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EXISTENCE OF GROUP NONEXPANSIVE RETRACTIONS AND ERGODIC THEOREMS IN TOPOLOGICAL GROUPS

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Abstract. Suppose that G is a topological group and C a compact subset of G. In this paper we define group nonexpansive mappings and then we consider $\mathcal{S} = \{T_i : i \in I\}$ as a family of the group nonexpansive mappings on C. Also we study the existence of group nonexpansive retractions P_i from C onto Fix($\mathcal{S}$) such that $P_i T_i = T_i P_i = P_i$.
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1. Introduction

A topological group G is a set endowed with two structures, a group structure and a topological structure. Specifically, G is both an abstract group and a topological space such that the two maps

$$G \times G \to G : (x, y) \mapsto xy$$
$$G \to G : x \mapsto x^{-1}$$

are assumed to be continuous. Also, the Hausdorff condition will be imposed.

Let D be a subset of B where B is a subset of a topological group G. A mapping P is called a retraction of B onto D, if $Px = x$ for each $x \in D$.

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [1]: Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $\mathrm{Fix}(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesaro means

$$S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$$

converge weakly to some $y \in \mathrm{Fix}(T)$. In Baillon's theorem, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $\mathrm{Fix}(T)$ such that $PT^n = T^n P = P$ for all positive integers n and $Px \in \overline{co}\{T^n x : n = 1, 2, ...\}$ for each $x \in C$. Takahashi [13] proved the existence of such retractions, "ergodic retractions", for non-commutative semigroups of nonexpansive mappings in a Hilbert space: If S is an amenable semigroup, C is a closed, convex subset of a Hilbert space H and $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, then there exists a nonexpansive retraction P from C onto $\mathrm{Fix}(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$. These results were extended to uniformly convex Banach spaces for commutative semigroups in [4] and for non-commutative amenable semigroups in [6, 5] and recently for a family of Q-nonexpansive mappings in locally convex spaces in [3]. For other results we refer the reader to [11, 9, 10, 12].

In this paper, first we define group nonexpansive mappings. Then we establish some ergodic retractions for topological groups, based on the definition. Also we present a family of desired retractions by removing "convexity" in the above mentioned theorems in the topological group setting.

2. PRELIMINARIES

In this section, we introduce our definition and give some examples:

Definition 2.1. Suppose that G is a topological group and $C \subset G$. A mapping $T : C \to C$ is said to be group nonexpansive if for each $x, y \in C$ and each closed neighborhood $U \in \mathfrak{B}_e$ (where $\mathfrak{B}_e$ is a local base in e (identity element)) that $xy^{-1} \in U$ then we have $Tx(Ty)^{-1} \in U$.

Example 2.2. Our group nonexpansiveness is more general than nonexpansiveness i.e. every nonexpansive mapping is a group nonexpansive mapping if the topological group is a normed vector space. Indeed, let E be a normed vector space and $C \subset E$. Consider the closed neighborhood $U = \{z \in E : \|z\| \leq r\} \in \mathfrak{B}_0$ (where $\mathfrak{B}_0$ is a local base in 0) that $x - y \in U$ for a positive number r. If T is a nonexpansive mapping on C then we have

$$\|Tx - Ty\| \leq \|x - y\|,$$

and hence $Tx - Ty \in U$. Thus T is a group nonexpansive.

In the following example we consider a case that some group nonexpansive mappings are nonexpansive mapping.

Example 2.3. Let G be metric topological group with a right invariant metric (that is, $d(yx, zx) = d(y, z)$ for all $x, y, z \in G$). Let T be a group nonexpansive mapping from G into G. Consider the neighbourhoods $N_{d(xy^{-1},e)+\frac{1}{n}}(e)$ of e with $xy^{-1} \in N_{d(xy^{-1},e)+\frac{1}{n}}(e)$ for each $n \in \mathbb{N}$. Then we have $Tx(Ty)^{-1} \in N_{d(xy^{-1},e)+\frac{1}{n}}(e)$

for each $n \in \mathbb{N}$, and hence

$$d(Tx(Ty)^{-1}, e) \leq d(xy^{-1}, e) + \frac{1}{n},$$

for each $n \in \mathbb{N}$. Therefore

$$d(Tx(Ty)^{-1}, e) \leq d(xy^{-1}, e),$$

and since d is right invariant, we have

$$d(Tx, Ty) \leq d(x, y).$$

Then we conclude that T is a nonexpansive mappings in the sense of nonexpansive mappings in metric spaces.

3. Main results

Let G be a topological group. In this section, we study the existence of group nonexpansive retractions onto the set of common fixed points of a family of group nonexpansive mappings that commute with the mappings. A group nonexpansive retraction that commutes with the mappings is usually called an ergodic retraction.

First, we prove the following theorem which is the main result of this section and will be essential in the sequel.

Theorem 3.1. *Let G be a topological group and let C be a compact subset of G. Suppose that $\mathcal{S} = \{T_i : i \in I\}$ is a family of the group nonexpansive mappings on C such that $Fix(\mathcal{S}) \neq \emptyset$ and for every $\alpha \in I$, there exists a subnet $\{T_\alpha^{n_\gamma}\}$ of the sequence $\{T_\alpha^n\}$ such that $\lim_\gamma T_\alpha^{n_\gamma} x = \lim_\gamma T_\alpha^{n_\gamma - 1} x$ for each $x \in C$. Also suppose for every nonempty compact $\mathcal{S}$-invariant subset K of C, $K \cap Fix(\mathcal{S}) \neq \emptyset$. Then, for each $i \in I$, there exists a group nonexpansive retraction P_i from C onto $Fix(\mathcal{S})$, such that $P_i T_i = T_i P_i = P_i$ and every closed $\mathcal{S}$-invariant subset of C is also P_i-invariant.*

Proof. Let C^C be the product space with the product topology induced by the relative topology on C. Now for a fixed $\alpha \in I$, consider the following set

$$\mathfrak{R} = \{T \in C^C : T \text{ is group nonexpansive}, T \circ T_\alpha = T$$
$$\text{and every closed } \mathcal{S}\text{-invariant subset of } C \text{ is also } T\text{-invariant}\}.$$

From the fact that G is Hausdorff, for each $z \in \mathrm{Fix}(\mathcal{S})$, the singleton set $\{z\}$ is a closed $\mathcal{S}$-invariant subset of C, and then for each $T \in \mathfrak{R}$, $Tz = z$. Fix $z_0 \in \mathrm{Fix}(\mathcal{S})$ and let for each $x \in C$,

$$C_x := \{y \in C : \text{for each closed neighborhood } U \text{ of } e \text{ that } xz_0^{-1} \in U \text{ then } yz_0^{-1} \in U\}.$$

For all $x \in C$ and $T \in \mathfrak{R}$, we have that $T(x) \in C_x$. Since T is group nonexpansive for a closed neighborhood U of e, if $xz_0^{-1} \in U$ then $T(x)z_0^{-1} = T(x)(T(z_0))^{-1} \in U$. Hence $\mathfrak{R} \subseteq \prod_{x \in C} C_x$, where $\prod_{x \in C} C_x$ is the Cartesian product of sets C_x for all $x \in C$. Let $\{y_\beta\}$ be a net in C_x such that $y_\beta \to y$. Consider a closed neighborhood U of e such that $xz_0^{-1} \in U$. Then we have $y_\beta z_0^{-1} \in U$. From the fact that the mapping $(x, y) \mapsto xy$ is continuous we have $y_\beta z_0^{-1} \to yz_0^{-1}$ and since U is closed we conclude that $yz_0^{-1} \in U$, and therefore C_x is closed and since C is compact we conclude that C_x is compact. By Tychonoff's theorem, we know that when C_x is given the relative topology and $\prod_{x \in C} C_x$ is given the corresponding product topology, $\prod_{x \in C} C_x$ is compact. Next we prove that $\mathfrak{R}$ is closed in $\prod_{x\in C} C_x$. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a net in $\mathfrak{R}$ which converges to T_0 in $\prod_{x\in C} C_x$. Hence if $z \in \text{Fix}(\mathcal{S})$, then we have $T_\lambda z = z$ for each $\lambda \in \Lambda$ (because $T_\lambda \in \mathfrak{R}$) and $T_0 z = \lim_\lambda T_\lambda(z) = z$. From the fact that the mapping $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous, if we consider a closed neighborhood U of e that $xy^{-1} \in U$ then we have $T_0x(T_0y)^{-1} = \lim_\lambda T_\lambda x(T_\lambda y)^{-1} \in U$. Hence, T is group nonexpansive. Obviously, we have $T_0 \circ T_\alpha = T_0$ and every closed $\mathcal{S}$-invariant subset of C is also T_0-invariant. Therefore, $T_0 \in \mathfrak{R}$. Then $\mathfrak{R}$ is closed in $\prod_{x\in C} C_x$. Since $\prod_{x\in C} C_x$ is compact, hence $\mathfrak{R}$ is compact. Next, we show that $\mathfrak{R} \neq \emptyset$. Consider the mappings $S_n = T_\alpha^{n-1} \in \prod_{x\in C} C_x$ for each $n \in \mathbb{N}$. Then from the fact that $\prod_{x\in C} C_x$ is compact and using our condition, it has a convergent subnet $\{S_{n_\eta}\}$ such that $\lim_\eta T_\alpha^{n_\eta} x = \lim_\eta T_\alpha^{n_\eta - 1} x$ for each $x \in C$. Define for each $x \in C$, $T(x) = \lim_\eta S_{n_\eta} x$. We now check that $T \in \mathfrak{R}$. Note that, from the continuity of the mapping $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ and the group nonexpansiveness of S_{n_η} and closedness of U, T is group nonexpansive. Indeed, if $xy^{-1} \in U$ for any closed neighbourhood U of e and $x, y \in C$, then we have $Tx(Ty)^{-1} = \lim_\eta S_{n_\eta} x(\lim_\eta S_{n_\eta} y)^{-1} = \lim_\eta S_{n_\eta} x(S_{n_\eta} y)^{-1} \in U$. Moreover, $T(T_\alpha x) = \lim_\eta S_{n_\eta}(T_\alpha x) = \lim_\eta T_\alpha^{n_\eta} x = \lim_\eta T_\alpha^{n_\eta - 1} x = \lim_\eta S_{n_\eta}(x) = T(x)$. Finally, if D is a closed $\mathcal{S}$-invariant subset of C, it is clear that D is S_{n_η}-invariant and thus from the closedness of D, is T-invariant. Therefore, we have shown that $T \in \mathfrak{R} \neq \emptyset$.

Now define a preorder $\preceq$ in $\mathfrak{R}$ by $T_1 \preceq T_2$ if for each $U \in \mathfrak{B}_e$ that $T_2x(T_2y)^{-1} \in U$ we have $T_1x(T_1y)^{-1} \in U$, and by using a method similar to Bruck's method [2], we find a minimal element T_{min} in $\mathfrak{R}$. Indeed, using Zorn's Lemma, it is enough that we show that each linearly ordered subset of $\mathfrak{R}$ has a lower bound in $\mathfrak{R}$. Let $\{A_\lambda\}$ be a linearly ordered subset of $\mathfrak{R}$. Then the family of sets $\{T \in \mathfrak{R} : T \preceq A_\lambda\}$ is a linearly ordered subset of $\mathfrak{R}$ by inclusion. Taking into account the closeness proof of $\mathfrak{R}$ in $\prod_{x\in C} C_x$, these sets are closed in $\mathfrak{R}$, and hence compact. Then from the finite intersection property, there exists $R \in \bigcap_\lambda \{T \in \mathfrak{R} : T \preceq A_\lambda\}$ with $R \preceq A_\lambda$ for all λ. Then each linearly ordered subset of $\mathfrak{R}$ has a lower bound in $\mathfrak{R}$. We have shown that there exist a minimal element P_α in the following sense:

$$\text{if } T \in \mathfrak{R} \text{ and for each } U \in \mathfrak{B}_e \text{ that } P_\alpha x(P_\alpha y)^{-1} \in U \text{ then } Tx(Ty)^{-1} \in U,$$
$$\text{then for each } U' \in \mathfrak{B}_e \text{ that } Tx(Ty)^{-1} \in U' \text{ we have } P_\alpha x(P_\alpha y)^{-1} \in U'. \qquad (*)$$

Next we prove that $P_\alpha x \in \text{Fix}(\mathcal{S})$ for every $x \in C$. For a given $x \in C$, consider $K := \{T(P_\alpha x) : T \in \mathfrak{R}\}$. From the fact that $\mathfrak{R}$ is compact, from Proposition 3.3.18 and Definition 3.3.19 in [8], we conclude that K is a nonempty compact subset of C. Now we have $S(K) \subset K$ for each $S \in \mathcal{S}$, because $STT_\alpha = ST$ for each $T \in \mathfrak{R}$ hence $ST \in \mathfrak{R}$ i.e, K is $\mathcal{S}$-invariant.

From our assumption $K \cap \text{Fix}(\mathcal{S}) \neq \emptyset$. Then there exists $L \in \mathfrak{R}$ such that $L(P_\alpha x) \in \text{Fix}(\mathcal{S})$. Suppose that $y = L(P_\alpha x)$. Since $P_\alpha, L \in \mathfrak{R}$ and the set $\{y\}$ is $\mathcal{S}$-invariant, we have $P_\alpha(y) = L(y) = y$, and since L is group nonexpansive, P_α is minimal and $L(P_\alpha x)(L(P_\alpha y))^{-1} = L(P_\alpha x)y^{-1} = yy^{-1} = e \in U$, for each $U \in \mathfrak{B}_e$ and then we have $(P_\alpha x)y^{-1} = P_\alpha x(P_\alpha y)^{-1} \in U$ for each $U \in \mathfrak{B}_e$ (to see this consider LP_α instead of T in $(*)$) and by (vi) of Corollary 1.11 in [7], $(P_\alpha x)y^{-1} = e$, hence $P_\alpha x = y \in \text{Fix}(\mathcal{S})$ and this holds for each $x \in C$.

Since $P_\alpha \in \mathfrak{R}$, $T_\alpha \in \mathcal{S}$ and $\{P_\alpha x\}$ is $\mathcal{S}$-invariant for each $x \in C$, hence, it must be P_α-invariant for each $x \in C$. Then we conclude that $P_\alpha^2 = P_\alpha$ and $P_\alpha T_\alpha = T_\alpha P_\alpha = P_\alpha$. As a consequence of Theorem 3.1, we establish an ergodic retraction by a group nonexpansive retraction.

Theorem 3.2 *Let G be a topological group and let C be a compact subset of G. Suppose that $\mathcal{S} = \{T_i : i \in I\}$ is a family of group nonexpansive mappings on C such that Fix($\mathcal{S}$) $\neq \emptyset$ and for every $\alpha \in I$, there exists a subnet $\{T_\alpha^{n_\gamma}\}$ of the sequence $\{T_\alpha^n\}$ such that $\lim_\gamma T_\alpha^{n_\gamma} x = \lim_\gamma T_\alpha^{n_\gamma - 1} x$ for each $x \in C$. Also suppose for every nonempty compact $\mathcal{S}$-invariant subset K of C, $K \cap$ Fix($\mathcal{S}$) $\neq \emptyset$. If there is a group nonexpansive retraction R from C onto Fix($\mathcal{S}$), then for each $i \in I$, there exists a group nonexpansive retraction P_i from C onto Fix($\mathcal{S}$), such that $P_i T_i = T_i P_i = P_i$, and every closed $\mathcal{S} \cup \{R\}$-invariant subset of C is also P_i-invariant.*

Proof. Set $\mathcal{S}' := \mathcal{S} \cup \{R\}$ and $\mathfrak{R}' = \{T \in C^C : T$ is group nonexpansive, $T \circ T_\alpha = T$ and every closed $\mathcal{S}'$-invariant subset of C is also T -invariant$\}$,
and we get that Fix($\mathcal{S}'$) = Fix($\mathcal{S}$) and by replacing $\mathcal{S}$ with $\mathcal{S}'$ and $\mathfrak{R}$ with $\mathfrak{R}'$ in the proof of Theorem 3.1, we find a minimal element P_α in the sense of ($*$). Now we have $R \circ T \in \mathfrak{R}'$ for each $T \in \mathfrak{R}'$. Indeed, $R \circ T \circ T_\alpha = R \circ T$ for each $T \in \mathfrak{R}'$ and because $R \in \mathcal{S}'$, we have that every closed $\mathcal{S}'$-invariant subset of C is also R -invariant, and therefore is $R \circ T$-invariant for each $T \in \mathfrak{R}'$. Hence for each $x \in C$, the set $K = \{T(P_\alpha x) : T \in \mathfrak{R}'\}$ is an R-invariant subset of C for each $T \in \mathfrak{R}'$. Therefore from the fact that $R(K) \subset K \cap R(C) = K \cap \text{Fix}(\mathcal{S})$, we have $K \cap \text{Fix}(\mathcal{S}') = K \cap \text{Fix}(\mathcal{S}) \neq \emptyset$. Now by repeating the reasoning used in Theorem 3.1, we get the desired result.

As an application of Theorem 3.2, we have the following result:

Theorem 3.3. *Let G be a topological group with the topology τ and let C be a compact subset of G. Suppose that $\mathcal{S} = \{T_i : i \in I\}$ is a family of group nonexpansive mappings on C such that Fix($\mathcal{S}$) $\neq \emptyset$ and for every $\alpha \in I$, there exists a subnet $\{T_\alpha^{n_\gamma}\}$ of the sequence $\{T_\alpha^n\}$ such that $\lim_\gamma T_\alpha^{n_\gamma} x = \lim_\gamma T_\alpha^{n_\gamma - 1} x$ for each $x \in C$. Consider the following assumptions:*
(a) Suppose for every nonempty compact $\mathcal{S}$-invariant subset K of C, $K \cap$ Fix($\mathcal{S}$) $\neq \emptyset$,
(b) there exists a group nonexpansive retraction R from C onto Fix($\mathcal{S}$).
Let $\{P_i\}_{i \in I}$ be the family of retractions obtained in the above Theorem. Then for each $x \in C$,

$$\overline{\{T_i^n x : i \in I, n \in \mathbb{N}\}}^\tau \cap Fix(\mathcal{S}) \subseteq \overline{\{P_i(x) : i \in I\}}^\tau.$$

Proof. Let $g \in \overline{\{T_i^n x : i \in I, n \in \mathbb{N}\}}^\tau \cap \text{Fix}(\mathcal{S})$. Then for each $U \in \mathfrak{B}_e$, there exists $i \in I$ and $n \in \mathbb{N}$ such that $T_i^n x g^{-1} \in U$. From our assumptions and using Theorems 3.1 and 3.2, there exists a group nonexpansive retraction P_i such that $P_i = P_i T_i$ and since from Theorems 3.1 and 3.2 every closed $\mathcal{S}$-invariant or $\mathcal{S} \cup \{R\}$-invariant subset of C is also P_i-invariant then we have $P_i g = g$ for each $i \in I$. Hence from the fact that P_i is group nonexpansive and since $T_i^n x g^{-1} \in U$ then we have,

$$(P_i x) g^{-1} = (P_i T_i^n x)(P_i T_i^n g)^{-1} \in U,$$

and then we conclude $g \in \overline{\{P_i(x) : i \in I\}}^\tau$.

4. Examples and applications

Recall every locally convex space is a topological group by the topology generated by a family of seminorms. Theorem 3.1 extends and generalizes Theorem 4.1(a) in [3], by removing the "convex" and "separated" conditions as follows:

Corollary 4.1. *Suppose that Q is a family of seminorms on a locally convex space E which determines the topology of E. Let C be a compact subset of E. Suppose that $\mathcal{S} = \{T_i : i \in I\}$ is a family of Q-nonexpansive mappings on C such that $Fix(\mathcal{S}) \neq \emptyset$ and for every $\alpha \in I$, there exists a subnet $\{T_\alpha^{n_\gamma}\}$ of the sequence $\{T_\alpha^n\}$ such that $\lim_\gamma T_\alpha^{n_\gamma} x = \lim_\gamma T_\alpha^{n_\gamma - 1} x$ for each $x \in C$. If for every nonempty compact $\mathcal{S}$-invariant subset K of C, $K \cap Fix(\mathcal{S}) \neq \emptyset$, then, for each $i \in I$, there exists a group nonexpansive retraction P_i from C onto $Fix(\mathcal{S})$, such that $P_i T_i = T_i P_i = P_i$ and every closed $\mathcal{S}$-invariant subset of C is also P_i-invariant.*

Theorem 3.1 extends and generalizes Theorem 2.1(a) in [11], by removing the "convex" condition as follows:

Corollary 4.2. *Let C be a compact subset of a Banach space E. Suppose that $\mathcal{S} = \{T_i : i \in I\}$ is a family of nonexpansive mappings on C such that $Fix(\mathcal{S}) \neq \emptyset$ and for every $\alpha \in I$, there exists a subnet $\{T_\alpha^{n_\gamma}\}$ of the sequence $\{T_\alpha^n\}$ such that $\lim_\gamma T_\alpha^{n_\gamma} x = \lim_\gamma T_\alpha^{n_\gamma - 1} x$ for each $x \in C$. If for every nonempty compact $\mathcal{S}$-invariant subset K of C, $K \cap Fix(\mathcal{S}) \neq \emptyset$, then, for each $i \in I$, there exists a group nonexpansive retraction P_i from C onto $Fix(\mathcal{S})$, such that $P_i T_i = T_i P_i = P_i$ and every closed $\mathcal{S}$-invariant subset of C is also P_i-invariant.*

In the following example for Theorem 3.1, we present a family of retractions without assuming a convexity condition on K.

Example 4.3. Let $G = \mathbb{R}$ with the usual topology and $C = [0, 2]$. Suppose that $\mathcal{S} = \{T_n : n = 2, 3, 4, \cdots\}$ is a family of the group nonexpansive mappings on C such that

$$T_n(x) = \begin{cases} x, & x \in [0, \frac{2n}{2n-1}]; \\ (\frac{1}{n} - 1)x + 2, & x \in (\frac{2n}{2n-1}, 2]. \end{cases}$$

First note that $T_n^2 x = T_n x$ for each $x \in [0, 2]$. Indeed if $x \in [0, \frac{2n}{2n-1}]$ then

$$T_n^2 x = T_n(T_n x) = x = T_n x.$$

Next, let $x \in (\frac{2n}{2n-1}, \frac{n}{n-1})$ and then $1 < (\frac{1}{n} - 1)x + 2 < \frac{2n}{2n-1}$ so

$$T_n^2 x = T_n((\frac{1}{n} - 1)x + 2) = \left(\frac{1}{n} - 1\right)x + 2 = T_n x.$$

Finally, if $x \in [\frac{n}{n-1}, 2]$, then we have $0 \leq (\frac{1}{n} - 1)x + 2 \leq 1$, and hence

$$T_n^2 x = T_n(T_n x) = T_n\left(\left(\frac{1}{n} - 1\right)x + 2\right) = \left(\frac{1}{n} - 1\right)x + 2 = T_n x,$$

so $T_n^2 = T_n$. Hence, $\lim_{m\to\infty} T_n^m x = \lim_{m\to\infty} T_n^{m-1} x = \lim_{m\to\infty} T_n x = T_n x$ for each $x \in [0, 2]$. Then the condition in Theorem 3.1 is true.

Next we show that T_n is group nonexpansive. Let $x \in (\frac{2n}{2n-1}, 2]$ and $y \in [0, \frac{2n}{2n-1}]$. Note $2(y-1) \le \frac{2}{2n-1} \le \frac{1}{n}x$ so $-\frac{1}{n}x \le -2(y-1)$, $-\frac{1}{n}x - 2 \le -2y$ and hence

$$-2(x-y) \le \left(\frac{1}{n} - 2\right)x + 2 \le 0,$$

and therefore since $x - y \ge 0$ we have

$$\begin{aligned}|T_n(x) - T_n(y)| &= \left|\left(\frac{1}{n} - 1\right)x + 2 - y\right| = \left|x - y + \left(\frac{1}{n} - 2\right)x + 2\right| \\ &\le |x - y - 2(x-y)| = |x - y|.\end{aligned}$$

The other cases are easy. Hence T_n is a group nonexpansive mapping, for each $n = 2, 3, \ldots$. Note $\mathrm{Fix}(\mathcal{S}) = [0,1]$. Also for every nonempty compact $\mathcal{S}$-invariant subset K of C, $K \cap \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Indeed, since K is $\mathcal{S}$-invariant, then for each $x \in K \cap [1,2]$, there exists a $n_1 \in \mathbb{N}$ such that $x \in [\frac{2n_1}{2n_1-1}, 2]$ and $T_n(x) = (\frac{1}{n} - 1)x + 2 \in K$ for each $n \ge n_1$. Let $n \to \infty$ (note K is closed) so $-x + 2 \in K$. However $-x + 2 \in [0, \frac{2n_1-2}{2n_1-1}] \subset [0,1]$, so $K \cap \mathrm{Fix}(\mathcal{S}) = K \cap [0,1] \neq \emptyset$. We now show that $1 \in K$. Indeed, $-x + 2 \in K$, for each $x \in (1,2]$, because if $x \in (1,2]$ then there exists an integer $n_2 \in \mathbb{N}$ such that $x \in (\frac{2n_2}{2n_2-1}, 2]$ and therefore for each $n \ge n_2$, $T_n(x) = (\frac{1}{n} - 1)x + 2 \in K$, so let $n \to \infty$ and we have $-x + 2 \in K$ for each $x \in (1,2]$. Put $x = 0.1, 0.01, 0.001, \ldots$ and we get $0.9, 0.99, 0.999, \ldots \in K$, and hence since K is closed, $1 \in K$.

Now define P_n by:

$$P_n(x) = \begin{cases} x, & x \in [0,1]; \\ 1, & x \in (1, \frac{n}{n-1}); \\ (\frac{1}{n} - 1)x + 2, & x \in [\frac{n}{n-1}, 2]. \end{cases}$$

First, we show that for each $n = 2, 3, \cdots$, $P_n^2 = P_n$. Let $x \in [0,1]$ and then

$$P_n^2 x = P_n(P_n x) = P_n x.$$

Next, let $x \in (1, \frac{n}{n-1})$ and then $P_n^2 x = P_n(P_n x) = P_n(1) = 1 = P_n x$. Finally, if $x \in [\frac{n}{n-1}, 2]$, then we have $0 \le (\frac{1}{n} - 1)x + 2 \le 1$, and hence

$$P_n^2 x = P_n(P_n x) = P_n\left(\left(\frac{1}{n} - 1\right)x + 2\right) = \left(\frac{1}{n} - 1\right)x + 2 = P_n x,$$

so $P_n^2 = P_n$.

Next we show that for each $n = 2, 3, \cdots$, P_n is a group nonexpansive retraction from C onto $\mathrm{Fix}(\mathcal{S})$. If $x \in [\frac{n}{n-1}, 2]$ and $y \in (1, \frac{n}{n-1})$, we have $\frac{x}{n} \ge \frac{1}{n-1}$ and then $y - \frac{x}{n} \le 1$, so $x - \frac{x}{n} - 1 \le x - y$ and hence we get $(\frac{1}{n} - 1)x + 1 \le x - y$, so

$$|P_n(x) - P_n(y)| \le |x - y|.$$

If $x \in [\frac{n}{n-1}, 2]$ and $y \in [0,1]$, then we have $2(y-1) \le \frac{1}{n}x$ and then $-\frac{1}{n}x \le -2(y-1)$, so $-\frac{1}{n}x - 2 \le -2y$, and hence

$$-2(x-y) \le \left(\frac{1}{n} - 2\right)x + 2 \le 0,$$

and therefore since $x-y\geq 0$ we have

$$|P_n(x)-P_n(y)| = \left|\left(\frac{1}{n}-1\right)x+2-y\right| = \left|x-y+\left(\frac{1}{n}-2\right)x+2\right| \\ \leq |x-y-2(x-y)| = |x-y|.$$

The other cases are easy. Hence P_n is a group nonexpansive mapping. Next to show $P_nT_n = T_nP_n = P_n$. First we prove $T_nP_n = P_n$. The case $x\in[0,1]$ is clear. Let $x\in(1,\frac{n}{n-1})$. Then we have $T_nP_nx = T_n(1) = 1 = P_nx$. Finally, let $x\in[\frac{n}{n-1},2]$. Then we have $\frac{2}{n}\leq(\frac{1}{n}-1)x+2\leq 1$ so

$$T_nP_nx = T_n\left(\left(\frac{1}{n}-1\right)x+2\right) = \left(\frac{1}{n}-1\right)x+2 = P_nx.$$

Next we show $P_nT_n = P_n$. Let $x\in(0,\frac{2n}{2n-1})$. Clearly we have $P_nT_nx = P_nx$. Let $x\in(\frac{2n}{2n-1},2]$. Then we have

$$P_nT_nx = P_n((\frac{1}{n}-1)x+2) = P_nx;$$

to see this we consider two cases; (a): if $x\in(\frac{2n}{2n-1},\frac{n}{n-1}]$, then we have

$$1\leq(\frac{1}{n}-1)x+2\leq\frac{2n}{2n-1}<\frac{n}{n-1},$$

so

$$P_nT_nx = P_n\left(\left(\frac{1}{n}-1\right)x+2\right) = 1 = P_nx;$$

(b): if $x\in(\frac{n}{n-1},2]$, then we have $0\leq(\frac{1}{n}-1)x+2\leq 1$, so

$$P_nT_nx = P_n\left(\left(\frac{1}{n}-1\right)x+2\right) = \left(\frac{1}{n}-1\right)x+2 = P_nx.$$

Now, we show that every closed $\mathcal{S}$-invariant subset K of C is also P_n-invariant for each $n=2,3,\cdots$. First if $x\in K\cap[0,1]$ then $P_nx = x\in K$. Next if $x\in K\cap(1,\frac{n}{n-1})$ then $P_nx = 1\in K$. Finally, if $x\in K\cap[\frac{n}{n-1},2]$, then $x\in K\cap[\frac{2n}{2n-1},2]$, and hence from the fact that $P_nx = (\frac{1}{n}-1)x+2 = T_nx\in K$, we have $P_nx\in K$. Therefore, K is also P_n-invariant for each $n=2,3,\cdots$.

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