $y^* \in X$. Since X has an upper comparable property there exists $z \in X$ such that $x^* \preceq z$ and $y^* \preceq z$. First, we assume $x^* \prec z$ and $y^* \prec z$. In this case, because the operator A is non-decreasing we have $Ax^* \preceq Az$ it implies $Ax^* + By \preceq Az + By$ that is $fx^* \preceq fz$. Since x^* is a fixed point of f we have $x^* \preceq fz$. Proceeding by induction, we obtain $x^* \preceq z_n$ for all $n \ge 1$ where $z_n := f^n z$. For definiteness we assume $x^* \ne z_n$ for all $n \ge 1$. By using hypothesis (a) we get

$$\begin{aligned}\psi(\|x^* - z_{n+1}\|) &= \psi(\|fx^* - fz_n\|)\\ &= \psi(\|Ax^* - Az_n\|) \le \theta(\|x^* - z_n\|) - \varphi(\|x^* - z_n\|).\end{aligned} \tag{3.16}$$

On the other hand we have

$$\psi(\|x^* - z_n\|) - \theta(\|x^* - z_n\|) + \varphi(\|x^* - z_n\|) > 0$$

since $\|x^* - z_n\| > 0$. It implies

$$\frac{\psi(\|x^* - z_{n+1}\|)}{\psi(\|x^* - z_n\|)} \le \frac{\theta(\|x^* - z_n\|) - \varphi(\|x^* - z_n\|)}{\psi(\|x^* - z_n\|)} < 1.$$

Thus

$$\psi(\|x^* - z_{n+1}\|) < \psi(\|x^* - z_n\|). \tag{3.17}$$

Since ψ is a generalized altering distance function we get

$$\|x^* - z_{n+1}\| < \|x^* - z_n\|. \tag{3.18}$$

Hence the sequence $\big(\|x^* - z_n\|\big)_n$ is decreasing and bounded below. The same manner as in the proof of relation (3.4) it can be shown that

$$\lim_{n\to\infty} \|x^* - z_n\| = 0.$$

Similarly, we can deduce that

$$\lim_{n\to\infty} \|y^* - z_n\| = 0.$$

On the other hand we have

$$\|x^* - y^*\| \le \|x^* - z_n\| + \|y^* - z_n\|.$$

Taking the limit as $n \to \infty$ we get $x^* = y^*$. Now we assume $x^* = z$ and $y^* \prec z$ that is $y^* \prec x^*$. It implies $\|x^* - y^*\| > 0$. By using hypothesis (a) we obtain

$$\begin{aligned}\psi(\|x^* - y^*\|) &= \psi(\|fx^* - fy^*\|)\\ &= \psi(\|Ax^* - Ay^*\|) \le \theta(\|x^* - y^*\|) - \varphi(\|x^* - y^*\|).\end{aligned} \tag{3.19}$$

On the other hand we have $\psi(\|x^* - y^*\|) - \theta(\|x^* - y^*\|) + \varphi(\|x^* - y^*\|) > 0$ since $\|x^* - y^*\| > 0$. But this contradicts the inequality (3.19). Hence $x^* = y^*$. The case $x^* \prec z$ and $y^* = z$ is similar to the above case and the case $x^* = z$ and $y^* = z$ is trivial. So, in all cases, we have shown that $x^* = y^*$. Hence f has a unique fixed point, that is, there exists a unique $x^* \in X$ such that

$$x^* = Ax^* + By. \tag{3.20}$$

Hypothesis (e) implies that $x^* \in S$. From the equality (3.20) it follows that

$$(I - A)x^* = By \text{ for all } y \in S.$$

Since the operator $(I-A)^{-1}$ exists and continuous we have $x^* = (I-A)^{-1}By \in S$ for all $y \in S$. Now according to (d) the set BS is contained in a compact subset of P, while $(I - A)^{-1}$ is continuous, and so $(I - A)^{-1}BS$ is contained in a compact subset of P. (For the proof of this in general metric spaces, see [16] pp. 412). From hypothesis (e) and equality (3.20) it follows that $(I-A)^{-1}BS$ is contained in a compact subset of the closed set S. By Schauder's second theorem (see [20] pp. 25) the operator $(I-A)^{-1}B$ has a fixed point in S, that is, there exists $z^* \in S$ such that $z^* = (I - A)^{-1}Bz^*$. This implies $Az^* + Bz^* = z^*$. Theorem 3.2 is proved. □

Remark 3.3. Note that

- the main idea of the proof of the existence of a fixed point of the mapping f has been borrowed from [19];
- in the case of ψ-altering distance function, the (ψ, θ, φ)-weak contraction condition has been successfully applied in multidimensional fixed point theorems and their applications to the system of matrices equations and nonlinear integral equations (see, for instance [1]-[2]);
- the generalized (ψ, θ, φ)-weak contraction condition extends the notion of D-contraction condition which is defined by a dominating function or, in short, D-function (see, for instance [6] and [8, 9]).
- Theorem 3.2 extends the main theorem of [5].

4. Applications of Theorem 3.2

In this section, by applying Theorem 3.2 we study the existence of a solution of the fractional hybrid differential equation (1.6) under the following assumptions.

Hypothesis for FHDE (1.6). We assume:

(H1) The function $F_t : \mathbb{R} \to \mathbb{R}$ defined as $F_t(x) := x - f(t,x)$ is strictly increasing in $\mathbb{R}$ for all $t \in J$.

(H2) The function $f(t,\cdot)$ satisfies the following weak contraction condition
$$|f(t,x) - f(t,y)| \leq \arctan\big(|x-y|\big)$$
for all $t \in J$ and $x, y \in \mathbb{R}$.

(H3) The function $f(t,0)$ satisfies
$$f(t,0) - f(t_0,x_0) + x_0 \geq 0$$
for all $t \in J$.

(H4) The function g is non-negative and there exists a continuous function $h \in C(J,\mathbb{R})$ such that
$$0 \leq g(t,x,y) \leq h(t)$$
for almost all $t \in J$ and for all $x, y \in \mathbb{R}$.

The following lemma is useful in what follows.

Lemma 4.1 ([17])**.** *Assume that hypothesis (H1) holds. Then, for any $y \in C(J,\mathbb{R})$ and $\alpha \in (0,1)$ the function $x \in C(J,\mathbb{R})$ is a solution of FHDE*
$$D^\alpha[x(t) - f(t,x(t))] = y(t), \ t \in J, \tag{4.1}$$
with the initial condition $x(t_0) = x_0$ if and only if $x(t)$ satisfies the hybrid integral equation (in short, HIE)
$$x(t) = x_0 - f(t_0,x_0) + f(t,x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} y(s)ds, \ t \in J. \tag{4.2}$$

Now, we are in a position to prove the following existence theorem for FHDE (1.6).

Theorem 4.2. *Assume that hypotheses (H1)-(H4) hold. Then FHDE (1.6) has a solution in $C(J,\mathbb{R})$.*

Proof. Set $X = C(J,\mathbb{R})$. Let $\|\cdot\|$ be the maximum norm in X, that is,
$$\|x\| = \max_{t\in J} |x(t)|.$$
Clearly, X is a Banach space with respect to this norm. We claim that the space X has an upper comparable property. Indeed, for any $x(t), y(t) \in X$ we can find $z(t) \in X$ as $z(t) = \max_{t\in J}\{x(t), y(t)\}$. It is clear that $x(t) \leq z(t)$ and $y(t) \leq z(t)$. Define a subset S of X as follows.
$$S = \Big\{x \in X : \ \ \|x\| \leq M\Big\}$$
where $M = |x_0 - f(t_0,x_0)| + \frac{\pi}{2} + L + \frac{|J|^\alpha}{\Gamma(\alpha+1)}\|h\|$ and $L = \max_{t\in J} f(t,0)$. Clearly, S is a closed, convex and bounded subset of the Banach space X.
Define $P = \big\{x \in X : \ \ x \geq 0\big\}$. It is obvious that the set P is a positive cone and normal in X. For a given $y \in S$, consider the following generalized fractional

hybrid differential equation involving the Riemann-Liouville differential and integral operators orders of $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^\alpha[x(t) - f(t, x(t))] = g(t, y(t), I^\beta(y(t))), \text{ a.e. } t \in J, \ \beta > 0, \\ x(t_0) = x_0 \end{cases} \tag{4.3}$$

where $J = [t_0, t_0 + a]$, for some fixed $t_0, a \in \mathbb{R}^+$ and $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in \mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Assume f and g satisfy the assumptions (H1)-(H4). Then by Lemma 4.1 the equation (4.3) is equivalent to the nonlinear HIE

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^\beta(y(s)))ds. \tag{4.4}$$

Define operators $A : X \to X$ and $B : S \to X$ by

$$Ax(t) = x_0 - f(t_0, x_0) + f(t, x(t)), \quad t \in J, \tag{4.5}$$

and

$$By(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^\beta(y(s)))ds, \quad t \in J. \tag{4.6}$$

Then HIE (4.4) is transformed into the operator equation as

$$x(t) = Ax(t) + By(t), \quad t \in J. \tag{4.7}$$

We will show that the operators A and B satisfy all hypotheses of Theorem 3.2. First, we show that the operator A satisfies hypothesis (a) of Theorem 3.2 with

$$\psi(t) = t, \ \ \theta(t) = \arctan(t) \ \text{ and } \ \varphi(t) = 0.$$

Let $x, y \in X$. Then by hypothesis (H2) we have

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \le \arctan(|x(t) - y(t)|) \le \arctan(\|x - y\|).$$

Taking maximum over t, we obtain

$$\|Ax - Ay\| \le \arctan(\|x - y\|).$$

One can easily see that $\psi(t) - \theta(t) + \varphi(t) = t - \arctan(t) > 0$ for all $t > 0$ and $\theta(0) = \varphi(0) = 0$. Hence the operator A satisfies hypothesis (a). Next, we show that A satisfies hypothesis (b). Let $x_0(t) \equiv 0$. Then by (H3) we have

$$Ax_0(t) = x_0 - f(t_0, x_0) + f(t, 0) \ge 0 = x_0(t).$$

Now we show that $(I - A)^{-1}$ exists and continuous. By hypotheses (H1) and (H2) the function $F_t(x) = x - f(t, x)$ is strictly increasing and continuous for all $t \in J$. Therefore F_t^{-1} exists and continuous for all $t \in J$. Consider the operator $T : X \to X$ defined as $Tx(t) = F_t^{-1}(x(t))$. One can easily see that

$$(I - A) \circ T = F_t \circ F_t^{-1} = I \ \text{ and } \ T \circ (I - A) = F_t^{-1} \circ F_t = I.$$

Hence T is the inverse of $I - A$ and continuous since F_t^{-1} is continuous. Next, we show that the operator B satisfies hypothesis (d) of Theorem 3.2. By (H4) the function $g(\cdot, x, y)$ is almost everywhere non-negative thus for any $y \in S$ we get

$$By(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^\beta(y(s)))ds \ge 0.$$

Hence BS is a subset of the positive cone P. Now, we show that B is continuous on S. Let (y_n) be a sequence in S converging to a point $y \in S$. Then, by Lebesgue dominated convergence theorem we have

$$\begin{aligned}\lim_{n\to\infty} By_n &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y_n(s), I^{\beta}(y_n(s)))ds \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \lim_{n\to\infty} g(s, y_n(s), I^{\beta}(y_n(s)))ds \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds = By(t)\end{aligned}$$

for all $t \in J$. Hence the operator B is continuous. Now, we show that B is a compact operator on S. It is enough to show that BS is a uniformly bounded and equicontinuous set in S. By hypothesis (H4) we have

$$\begin{aligned}|By(t)| &= \Big|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds\Big| \\ &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big|g(s, y(s), I^{\beta}(y(s)))\Big|ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} h(s)ds \le \frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\|h\|\end{aligned}$$

for $t \in J$. Taking maximum over t we get

$$\|By\| \le \frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\|h\|.$$

Thus the operator B is uniformly bounded on S. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $x \in S$, one has

$$\begin{aligned}|Bx(t_1) - Bx(t_2)| &= \Big|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds\Big| \\ &\le \Big|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_2-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds\Big| \\ &\quad + \Big|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_2-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds\Big| \\ &\le \frac{\|h\|}{\Gamma(\alpha+1)}\Big[|(t_2-t_0)^{\alpha} - (t_1-t_0)^{\alpha} + (t_2-t_1)^{\alpha}| + (t_2-t_1)^{\alpha}\Big].\end{aligned}$$

Hence, for any ε, there exists a $\delta > 0$ such that

$$|t_1 - t_2| \leq \delta \quad \Rightarrow \quad |Bx(t_1) - Bx(t_2)| \leq \varepsilon,$$

for $t_1, t_2 \in J$ and for all $x \in S$. This shows that BS is an equicontinuous set in X. Hence according to Arzela-Ascoli theorem the set BS is compact . Finally, we show that hypothesis (e) of Theorem 3.2 is satisfied. Let $x \in X$ and $y \in S$ satisfy the equation $x = Ax + By$. Then, by hypothesis (H2), we have

$$\begin{aligned}
|Ax(t) - By(t)| &\leq |Ax(t)| + |By(t)| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t))| \\
&+ \Big| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s))) ds \Big| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |g(s, y(s), I^{\beta}(y(s)))| ds \\
&\leq |x_0 - f(t_0, x_0)| + \arctan(\|x\|) + L + \frac{|J|^{\alpha}}{\Gamma(\alpha+1)} \|h\| \\
&\leq |x_0 - f(t_0, x_0)| + \frac{\pi}{2} + L + \frac{|J|^{\alpha}}{\Gamma(\alpha+1)} \|h\|.
\end{aligned}$$

Taking maximum over t, we have

$$\|x\| = \max_{t \in J} |Ax(t) - By(t)| \leq |x_0 - f(t_0, x_0)| + \frac{\pi}{2} + L + \frac{|J|^{\alpha}}{\Gamma(\alpha+1)} \|h\| = M.$$

Hence $x \in S$. Thus all hypotheses of Theorem 3.2 are satisfied and so the operator $A + B$ has a fixed point in S, that is, there exists $z^* \in S$ such that $Az^* + Bz^* = z^*$. As a result, FHDE (1.6) has a solution in S. This completes the proof of Theorem 4.2.

5. Illustrative example

Let $J = [0, 1]$. Denote by X the set of continuous and non-negative functions $f : J \to [0, \infty)$. In X consider the following fractional hybrid differential equation:

$$\begin{cases} D^{\frac{1}{2}} \big[x(t) - \tanh(t) \arctan\big(x(t) + 1\big) \big] = t^2 e^t |\sin\big(x(t)\big)| \frac{I^{\beta}\left(x(t)\right)}{1 + I^{\beta}\left(x(t)\right)}, \\ x(0) = 0, \end{cases} \tag{5.1}$$

where $t \in J$ and $\beta > 0$. Observe that this equation is a special case of the FHDE (1.6) if we set

$$f(t, x(t)) = \tanh(t) \arctan\big(x(t) + 1\big)$$

and

$$g(t, x(t), I^{\beta}(x(t))) = t^2 e^t |\sin\big(x(t)\big)| \frac{I^{\beta}\big(x(t)\big)}{1 + I^{\beta}\big(x(t)\big)}.$$

We show that the equation (5.1) satisfies the hypothesis (H1)-(H4). We claim that the function

$$F_t(x) := x - f(t,x) = x - \tanh(t)\arctan(x+1)$$

is strictly increasing in $\mathbb{R}^+ \cup \{0\}$ for all $t \in J$. In order to show that F_t is strictly increasing it is sufficient to show that the partial derivative $\partial F_t/\partial x$ is positive for all $t \in J$. Indeed, since $\tanh(t) < 1$ for $t \in J$ we have

$$\frac{\partial F_t}{\partial x} = 1 - \frac{\tanh(t)}{1+(x+1)^2} > 0.$$

Hence, the hypothesis (H1) is satisfied. Next we show that f satisfies the hypothesis (H2). Let $x \in \mathbb{R}^+ \cup \{0\}$ and $\delta > 0$. One can see

$$\begin{aligned} |f(t,x+\delta) - f(t,x)| &= \tanh(t)\Big|\arctan(x+\delta+1) - \arctan(x+1)\Big| \\ &\leq \tanh(t) \sup_{x\in\mathbb{R}^+\cup\{0\}} \Big|\arctan(x+\delta+1) - \arctan(x+1)\Big|. \end{aligned} \tag{5.2}$$

We estimate $g_\delta(x) := \arctan(x+\delta+1) - \arctan(x+1)$. Since the arctangent function is increasing by the arctangent subtraction formula we have

$$\begin{aligned} |g_\delta(x)| &= |\arctan(x+\delta+1) - \arctan(x+1)| \\ &= \Big|\arctan\Big(\frac{\delta}{1+(x+\delta+1)(x+1)}\Big)\Big| \leq \arctan(\delta). \end{aligned} \tag{5.3}$$

For any $x, y \in \mathbb{R}^+ \cup \{0\}$ with $x < y$ by setting $\delta := y - x$ and combining the relations (5.2) and (5.3) we get

$$\begin{aligned} |f(t,x) - f(t,y)| &= |f(t,x+\delta) - f(t,x)| \\ &= \tanh(t)\Big|\arctan(x+\delta+1) - \arctan(x+1)\Big| \\ &\leq \tanh(t) \sup_{x\in\mathbb{R}^+\cup\{0\}} \Big|\arctan(x+\delta+1) - \arctan(x+1)\Big| \\ &\leq \tanh(t)\arctan(\delta) \leq \arctan(|x-y|) \end{aligned} \tag{5.4}$$

since $\tanh(t) < 1$ for $t \in J$. Hence, f satisfies the hypothesis (H2). It is easy to check hypothesis (H3) because

$$f(t,0) - f(0,0) = \tanh(t)\arctan(1) - \tanh(0)\arctan(1) = \frac{\pi}{4}\tanh(t) \geq 0.$$

Finally, we show that g satisfies hypothesis (H4) with $h(t) = t^2e^t$. It is easy to see that $I^\beta\big(x(t)\big) \geq 0$ since $x(t) \geq 0$. This implies

$$g(t, x(t), I^\beta(x(t))) = t^2e^t|\sin\big(x(t)\big)|\frac{I^\beta\big(x(t)\big)}{1+I^\beta\big(x(t)\big)} \geq 0.$$

On the other hand we have

$$|\sin\big(x(t)\big)| \leq 1 \quad \text{and} \quad \frac{I^\beta\big(x(t)\big)}{1+I^\beta\big(x(t)\big)} \leq 1$$

which implies that

$$g(t, x(t), I^\beta(x(t))) = t^2 e^t |\sin\big(x(t)\big)| \frac{I^\beta\big(x(t)\big)}{1 + I^\beta\big(x(t)\big)} \le t^2 e^t.$$

Thus

$$0 \le g(t, x(t), I^\beta(x(t))) \le t^2 e^t.$$

So, g satisfies hypothesis (H4). Hence all (H1)-(H4) hypotheses are satisfied. Thus by Theorem 4.2 we conclude that the hybrid differential equation (5.1) has a solution. □

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