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COUPLED BEST PROXIMITY POINTS FOR CYCLIC CONTRACTIVE MAPS AND THEIR APPLICATIONS

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Abstract. We enrich the known results about coupled fixed points and coupled best proximity points. We generalize the notion of ordered pairs of cyclic contraction maps and we obtain sufficient conditions for the existence and uniqueness of best proximity points. We get a priori and a posteriori error estimates for the coupled fixed points and for the coupled best proximity points, provided that the underlying Banach space has modulus of convexity of power type in the case of best proximity points, obtained by sequences of successive iterations. We illustrate the main result with an example.
Key Words and Phrases Coupled best proximity points, uniformly convex Banach space, modulus of convexity, a priori error estimate, a posteriori error estimate, system of linear equations.
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1. Introduction

The Banach contraction principle states that in a complete metric space (X, ρ) any contraction map $T : X \to X$ has a fixed point, i.e. $\min\{\rho(x, Tx) : x \in X\} = 0$. A lot of results in modelling real world processes in applied mathematics lead to the problem of finding $\min\{\rho(x, Tx) : x \in X\}$. It may happen that the above minimum is greater than zero. One approach for solving the above mentioned problems uses the notion of a best proximity point is introduced in [14], where a sufficient condition for the existence and the uniqueness of best proximity points in uniformly convex Banach spaces is obtained.

A constructed model may depends on two parameters, i.e. $F : X \times X \to X$. The notion of coupled fixed points [17] and of a coupled best proximity points for an ordered pair (F, G), $F : A \times A \to B$, $G : B \times B \to A$, where $A, B \subset X$ [18, 25], is relevant in this context. Deep results in the theory of coupled fixed points can be found for example in [3, 4, 6].

There are many problems about fixed points and best proximity points that are not easy to be solved or can not be solved exactly. One of the advantages of the Banach fixed point theorem is the error estimates of the successive iterations and the rate of convergence. That is why an estimation of the error when an iterative process is used is of interest, when fixed points or best proximity points are investigated. An extensive study about approximations of fixed points can be found in [2].

A first result in the approximation of the sequence of successive iterations, which converges to the best proximity point for cyclic contractions, is obtained in [27]. This result was expanded for a coupled best proximity point in [21].

We have tried to enrich the known results about coupled best proximity points for order pairs of cyclic contraction maps (F,G), by proving that the coupled best proximity points $(x,y)\in A\times A$ reduce to the point $(x,x)\in A\times A$.

In order to get a general result for the existence of coupled best proximity points $(x,y)\in A\times A$ with $x\neq y$, we needed to consider an ordered pair of an order pair of maps $((F,f),(G,g))$, such that $F:A_1\times A_2\to B_1$, $f:A_1\times A_2\to B_2$, $G:B_1\times B_2\to A_1$, $g:B_1\times B_2\to A_2$, where $A_1,A_2,B_1,B_2\subset X$.

2. Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the best proximity points. Let (X,ρ) be a metric space. We define a distance between two subsets $A,B\subset X$ by $\operatorname{dist}(A,B)=\inf\{\rho(x,y):x\in A,y\in B\}$.

The concept of a coupled best proximity point theorem is introduced in [25].

Definition 2.1. ([25]) Let A and B be nonempty subsets of a metric space (X,ρ), $F:A\times A\to B$. An ordered pair $(x,y)\in A\times A$ is called a coupled best proximity point of F if

$$\rho(x,F(x,y))=\rho(y,F(y,x))=\operatorname{dist}(A,B).$$

Definition 2.2. ([17]) Let A and B be nonempty subsets of a metric space (X,ρ), $F:A\times A\to A$. An ordered pair $(x,y)\in A\times A$ is said to be a coupled fixed point of F in A if $x=F(x,y)$ and $y=F(y,x)$.

It is easy to see that if $A=B$ in Definition 2.1, then a coupled best proximity point reduces to a coupled fixed point.

The notion of an order pair (F,G) of cyclic contraction maps, which generalizes the notion of a cyclic contraction map [14], is introduced in [25].

Definition 2.3. ([18, 25]) Let A and B be nonempty subsets of a metric space (X,ρ), $F:A\times A\to B$ and $G:B\times B\to A$. The ordered pair (F,G) is said to be a cyclic contraction if there exist non-negative numbers α,β, such that $\alpha+\beta<1$ and there holds the inequality

$$\rho(F(x,y),G(u,v))\leq\alpha\rho(x,u)+\beta\rho(y,v)+(1-(\alpha+\beta))d(A,B)$$

for all $(x,y)\in A\times A$ and $(u,v)\in B\times B$.

If $\alpha=\beta$ in the above definition we get the definition from [25].

Just to fit some of the formulas in the text field let us denote $d_x=d(A_x,B_x)$ and $d_y=d(A_y,B_y)$.

Definition 2.4. Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ), $F:A_x\times A_y\to B_x$, $f:A_x\times A_y\to B_y$, $G:B_x\times B_y\to A_x$ and $g:B_x\times B_y\to A_y$. The ordered pair of orderer pairs $((F,f),(G,g))$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$, such that $\max\{\alpha+\gamma, \beta+\delta\} < 1$ and there holds the inequality

$$\begin{array}{rcl} S_1 & = & \rho(F(x,y), G(u,v)) + \rho(f(z,w), g(t,s)) \\ & \leq & \alpha\rho(x,u) + \beta\rho(y,v) + \gamma\rho(z,t) + \delta\rho(w,s) \\ & & +(1-(\alpha+\gamma))d_x + (1-(\beta+\delta))d_y \end{array} \tag{2.1}$$

for all $(x,y), (z,w) \in A_x \times A_y$ and $(u,v), (t,s) \in B_x \times B_y$.

Definition 2.5. Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X, ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$. An ordered pair $(\xi, \eta) \in A_x \times A_y$ is called a coupled best proximity point of (F, f) if

$$\rho(\xi, F(\xi,\eta)) = \operatorname{dist}(A_x, B_x) \quad \text{and} \quad \rho(\eta, f(\xi,\eta)) = \operatorname{dist}(A_y, B_y).$$

Definition 2.6. Let A_x, A_y, B_x and B_y be nonempty subsets of X. Let $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. For any pair $(x,y) \in A_x \times A_y$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and

$$\begin{array}{ll} x_{2n+1} = F(x_{2n}, y_{2n}), & y_{2n+1} = f(x_{2n}, y_{2n}) \\ x_{2n+2} = G(x_{2n+1}, y_{2n+1}), & y_{2n+2} = g(x_{2n+1}, y_{2n+1}) \end{array}$$

for all $n \geq 0$.

Everywhere, when considering the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ we will assume that they are the sequences defined in Definition 2.6.

Let us put $A_x = A_y = A$, $B_x = B_y = B$, $D = d(A,B)$, $f(x,y) = F(y,x)$, $g(x,y) = G(y,x)$, $z = y$, $w = x$, $t = v$, $s = u$, $\gamma = \beta$ and $\delta = \alpha$ in Definition 2.4. Then

$$\begin{array}{rcl} S_2 & = & 2\rho(F(x,y), G(u,v)) \\ & = & \rho(F(x,y), G(u,v)) + \rho(f(y,x), g(v,u)) \\ & = & \rho(F(x,y), G(u,v)) + \rho(f(z,w), g(t,s)) \\ & \leq & \alpha\rho(x,u) + \beta\rho(y,v) + \gamma\rho(z,t) + \delta\rho(w,s) + (2-(\alpha+\beta+\gamma+\delta))D \\ & = & \alpha\rho(x,u) + \beta\rho(y,v) + \beta\rho(y,v) + \alpha\rho(x,u) + (2-2(\alpha+\beta))D \\ & = & 2(\alpha\rho(x,u) + \beta\rho(y,v) + (1-(\alpha+\beta))D, \end{array} \tag{2.2}$$

which is the condition from Definition 2.3, because $\alpha+\beta = \max\{\alpha+\gamma, \beta+\delta\} < 1$. Thus the ordered pairs (F, G) of cyclic contractions is a particular case of the orderer pairs $((F,f),(G,g))$ cyclic contractions.

The best proximity results need norm-structure of the space X.

When we investigate a Banach space $(X, \|\cdot\|)$, we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$ i.e. $\rho(x,y) = \|x-y\|$. We will denote the unit sphere and the unit ball of a Banach space $(X, \|\cdot\|)$ by S_X and B_X respectively.

The assumption that the Banach space $(X, \|\cdot\|)$ is uniformly convex plays a crucial role in the investigation of best proximity points.

Definition 2.7. Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0, 2]$ we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The space $(X, \|\cdot\|)$ is then called a uniformly convex space.

The next two lemmas a crucial in the investigations of best proximity points in uniformly convex Banach spaces.

Lemma 2.8. ([14]) *Let A be a nonempty closed, convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A,B)$*;*
2) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A,B)$*;*
then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.9. ([14]) *Let A be a nonempty closed, convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A,B)$*;*
2) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$,

$$\|x_n - y_n\| \leq \operatorname{dist}(A,B) + \varepsilon,$$

then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, holds

$$\|x_m - z_n\| \leq \varepsilon.$$

For obtaining error estimates for the sequence of successive iterations that approximates the best proximity point, which is generated by a cyclic contraction, the modulus of convexity $\delta_{(X,\|\cdot\|)}$ is used [27].

For any uniformly convex Banach space X there holds the inequality

$$\left\| \frac{x+y}{2} - z \right\| \leq \left(1 - \delta_X\left(\frac{r}{R}\right)\right) R \tag{2.3}$$

for any $x, y, z \in X$, $R > 0$, $r \in [0, 2R]$, $\|x - z\| \leq R$, $\|y - z\| \leq R$ and $\|x - y\| \geq r$ [14].

If $(X, \|\cdot\|)$ is a uniformly convex Banach space, then $\delta_X(\varepsilon)$ is a strictly increasing function. Therefore if $(X, \|\cdot\|)$ is a uniformly convex Banach space, then there exists the inverse function δ^{-1} of the modulus of convexity. If there exist constants $C > 0$ and $q > 0$, such that the inequality $\delta_{\|\cdot\|}(\varepsilon) \geq C\varepsilon^q$ holds for every $\varepsilon \in (0,2]$, we say that the modulus of convexity is of power type q. It is well known that for any Banach space and for any norm there holds the inequality $\delta(\varepsilon) \leq K\varepsilon^2$. The modulus of convexity with respect to the canonical norm $\|\cdot\|_p$ in ℓ_p or L_p is

$$\delta_{\|\cdot\|_p}(\varepsilon) = 1 - \sqrt[p]{1 - \left(\frac{\varepsilon}{2}\right)^p}$$

for $p \geq 2$ and for $1 < p < 2$ the modulus of convexity $\delta_{\|\cdot\|_p}(\varepsilon)$ is the solution of the equation

$$\left(1 - \delta + \frac{\varepsilon}{2}\right)^p + \left|1 - \delta - \frac{\varepsilon}{2}\right|^p = 2.$$

It is well known that the modulus of convexity with respect to the canonical norm in ℓ_p or L_p is of power type and there hold the inequalities $\delta_{\|\cdot\|_p}(\varepsilon) \geq \frac{\varepsilon^p}{p2^p}$ for $p \geq 2$ and $\delta_{\|\cdot\|_p}(\varepsilon) \geq \frac{(p-1)\varepsilon^2}{8}$ for $p \in (1,2)$ [24].

An extensive study of the Geometry of Banach spaces can be found in [1, 13, 16].

3. Comments on the known results about coupled best proximity points

We would like to pay attention to the known examples for using the technique of best proximity points in solving of systems of linear equations.

Theorem 3.1. [18, 22, 25] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Let $F : A \times A \to B$, $G : B \times B \to A$ and the ordered pair (F, G) be a cyclic contraction. Then F has a unique coupled best proximity point $(\xi, \eta) \in A \times A$ and G has a unique coupled best proximity point $(\zeta, \varsigma) \in B \times B$, (i.e. $\|\xi - F(\xi, \eta)\| = \|\eta - F(\eta, \xi)\| = d$ and $\|\zeta - G(\zeta, \varsigma)\| = \|\varsigma - G(\varsigma, \zeta)\| = d$). Moreover there hold*

$$\begin{array}{c} G(F(\xi,\eta),F(\eta,\xi)) = \xi,\ G(F(\eta,\xi),F(\xi,\eta)) = \eta, \\ F(G(\zeta,\varsigma),G(\varsigma,\zeta)) = \zeta,\ F(G(\varsigma,\zeta),F(\zeta,\varsigma)) = \varsigma \end{array} \tag{3.1}$$

and

$$\zeta = F(\xi,\eta),\ \varsigma = F(\eta,\xi),\ \xi = G(\zeta,\varsigma),\ \eta = G(\varsigma,\zeta). \tag{3.2}$$

For any arbitrary point (x, y) there hold

$$\lim_{n\to\infty} x_{2n} = \xi,\ \lim_{n\to\infty} y_{2n} = \eta,\ \lim_{n\to\infty} x_{2n+1} = \zeta,\ \lim_{n\to\infty} y_{2n+1} = \varsigma$$

and

$$\|\xi - \zeta\| + \|\eta - \varsigma\| = 2\mathrm{dist}(A, B).$$

As it is pointed out in [21] the space $(\mathbb{R}, |\cdot|)$ is a uniformly convex Banach space. The next two examples are solved with the help of ordered pairs of cyclic contractions and coupled best proximity points [22, 25]. Error estimates of the sequences of successive iterations are obtained in [21].

Example 3.2. [20, 22, 25] Let us consider the space $\mathbb{R}$, endowed with the canonical norm $|\cdot|$ and $A = [1, 2]$. We search for the solutions of the system

$$\left\{ \begin{array}{rcl} 5x + y & = & 6 \\ 5y + x & = & 6, \end{array} \right. \tag{3.3}$$

which belong to the set A.

Solution. Let us denote $B = [-2, -1]$. Let $F : A \times A \to B$ and $G : B \times B \to A$ be defined by

$$F(x, y) = \frac{-x - y - 2}{4} \text{ and } G(x, y) = \frac{-x - y + 2}{4}.$$

It is easy to observe that the pair $(\xi, \eta) \in A \times A$ is a solution of (3.3) if and only if $|\xi - F(\xi, \eta)| = |\eta - F(\eta, \xi)| = \mathrm{dist}(A, B)$. Thus the solution $(2, 2)$ of (3.3) can be obtained if we can solve the problem for the coupled best proximity points (ξ, η) of F in $A \times A$. □

Example 3.3. [18, 20, 22] Let us consider the space $\mathbb{R}$ endowed with the canonical norm $|\cdot|$, $A=[1,2]$. We search for the solutions of the system

$$\left\{\begin{array}{rcl} 8x+3y &=& 11 \\ 8y+3x &=& 11, \end{array}\right. \tag{3.4}$$

which belong to the set A.

Solution. Let us denote $B=[-2,-1]$. Let $F:A\times A\to B$ and $G:B\times B\to A$ be defined by

$$F(x,y)=\frac{-2x-3y-1}{6} \text{ and } G(x,y)=\frac{-2x-3y+1}{6}.$$

It is easy to observe that the pair $(\xi,\eta)\in A\times A$ is a solution of (3.4) if and only if $|\xi-F(\xi,\eta)|=|\eta-F(\eta,\xi)|=\operatorname{dist}(A,B)$. Thus the solution $(2,2)$ of (3.4) can be obtained if we can solve the problem for the coupled best proximity points (ξ,η) of F in $A\times A$. □

It is interesting to see that in both examples $\xi=\eta$. It turns out that this is not just a coincidence.

The next theorem enriches Theorem 3.1 by showing that $\xi=\eta$ and $\zeta=\varsigma$.

Theorem 3.4. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Let $F:A\times A\to B$, $G:B\times B\to A$ and the ordered pair (F,G) be a cyclic contraction. Then F has a unique coupled best proximity point $(\xi,\eta)\in A\times A$ and $\xi=\eta$ and G has a unique coupled best proximity point $(\zeta,\varsigma)\in B\times B$.*

Proof. By Theorem 3.1 it follows that there exists a unique coupled best proximity point (ξ,η) of F. From Theorem 3.1 it follows that $\xi=G(F(\xi,\eta),F(\eta,\xi))$ and $\eta=G(F(\eta,\xi),F(\xi,\eta))$. Therefore we get

$$\begin{array}{rcl} \|\eta-F(\xi,\eta)\| &=& \|G(F(\eta,\xi),F(\xi,\eta))-F(\xi,\eta)\| \\ &\le& \alpha\|(\xi,F(\eta,\xi)\|+\beta\|(\eta-F(\xi,\eta)\|+(1-(\alpha+\beta))\operatorname{dist}(A,B) \end{array}$$

and

$$\begin{array}{rcl} \|\xi-F(\eta,\xi)\| &=& \|G(F(\xi,\eta),F(\eta,\xi))-F(\eta,\xi)\| \\ &\le& \alpha\|\eta-F(\xi,\eta)\|+\beta\|\xi-F(\eta,\xi)\|+(1-(\alpha+\beta))\operatorname{dist}(A,B). \end{array}$$

Summing the last two inequalities we get

$$\|\eta-F(\xi,\eta)\|+\|\xi-F(\eta,\xi\|\le 2\operatorname{dist}(A,B).$$

Consequently $\|\eta-F(\xi,\eta)\|=\|\xi-F(\eta,\xi)\|=\operatorname{dist}(A,B)$. Using the uniform convexity of $(X,\|\cdot\|)$, the equalities $\|\xi-F(\xi,\eta)\|=\|\eta-F(\eta,\xi)\|=\operatorname{dist}(A,B)$ and Lemma 2.8 we get that $\xi=\eta$.

It can be proven in a similar fashion that $\zeta=\varsigma$. □

The same observation is made in [3, 8] for coupled fixed points in partially ordered metric spaces.

Theorem 3.5. (*Theorem 2.4* [8]) *Let (X,ρ) be a partially ordered set and ρ be a metric such that (X,ρ) be a complete metric space. Let $F:X\times X\to X$ be a* *continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$\rho(F(x, y), F(u, v)) \leq \frac{k}{2}(\rho(x, u) + \rho(y, v)) \tag{3.5}$$

all $x \geq u$ and all $y \leq v$. If there are $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exists a coupled fixed point $(x, y) \in X \times X$. If in addition every pair of elements of X has an upper bound or a lower bound in X, then $x = y$.

Let us point out that condition (3.5) does not hold for all $x, y \in X$. That is why the additional condition is needed to assure that $x = y$. If (3.5) holds for all $x, y \in X$, then we get right away that $x = y$. Indeed there holds the inequality $\rho(x, y) = \rho(F(x, y), F(y, x) \leq \frac{k}{2}(\rho(x, y) + \rho(y, x)) = k\rho(x, y)$. Because of $k \in [0, 1)$ it follows that $x = y$.

The next result enriches the results from [21] by proving that the coupled fixed point (ξ, η) in $A \cap B$ satisfies $\xi = \eta$.

Theorem 3.6. *Let A and B be nonempty closed subsets of a complete metric space (X, ρ) and $F : A \times A \to B$ and $G : B \times B \to A$. Let there exist $\alpha, \beta > 0$, $\alpha + \beta < 1$, such that*

$$\rho(F(x, y), G(u, v)) \leq \alpha\rho(x, u) + \beta\rho(y, v) \tag{3.6}$$

for all $x, y \in A$ and $u, v \in B$. Then there exists a unique pair (ξ, η) in $A \cap B$, which is a common coupled fixed point for the maps F and G. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ for any arbitrary initial guess $(x, y) \in A \times A$, defined in Definition 2.6 converge to ξ and η respectively and moreover $\xi = \eta$.

Proof. Theorem 3.6 is proven [21] except for $\xi = \eta$. By [21] there exists (ξ, η) in $A \cap B$, which is the unique coupled fixed point for the map F and (ξ, η) is a coupled fixed point for the map G too. Then

$$\rho(\xi, \eta) = \rho(F(\xi, \eta), G(\eta, \xi) \leq \alpha\rho(\xi, \eta) + \beta\rho(\eta, \xi)) = (\alpha + \beta)\rho(\xi, \eta).$$

Because of $\alpha + \beta \in [0, 1)$ it follows that $\xi = \eta$. □

4. Main result

Just to fit some of the formulas in the text field we will denote

$$P_{n,m}(x, y) = \|x_n - x_m\| + \|y_n - y_m\|$$

and

$$W_{n,m}(x, y) = P_{n,m}(x, y) - (d_x + d_y) = \|x_n - x_m\| + \|y_n - y_m\| - (d_x + d_y),$$

where $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences defined in Definition 2.6 and

$$k = \max\{\alpha + \gamma, \beta + \delta\},$$

where $\alpha, \beta, \gamma, \delta$ are the constants from Definition 2.4.

Theorem 4.1. *Let A_x, A_y, B_x and B_y be nonempty convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let the ordered pair $((F, f),(G, g))$ be a cyclic contraction. Then (F, f) has a unique coupled best proximity point $(\xi, \eta) \in A_x \times A_y$ and (G, g) has a unique coupled best proximity point $(\zeta, \varsigma) \in B_x \times B_y$, (i.e. $\|\xi - F(\xi, \eta)\| = d_x$, $\|\eta - f(\xi, \eta)\| = d_y$ and $\|\zeta - G(\zeta, \varsigma)\| = d_x$, $\|\varsigma - g(\zeta, \varsigma)\| = d_y$). Moreover $\zeta = F(\xi, \eta)$, $\varsigma = f(\xi, \eta)$, $\xi = G(\zeta, \varsigma)$ and $\eta = g(\zeta, \varsigma)$. For any arbitrary point $(x, y) \in A \times A$ there hold*

$$\lim_{n\to\infty} x_{2n} = \xi, \ \lim_{n\to\infty} y_{2n} = \eta, \ \lim_{n\to\infty} x_{2n+1} = \zeta, \ \lim_{n\to\infty} y_{2n+1} = \varsigma$$

and

$$\|\xi - \zeta\| + \|\eta - \varsigma\| = d_x + d_y.$$

Moreover there hold

$$\begin{array}{c} G(F(\xi, \eta), f(\xi, \eta)) = \xi, \ g(F(\xi, \eta), f(\xi, \eta)) = \eta, \\ F(G(\zeta, \varsigma), g(\zeta, \varsigma)) = \zeta, \ f(G(\zeta, \varsigma), g(\zeta, \varsigma)) = \varsigma. \end{array} \tag{4.1}$$

If in addition $(X, \|\cdot\|)$ has a modulus of convexity of power type with constants $C > 0$ and $q > 1$, then

(i) *a priori error estimates hold*

$$\|\xi - x_{2m}\| \le P_{0,1}(x, y) \sqrt[q]{\frac{W_{0,1}(x, y)}{Cd_x}} \cdot \frac{\sqrt[q]{k^{2m}}}{1 - \sqrt[q]{k^2}}; \tag{4.2}$$

$$\|\eta - y_{2m}\| \le P_{0,1}(x, y) \sqrt[q]{\frac{W_{0,1}(x, y)}{Cd_y}} \cdot \frac{\sqrt[q]{k^{2m}}}{1 - \sqrt[q]{k^2}}; \tag{4.3}$$

(ii) *a posteriori error estimates hold*

$$\|\xi - x_{2n}\| \le P_{2n,2n-1}(x, y) \sqrt[q]{\frac{W_{2n,2n-1}(x, y)}{Cd_x}} \cdot \frac{\sqrt[q]{k}}{1 - \sqrt[q]{k^2}}. \tag{4.4}$$

$$\|\eta - y_{2n}\| \le P_{2n,2n-1}(x, y) \sqrt[q]{\frac{W_{2n,2n-1}(x, y)}{Cd_y}} \cdot \frac{\sqrt[q]{k}}{1 - \sqrt[q]{k^2}}. \tag{4.5}$$

Theorem 4.2. *Let A_x, A_y, B_x and B_y be nonempty subsets of a complete metric space (X, ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let there exist $\alpha, \beta, \gamma, \delta > 0$, $\max\{\alpha + \gamma, \beta + \delta\} < 1$, such that*

$$\rho(F(x, y), G(u, v)) + \rho(f(z, w), g(t, s)) \le \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(z, t) + \delta\rho(w, s) \tag{4.6}$$

for all $(x, y) \in A_x \times A_y$, $(u, v) \in B_x \times B_y$, $(z, w) \in A_x \times A_y$ and $(t, s) \in B_x \times B_y$. Then

(I) *There exists a unique pair (ξ, η) in $A \cap B$, which is a common coupled fixed point for the maps F and G. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 2.6 converge to ξ and η respectively.*

(II) *a priori error estimates hold*

$$\max\{\rho(x_n, \xi), \rho(y_n, \eta)\} \le \frac{k^n}{1 - k}(\rho(x_1, x_0) + \rho(y_1, y_0)); \tag{4.7}$$

(III) *a posteriori error estimates hold*

$$\max\{\rho(x_n,\xi),\rho(y_n,\eta)\}\le\frac{k}{1-k}(\rho(x_{n-1},x_n)+\rho(y_{n-1},y_n)); \tag{4.8}$$

(IV) *The rate of convergence for the sequences of successive iterations is given by*

$$\rho(x_n,\xi)+\rho(y_n,\eta)\le k\left(\rho(x_{n-1},\xi)+(y_{n-1},\eta)\right). \tag{4.9}$$

5. Auxiliary Results

In what follows we will use the notation $d=d_x+d_y$.

Lemma 5.1. *Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ), $F:A_x\times A_y\to B_x$, $f:A_x\times A_y\to B_y$, $G:B_x\times B_y\to A_x$ and $g:B_x\times B_y\to A_y$. Let the ordered pair $((F,f),(G,g))$ be a cyclic contraction. Then there holds*

$$\lim_{n\to\infty}\rho(x_n,x_{n+1})=d_x \text{ and } \lim_{n\to\infty}\rho(y_n,y_{n+1})=d_y$$

for arbitrary chosen $(x,y)\in A_x\times A_y$.

Proof. Let us choose arbitrary $(x,y)\in A_x\times A_y$ and let us define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$.
Using the cyclic contraction condition (2.1) we get that for all $n\in\mathbb{N}$ there holds

$$\begin{array}{rcl}S_3&=&\rho(x_{2n+1},x_{2n+2})+\rho(y_{2n+1},y_{2n+2})\\&=&\rho\left(F(x_{2n},y_{2n}),G(x_{2n+1},y_{2n+1})\right)+\rho\left(f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1})\right.\\&\le&\alpha\rho(x_{2n},x_{2n+1})+\beta\rho(y_{2n},y_{2n+1})+\gamma\rho(x_{2n},x_{2n+1})+\delta\rho(y_{2n},y_{2n+1})\\&&+(1-(\alpha+\gamma))d_x+(1-(\beta+\delta))d_y\\&=&(\alpha+\gamma)\rho(x_{2n},x_{2n+1})+(1-(\alpha+\gamma))d_x\\&&+(\beta+\delta)\rho(y_{2n},y_{2n+1})+(1-(\beta+\delta))d_y.\end{array}$$

Thus we get

$$\begin{array}{rcl}S_4&=&\rho(x_{2n+1},x_{2n+2})+\rho(y_{2n+1},y_{2n+2})-d\\&\le&(\alpha+\gamma)(\rho(x_{2n},x_{2n+1})-d_x)+(\beta+\delta)(\rho(y_{2n},y_{2n+1})-d_y)\\&\le&k\left(\rho(x_{2n},x_{2n+1})+\rho(y_{2n},y_{2n+1})-d\right)\\&\le&k^2(\rho(x_{2n-1},x_{2n})+\rho(y_{2n-1},y_{2n})-d)\\&\le&k^3(\rho(x_{2n-2},x_{2n-1})+\rho(y_{2n-2},y_{2n-1})-d)\\ \cdot\cdot&\cdot&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\le&k^{2n+1}(\rho(x_0,x_1)+\rho(y_0,y_1)-d)).\end{array} \tag{5.1}$$

After taking limit in (5.1), when $n\to\infty$ we get

$$\lim_{n\to\infty}(\rho(x_{2n+1},x_{2n+2})+\rho(y_{2n+1},y_{2n+2})-(d_x+d_y))=0$$

and thus from the inequalities $\rho(x_{2n+1},x_{2n+2})\ge d_x$ and $\rho(y_{2n+1},y_{2n+2})\ge d_y$ we obtain

$$\lim_{n\to\infty}\rho(x_{2n+1},x_{2n+2})=d_x \quad\text{and}\quad \lim_{n\to\infty}\rho(y_{2n+1},y_{2n+2})=d_y. \qquad\square$$

It is easy to see that inequality (5.1) holds also for indexes $m>n$, such that $n+m$ is an odd number

$$\rho(x_n,x_m)+\rho(y_n,y_m)-d\le k^n(\rho(x_0,x_{m-n})+\rho(y_0,y_{m-n})-d). \tag{5.2}$$

Lemma 5.2. *Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let the ordered pair $((F,f),(G,g))$ be a cyclic contraction. For any arbitrary chosen $(x,y) \in A_x \times A_y$ the sequences $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n+1}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ and $\{y_{2n+1}\}_{n=0}^{\infty}$ are bounded.*

Proof. Let $(x,y) \in A_x \times A_y$ be arbitrary chosen and fixed. From Lemma 5.1 we have that $\lim_{n\to\infty} \rho(x_{2n+1}, x_{2n+2}) = d_x$ and $\lim_{n\to\infty} \rho(y_{2n+1}, y_{2n+2}) = d_y$ and thus it will be enough to prove that only the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{y_{2n+1}\}_{n=0}^{\infty}$ are bounded.

Let as choose

$$M > \frac{d + (1+k)k^2(\rho(x_0,Tx_0) + \rho(y_0,Ty_0))}{1-k^2}.$$

Let us suppose that at least one of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{y_{2n+1}\}_{n=0}^{\infty}$ is not bounded. Then there exists $n_0 \in \mathbb{N}$, such that there hold

$$\rho(T^2x_0, T^{2n_0-1}x_0) + \rho(T^2y_0, T^{2n_0-1}y_0) \le M$$

and

$$\rho(T^2x_0, T^{2n_0+1}x_0) + \rho(T^2y_0, T^{2n_0+1}y_0) > M. \tag{5.3}$$

From inequality (5.3) after using (5.2) with $n = 2$ and $m = 2n_0 + 1$ we get

$$\begin{aligned}
S_5 &= \frac{M-d}{k^2} < \frac{\rho(T^2x_0, T^{2n_0+1}x_0) + \rho(T^2y_0, T^{2n_0+1}y_0) - d}{k^2} \\
&\le \rho(x_0, T^{2n_0-1}x_0) + \rho(y_0, T^{2n_0-1}y_0) - d \\
&\le \rho(x_0, T^2x_0) + \rho(y_0, T^2y_0) - d + \rho(T^2x_0, T^{2n_0-1}x_0) + \rho(T^2y_0, T^{2n_0-1}y_0) \\
&\le \rho(x_0, Tx_0) + \rho(y_0, Ty_0) + \rho(Tx_0, T^2x_0) + \rho(Ty_0, T^2y_0) - d + M \\
&\le \rho(x_0, Tx_0) + \rho(y_0, Ty_0) + k(\rho(x_0, Tx_0) + \rho(y_0, Ty_0) - d)) + M \\
&= (1+k)(\rho(x_0, Tx_0) + \rho(y_0, Ty_0)) + M,
\end{aligned}$$

which inequality can hold true only if the inequality

$$M \le \frac{d + (1+k)k^2(\rho(x_0,Tx_0) + \rho(y_0,Ty_0))}{1-k^2},$$

holds, which contradicts with the choice of M. □

Lemma 5.3. *Let A_x, A_y, B_x and B_y be nonempty convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let the ordered pair $((F,f),(G,g))$ be a cyclic contraction. Then for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that the inequality*

$$\|x_m - x_{n+1}\| + \|y_m - y_{n+1}\| < d + \varepsilon$$

holds for any $m > n > n_0$ and $m + n + 1$ be an odd number.

Proof. From Lemma 5.1 we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = d_x$$

and

$$\lim_{n\to\infty}\|y_n - y_{n+1}\| = \lim_{n\to\infty}\|y_{n+2} - y_{n+1}\| = d_y.$$

By Lemma 2.8 after using the uniform convexity of $(X, \|\cdot\|)$ it follows that $\lim_{n\to\infty}\|x_n - x_{n+2}\| = 0$ and $\lim_{n\to\infty}\|y_n - y_{n+2}\| = 0$.

Let us suppose that there exists $\varepsilon > 0$ such that for every $j \in \mathbb{N}$ there are $m_j > n_j + 1 \geq j$ so that $\|x_{m_j} - x_{n_j+1}\| + \|y_{m_j} - y_{n_j+1}\| \geq d + \varepsilon$. Let us choose m_j to be the smallest integer so that the above inequality is satisfied, i.e.

$$\|x_{m_j-2} - x_{n_j+1}\| + \|y_{m_j-2} - y_{n_j+1}\| < d + \varepsilon.$$

Thus we get

$$\begin{aligned} d + \varepsilon &\leq \|x_{m_j} - x_{n_j+1}\| + \|y_{m_j} - y_{n_j+1}\| \\ &\leq \|x_{m_j} - x_{m_j-2}\| + \|x_{m_j-2} - x_{n_j+1}\| \\ &\quad + \|y_{m_j} - y_{m_j-2}\| + \|y_{m_j-2} - y_{n_j+1}\| \\ &< \|x_{m_j} - x_{m_j-2}\| + \|y_{m_j} - y_{m_j-2}\| + d + \varepsilon. \end{aligned} \tag{5.4}$$

Letting $j \to \infty$ in (5.4) we get

$$\lim_{j\to\infty}(\|x_{m_j} - x_{n_j+1}\| + \|y_{m_j} - y_{n_j+1}\|) = d + \varepsilon.$$

Using the boundedness of the sequences $\{x_n\}_{n=0}^{\infty}$ and$\{y_n\}_{n=0}^{\infty}$ it follows that there exists $M \geq d_x + d_y$, such that the inequality

$$M \geq \|x_0 - x_{n_j-m_j+1}\| + \|y_0 - y_{n_j-m_j+1}\|$$

holds for every $k \in \mathbb{N}$. The inequality

$$\begin{aligned} S_6 &= \|x_{m_j} - x_{n_j+1}\| + \|y_{m_j} - y_{n_j+1}\| - d \\ &\leq k^{m_j}(\|x_0, x_{n_j-m_j+1}\| + \|y_0, y_{n_j-m_j+1}\| - d) \leq k^{m_j}(M - d) \end{aligned}$$

holds. For any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$, such that $k^j(M - d) < \varepsilon$ for every $j \geq j_0$. Therefore for any $m_j > n_j + 1 \geq j_0$ there holds $\|x_{m_j} - x_{n_j+1}\| + \|y_{m_j} - y_{n_j+1}\| < d + \varepsilon$, which is a contradiction. □

Lemma 5.4. *Let A_x, A_y, B_x and B_y be nonempty convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let the ordered pair $((F, f), (G, g))$ be a cyclic contraction. For an arbitrary chosen $(x, y) \in A_x \times A_y$ the sequences $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n+1}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ and $\{y_{2n+1}\}_{n=0}^{\infty}$ are Cauchy sequences.*

Proof. We will prove that $\{x_{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence. The proofs for the other three cases are similar. By Lemma 5.3 we have that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$, so that for all $2m > 2n + 1 \geq n_0$ holds the inequality

$$\|x_{2m} - x_{2n+1}\| + \|y_{2m} - y_{2n+1}\| < d + \varepsilon.$$

From the inequalities $d_x \leq \|x_{2m} - x_{2n+1}\|$ and $d_y \leq \|y_{2m} - y_{2n+1}\|$ it follows that the inequality $\|x_{2m} - x_{2n+1}\| < d_x + \varepsilon$ holds for all $2m > 2n + 1 \geq n_0$. From Lemma 5.1 it follows that $\lim_{n\to\infty}\|x_{2n} - x_{2n+1}\| = d_x$. According to Lemma 2.9 it follows that for every $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$, so that for all $m > n \geq N_0$ holds the inequality $\|x_{2m} - x_{2n}\| < \varepsilon$ and consequently $\{x_{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence. □

Lemma 5.5. *Let A_x, A_y, B_x and B_y be nonempty subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, let $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$ and the ordered pair $((F, f), (G, g))$ be a cyclic contraction. Then for any $1 \leq l \leq 2n$ there hold the inequalities*

$$\|x_{2n+1} - x_{2n}\| \leq k^l W_{2n+1-l,2n-l}(x, y) + d_x;$$

and

$$\|y_{2n+1} - y_{2n}\| \leq k^l W_{2n+1-l,2n-l}(x, y) + d_y;$$

Proof. By Lemma 5.1 we have the inequality

$$W_{2n+1,2n}(x, y) \leq kW_{2n,2n-1}(x, y).$$

and therefore $W_{2n+1,2n}(x, y) \leq k^l W_{2n+1-l,2n-l}(x, y)$.
Consequently using the inequalities $d_y \leq \|y_{2n+1} - y_{2n}\|$ and $d_x \leq \|x_{2n+1} - x_{2n}\|$ for any $n \in \mathbb{N}$ we get

$$\begin{aligned} \|x_{2n+1} - x_{2n}\| &\leq k^l W_{2n+1-l,2n-l}(x, y) + d_x + d_y - \rho(y_{2n+1}, y_{2n}) \\ &\leq k^l W_{2n+1-l,2n-l}(x, y) + d_x \end{aligned}$$

and

$$\begin{aligned} \|y_{2n+1} - y_{2n}\| &\leq k^l W_{2n+1-l,2n-l}(x, y) + d_x + d_y - \rho(x_{2n+1}, x_{2n}) \\ &\leq k^l W_{2n+1-l,2n-l}(x, y) + d_y. \end{aligned}$$

□

Lemma 5.6. *Let A_x, A_y, B_x and B_y be nonempty closed and convex subsets of a uniformly convex Banach space. Let $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$ and the ordered pair $((F, f), (G, g))$ be a cyclic contraction. Then there holds the inequalities*

$$\delta_{\|\cdot\|}\left(\frac{\|x_{2n+2} - x_{2n}\|}{d_x + k^l W_{2n+1-l,2n-l}(x, y)}\right) \leq \frac{k^l W_{2n+1-l,2n-l}(x, y)}{d_x + k^l W_{2n+1-l,2n-l}(x, y)}$$

and

$$\delta_{\|\cdot\|}\left(\frac{\|y_{2n+2} - y_{2n}\|}{d_y + k^l W_{2n+1-l,2n-l}(x, y)}\right) \leq \frac{k^l W_{2n+1-l,2n-l}(x, y)}{d_y + k^l W_{2n+1-l,2n-l}(x, y)}.$$

Proof. From Lemma 5.5 we have the inequalities

$$\|x_{2n+1} - x_{2n}\| \leq d_x + k^l W_{2n+1-l,2n-l}(x, y),$$

$$\begin{aligned} \|x_{2n+2} - x_{2n+1}\| &\leq d_x + k^{l+1} W_{2n+1-l,2n-l}(x, y) \\ &\leq d_x + k^l W_{2n+1-l,2n-l}(x, y) \end{aligned}$$

and

$$\begin{aligned} \|x_{2n+2} - x_{2n}\| &\leq \|x_{2n+2} - x_{2n+1}\| + \|x_{2n+1} - x_{2n}\| \\ &\leq 2\left(d_x + k^l W_{2n+1-l,2n-l}(x, y)\right). \end{aligned}$$

After a substitution in (2.3) with $x = x_{2n},\ y = x_{2n+2},\ z = x_{2n+1}x,$

$$R = d_x + k^l W_{2n+1-l,2n-l}(x, y) \text{ and } r = \|x_{2n+2} - x_{2n}\|$$

and using the convexity of the set A_x we get the chain of inequalities

$$\begin{aligned} d_x &\leq \left\|\tfrac{x_{2n}+x_{2n+2}}{2} - x_{2n+1}\right\| \\ &\leq \left(1-\delta_{\|\cdot\|}\left(\tfrac{\|x_{2n+2}-x_{2n}\|}{d_x+k^lW_{2n+1-l,2n-l}(x,y)}\right)\right)\left(d_x + k^lW_{2n+1-l,2n-l}(x,y)\right). \end{aligned} \tag{5.5}$$

and thereafter we obtain the inequality

$$\delta_{\|\cdot\|}\left(\frac{\|x_{2n+2}-x_{2n}\|}{d_y+k^lW_{2n+1-l,2n-l}(x,y)}\right) \leq \frac{k^lW_{2n+1-l,2n-l}(x,y)}{d_y+k^lW_{2n+1-l,2n-l}(x,y)}.$$

The proof that

$$\delta_{\|\cdot\|}\left(\frac{\|y_{2n+2}-y_{2n}\|}{d_x+k^lW_{2n+1-l,2n-l}(x,y)}\right) \leq \frac{k^lW_{2n+1-l,2n-l}(x,y)}{d_x+k^lW_{2n+1-l,2n-l}(x,y)}$$

can be done similarly. □

6. Proof of the Main Results

Proof of Theorem 4.1. For any initial guess $(x, y) \in A_x \times A_y$ it follows from Lemma 5.4 that the sequences $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n+1}\}_{n=0}^{\infty}$, $\{y_{2n}\}_{n=0}^{\infty}$ and $\{y_{2n+1}\}_{n=0}^{\infty}$ are Cauchy sequences. From the assumptions that $(X, \|\cdot\|)$ is a Banach space and A_x, A_y, B_x and B_y are closed it follows that there are $(\xi, \eta) \in A_x \times A_y$, so that

$$\lim_{n\to\infty} T^{2n}x_0 = \lim_{n\to\infty} x_{2n} = \xi \in A_x \text{ and } \lim_{n\to\infty} T^{2n}y_0 = \lim_{n\to\infty} y_{2n} = \eta \in A_y.$$

From the inequalities by using the continuity of the norm function $\|\cdot - \cdot\|$ and Lemma 5.1 we get

$$\begin{aligned} S_7 &= \|\xi - F(\xi,\eta)\| + \|\eta - f(\xi,\eta)\| - d \\ &= \lim_{n\to\infty}\|x_{2n} - F(\xi,\eta)\| + \lim_{n\to\infty}\|y_{2n} - f(\xi,\eta)\| - d \\ &= \lim_{n\to\infty}\|G(x_{2n-1},y_{2n-1}) - F(\xi,\eta)\| + \lim_{n\to\infty}\|g(x_{2n-1},y_{2n-1}) - f(\xi,\eta)\| - d \\ &\leq \lim_{n\to\infty}(\alpha\|x_{2n-1}-\xi\| + \beta\|y_{2n-1}-\eta\| + \gamma\|x_{2n-1}-\xi\| + \delta\|y_{2n-1}-\eta\|) \\ &\quad -(\alpha+\gamma))d_x - (\beta+\delta))d_y \\ &\leq \lim_{n\to\infty}((\alpha+\gamma)(\|x_{2n-1}-x_{2n}\| - d_x) + (\beta+\delta)(\|y_{2n-1}-y_{2n}\| - d_y)) = 0. \end{aligned}$$

Thus $\|\xi - F(\xi,\eta)\| = d_x$, $\|\eta - f(\xi,\eta)\| = d_y$. It can be proven in a similar fashion that $\|\zeta - G(\zeta,\varsigma)\| + \|\varsigma - g(\zeta,\varsigma)\| \leq d_x + d_y$ and consequently $\|\zeta - G(\zeta,\varsigma)\| = d_x$, $\|\varsigma - g(\zeta,\varsigma)\| = d_y$.

It has remained to prove that there holds (4.1). Indeed from the inequalities

$$\begin{aligned} S_8 &= \|G(F(\xi,\eta),f(\xi,\eta)) - F(\xi,\eta)\| + \|g(F(\xi,\eta),f(\xi,\eta)) - f(\xi,\eta)\| \\ &\leq \alpha\|\xi - F(\xi,\eta)\| + \beta\|\eta - f(\eta,\xi)\| + \gamma\|\xi - F(\xi,\eta)\| + \delta\|\eta - f(\xi,\eta)\| \\ &\quad +(1-(\alpha+\gamma))d_x + (1-(\beta+\delta))d_y = d \end{aligned}$$

it follows that

$$\|G(F(\xi,\eta),f(\xi,\eta)) - F(\xi,\eta)\| = d_x,\ \|g(F(\xi,\eta),f(\xi,\eta)) - f(\xi,\eta)\| = d_y.$$

From the assumption that (ξ,η) is a coupled best proximity pair for (F, f) i.e.

$$\|\xi - F(\xi,\eta)\| = d_x,\ \|\eta - f(\xi,\eta)\| = d_y$$

and the uniform convexity of $(X, \|\cdot\|)$ it follows that

$$G(F(\xi,\eta), f(\xi,\eta)) = \xi,\ \ g(F(\xi,\eta), f(\xi,\eta)) = \eta.$$

By similar arguments it can be proven that

$$F(G(\zeta,\varsigma), g(\zeta,\varsigma)) = \zeta \text{ and } f(G(\zeta,\varsigma), g(\zeta,\varsigma)) = \varsigma.$$

We will prove the uniqueness of the coupled best proximity points. Let us suppose that the coupled best proximity point (ξ,η) of (F,f) is not unique, i.e. there exists (ξ^*,η^*), such that

$$\|\xi^* - F(\xi^*,\eta^*)\| = d_x,\ \ \|\eta^* - f(\xi^*,\eta^*)\| = d_y$$

and $\|\xi-\xi^*\| + \|\eta-\eta^*\| > 0$. By similar arguments it can be proven that

$$G(F(\xi,\eta), f(\xi,\eta)) = \xi,\ \ g(F(\xi,\eta), f(\xi,\eta)) = \eta,$$

$$G(F(\xi^*,\eta^*), f(\xi^*,\eta^*)) = \xi^*,\ \ g(F(\xi^*,\eta^*), f(\xi^*,\eta^*)) = \eta^*.$$

Thus we get the inequalities

$$\begin{array}{rcl}
S_9 & = & \|\xi - F(\xi^*,\eta^*)\| + \|\eta - f(\xi^*,\eta^*)\| \\
& = & \|G(F(\xi,\eta), f(\xi,\eta)) - F(\xi^*,\eta^*)\| + \|g(F(\xi,\eta), f(\xi,\eta)) - f(\xi^*,\eta^*)\| \\
& \leq & \alpha\|\xi^* - F(\xi,\eta)\| + \beta\|\eta^* - f(\xi,\eta)\| + \gamma\|\xi^* - F(\xi,\eta)\| \\
& & +\delta\|\eta^* - f(\xi,\eta)\| + (1-(\alpha+\gamma))d_x + (1-(\beta+\delta))d_y \\
& \leq & (\alpha+\gamma)\|\xi^* - F(\xi,\eta)\| + (\beta+\delta)\|\eta^* - f(\xi,\eta)\| \\
& & +(1-(\alpha+\gamma))\|\xi^* - F(\xi,\eta)\| + (1-(\beta+\delta))\|\eta^* - f(\xi,\eta)\| \\
& \leq & \|\xi^* - F(\xi,\eta)\| + \|\eta^* - f(\xi,\eta)\|.
\end{array} \tag{6.1}$$

By similar calculations we obtain

$$\|\xi^* - F(\xi,\eta)\| + \|\eta^* - f(\xi,\eta)\| \leq \|\xi - F(\xi^*,\eta^*)\| + \|\eta - f(\xi^*,\eta^*)\|. \tag{6.2}$$

Consequently there holds

$$\|\xi^* - F(\xi,\eta)\| + \|\eta^* - f(\xi,\eta)\| = \|\xi - F(\xi^*,\eta^*)\| + \|\eta - f(\xi^*,\eta^*)\|. \tag{6.3}$$

We will show that $\|\xi^* - F(\xi,\eta)\| + \|\eta^* - f(\eta,\xi)\| = d$. Let us assume the contrary, i.e. $\|\xi^* - F(\xi,\eta)\| + \|\eta^* - f(\xi,\eta)\| > d$. Then there holds at least on of the inequalities $\|\xi^* - F(\xi,\eta)\| > d_x$ or $\|\eta^* - f(\xi,\eta)\| > d_y$. Then from the chain of inequalities

$$\begin{array}{rcl}
S_{10} & = & \|\xi - F(\xi^*,\eta^*)\| + \|\eta - f(\xi^*,\eta^*)\| \\
& = & \|G(F(\xi,\eta), f(\xi,\eta)) - F(\xi^*,\eta^*)\| + \|G(F(\xi,\eta), f(\xi,\eta)) - f(\xi^*,\eta^*)\| \\
& \leq & \alpha\|\xi^* - F(\xi,\eta)\| + \beta\|\eta^* - f(\xi,\eta) + \gamma\|\xi^* - F(\xi,\eta)\| \\
& & +\delta\|\eta^* - f(\xi,\eta) + (1-(\alpha+\gamma))d_x + (1-(\beta+\delta))d_y \\
& < & \|\xi^* - F(\xi,\eta)\| + \|\eta^* - F(\eta,\xi)\|
\end{array}$$

we get a contradiction with (6.3). Therefore

$$\|\xi^* - F(\xi,\eta)\| + \|\eta^* - F(\eta,\xi)\| = d_x + d_y.$$

From the above equalities, the uniform convexity of $(X, \|\cdot\|)$ and $\|\xi - F(\xi,\eta)\| = d_x$, $\|\eta - F(\eta,\xi)\| = d_y$ it follows that $(\xi^*,\eta^*) = (\xi,\eta)$.
The proof that $(\zeta,\varsigma) \in B \times B$ is a unique coupled best proximity point of G can be done in a similar fashion.

(i) From the uniform convexity of X is follows that $\delta_{\|\cdot\|}$ is strictly increasing and therefore there exists its inverse function $\delta^{-1}_{\|\cdot\|}$, which is strictly increasing too. From Lemma 5.6 we get

$$\|x_{2n} - x_{2n+2}\| \leq \left(d_x + k^l W_{2n+1-l,2n-l}(x,y)\right) \delta^{-1}_{\|\cdot\|} \left(\tfrac{k^l W_{2n+1-l,2n-l}(x,y)}{d_x + k^l W_{2n+1-l,2n-l}(x,y)}\right). \quad (6.4)$$

By the inequality $\delta_{\|\cdot\|}(t) \geq Ct^q$ it follows that $\delta^{-1}_{\|\cdot\|}(t) \leq \left(\frac{t}{C}\right)^{1/q}$. From (6.4) and the inequalities

$$d_x \leq d_x + (\max\{\alpha+\gamma, \beta+\delta\})^l W_{2n+1-l,2n-l}(x,y) \leq P_{2n-l,2n+1-l}(x,y)$$

we obtain

$$\begin{aligned}\|x_{2n} - x_{2n+2}\| &\leq \left(d_x + k^l W_{2n+1-l,2n-l}(x,y)\right) \sqrt[q]{\frac{k^l W_{2n+1-l,2n-l}(x,y)}{C\left(d_x + k^l W_{2n+1-l,2n-l}(x,y)\right)}} \\ &\leq P_{2n-l,2n+1-l}(x,y) \sqrt[q]{\frac{W_{2n+1-l,2n-l}(x,y)}{Cd_x}} \sqrt[q]{k^l}. \end{aligned} \quad (6.5)$$

We have proven that there exists a unique pair $(\xi,\eta) \in A_x \times A_y$, such that

$$\|\xi - F(\xi,\eta)\| = d_x$$

and ξ is a limit of the sequence $\{x_{2n}\}_{n=1}^{\infty}$ for any $(x,y) \in A_x \times A_y$.
After a substitution with $l = 2n$ in (6.5) we get the inequality

$$\begin{aligned} S_{11} &= \sum_{n=1}^{\infty} \|x_{2n} - x_{2n+2}\| \\ &\leq (\|x_0 - x_1\| + \|y_0 - y_1\|) \sqrt[q]{\frac{\|x_0 - x_1\| + \|y_0 - y_1\| - d}{Cd_x}} \sum_{n=1}^{\infty} \sqrt[q]{k^{2n}} \\ &= (\|x_0 - x_1\| + \|y_0 - y_1\|) \sqrt[q]{\frac{\|x_0 - x_1\| + \|y_0 - y_1\| - d}{Cd_x}} \cdot \frac{\sqrt[q]{k^2}}{1 - \sqrt[q]{k^2}} \end{aligned}$$

and consequently the series $\sum_{n=1}^{\infty}(x_{2n} - x_{2n+2})$ is absolutely convergent. Thus for any $m \in \mathbb{N}$ there holds

$$\xi = x_{2m} - \sum_{n=m}^{\infty} (x_{2n} - x_{2n+2})$$

and therefore we get the inequality

$$\begin{aligned}\|\xi - x_{2m}\| &\leq \sum_{n=m}^{\infty} \|x_{2n} - x_{2n+2}\| \\ &\leq (\|x_0 - x_1\| + \|y_0 - y_1\|) \sqrt[q]{\frac{\|x_0 - x_1\| + \|y_0 - y_1\| - d}{Cd_x}} \cdot \frac{\sqrt[q]{k^{2m}}}{1 - \sqrt[q]{k^2}}. \end{aligned}$$

The proof for $\|\eta - y_{2m}\|$ can be done in a similar fashion.

(ii) After a substitution with $l = 1 + 2i$ in (6.5) we obtain

$$\|x_{2n+2i} - x_{2n+2(i+1)}\| \leq P_{2n-1,2n}(x,y) \sqrt[q]{\frac{W_{2n-1,2n}(x,y)}{Cd_x}} \left(\sqrt[q]{k}\right)^{1+2i}. \quad (6.6)$$

From (6.6) we get that there holds the inequality

$$\begin{aligned}
S_{12} = \|x_{2n} - x_{2(n+m)}\| &\le \sum_{i=0}^{m-1} \|x_{2n+2i} - x_{2n+2(i+1)}\| \\
&\le \sum_{i=0}^{m-1} P_{2n-1,2n}(x,y) \sqrt[q]{\frac{W_{2n-1,2n}(x,y)}{Cd_x}} \sqrt[q]{k^{1+2i}} \\
&= P_{2n-1,2n}(x,y) \sqrt[q]{\frac{W_{2n-1,2n}(x,y)}{Cd_x}} \sum_{i=0}^{m-1} \sqrt[q]{k^{1+2i}} \\
&= P_{2n-1,2n}(x,y) \sqrt[q]{\frac{W_{2n-1,2n}(x,y)}{Cd_x}} \cdot \frac{1-\sqrt[q]{k^{2m}}}{1-\sqrt[q]{k^2}} \sqrt[q]{k} \qquad (6.7)
\end{aligned}$$

and after letting $m \to \infty$ in (6.7) we obtain the inequality

$$\|x_{2n} - \xi\| \le P_{2n,2n-1}(x,y) \sqrt[q]{\frac{W_{2n,2n-1}(x,y)}{Cd_x}} \frac{\sqrt[q]{k}}{1-\sqrt[q]{k^2}}.$$

The proof for $\|y_{2n} - \eta\|$ can be done in a similar fashion. □

Proof of Theorem 4.2. It is easy to observe that for any $n \in \mathbb{N}$ there hold the inequalities

$$\begin{aligned}
S_{13} &= \rho(x_{2n+1}, x_{2n}) + \rho(y_{2n+1}, y_{2n}) \\
&= \rho(F(x_{2n}, y_{2n}), G(x_{2n-1}, y_{2n-1}) + \rho(f(x_{2n}, y_{2n}), g(x_{2n-1}, y_{2n-1}) \\
&\le \alpha\rho(x_{2n}, x_{2n-1}) + \beta\rho(y_{2n}, y_{2n-1}) + \gamma\rho(x_{2n}, x_{2n-1}) + \delta\rho(y_{2n}, y_{2n-1}) \\
&= (\alpha+\gamma)\rho(x_{2n}, x_{2n-1}) + (\beta+\delta)\rho(y_{2n}, y_{2n-1}) \\
&\le k(\rho(x_{2n}, x_{2n-1}) + \rho(y_{2n}, y_{2n-l})).
\end{aligned}$$

Consequently

$$\rho(x_{2n+1}, x_{2n}) + \rho(y_{2n+1}, y_{2n}) \le k^l(\rho(x_{2n+1-l}, x_{2n-l}) + \rho(y_{2n+1-l}, y_{2n-l})). \qquad (6.8)$$

(I) Let $(x,y) \in A_x \times A_y$ be arbitrary chosen. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences defined in Definition 2.6. Then from (6.8), because one of $n+1$ or n is even and the other is an odd number, applied for $l = n$ we have

$$\max\{\rho(x_{n+1}, x_n), \rho(y_{n+1}, y_n)\} \le k^n(\rho(x_1, x_0) + \rho(y_1, y_0)).$$

Thus

$$\begin{aligned}
\rho(x_n, x_{n+m}) &\le \sum_{j=n}^{n+m-1} \rho(x_j, x_{j+1}) \le \sum_{j=n}^{n+m-1} k^j(\rho(x_1, x_0) + \rho(y_1, y_0)) \\
&\le k^n \frac{1-k^{n+m}}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0)).
\end{aligned} \qquad (6.9)$$

Since $k \in (0,1)$ it follows that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $A_x \cup B_x$. Consequently $\{x_n\}$ converges to some $\xi \in A_x \cap B_x$. However the sequence $\{x_n\}_{n=0}^{\infty}$ has an infinite number of terms in A_x and in B_x and therefore $\xi \in A_x \cap B_x$. So $A_x \cap B_x \neq \emptyset$.

The proof that the sequence $\{y_n\}_{n=0}^{\infty}$ converges to some $\eta \in A_y \cap B_y$ can be done in a similar fashion.

Now, we will prove that the pair (ξ, η) is a coupled fixed point of F. Let $\xi \in A_x \cap B_x$ and $\eta \in A_y \cap B_y$. WLOG we can assume that $(\xi, \eta) \in A_x \times A_y$. It follows that $(F(\xi, \eta), f(\xi, \eta)) \in B_x \times B_y$. Then by the triangle inequality and (4.6) we get the chain of inequalities

$$\begin{aligned} S_{14} &= \rho(\xi, F(\xi, \eta)) + \rho(\eta, f(\xi, \eta)) \\ &\leq \rho(\xi, x_{2n}) + \rho(x_{2n}, F(\xi, \eta)) + \rho(\eta, y_{2n}) + \rho(y_{2n}, f(\xi, \eta)) \\ &\leq \rho(\xi, x_{2n}) + \rho(G(x_{2n-1}, y_{2n-1}), F(\xi, \eta)) \\ &\quad + \rho(\eta, y_{2n}) + \rho(g(x_{2n-1}, y_{2n-1}), f(\xi, \eta)) \\ &\leq \rho(\xi, x_{2n}) + \alpha\rho(x_{2n-1}, \xi) + \beta\rho(y_{2n-1}, \eta) \\ &\quad + \rho(\eta, x_{2n}) + \gamma\rho(x_{2n-1}, \xi) + \delta\rho(y_{2n-1}, \eta). \end{aligned}$$

Taking the limit when $n \to \infty$, we obtain $\rho(\xi, F(\xi, \eta)) + \rho(\eta, F(\eta, \xi)) = 0$, i.e $\rho(\xi, F(\xi, \eta)) = 0$ and $\rho(\eta, F(\eta, \xi)) = 0$. Thus the pair (ξ, η) is a coupled fixed point of (F, f).

The proof that the pair (ξ, η) is a coupled fixed point of (G, g) can be done in a similar fashion by assuming that $(\xi, \eta) \in B_x \times B_y$.

We still have to prove that the pair (ξ, η) is the unique coupled fixed point of (F, f). Arguing by contradiction, suppose there exists $(\xi^*, \eta^*) \in (A_x \cup B_x) \times (A_y \cup B_y)$ such that $(\xi^*, \eta^*) \neq (\xi, \eta)$ and $\xi^* = F(\xi^*, \eta^*)$, $\eta^* = f(\xi^*, \eta^*)$. If we suppose that $\xi^* \in A_x$ then by the definition of a coupled fixed point it follows that $\eta^* \in A_y$ and therefore $\xi^* = F(\xi^*, \eta^*) \in B_x$ and $\eta^* = F(\eta^*, \xi^*) \in B_x$. A similar argument holds if we assume that $\xi^* \in B_x$ and $\eta^* \in B_y$. Thus we can assume that if the pair (ξ^*, η^*) is a coupled fixed point of (F, f) then $(\xi^*, \eta^*) \in (A_x \cap B_x) \times (A_y \cap B_y)$. From (4.6), using the observation that (ξ, η) and (ξ^*, η^*) are coupled fixed points and for (G, g), we have the inequalities

$$\begin{aligned} \rho(\xi^*, \xi) + \rho(\eta^*, \eta) &= \rho(F(\xi^*, \eta^*), G(\xi, \eta) + \rho(f(\eta^*, \xi^*), g(\eta, \xi) \\ &\leq \alpha\rho(\xi^*, \xi) + \beta\rho(\eta^*, \eta) + \gamma\rho(\xi^*, \xi) + \delta\rho(\eta^*, \eta) \\ &= (\alpha + \gamma)\rho(\xi^*, \xi) + (\beta + \delta)\rho(\eta^*, \eta) < \rho(\xi^*, \xi) + \rho(\eta^*, \eta). \end{aligned}$$

It results that $\rho(\xi^*, \xi) = \rho(\eta^*, \eta) = 0$, which is a contradiction and therefore the pair (ξ, η) is the unique coupled fixed point of (F, f).

The proof that the pair (ξ, η) is the unique coupled fixed point of (G, g) can be done in a similar way.

(II) Letting $m \to \infty$ in (6.9) we obtain the a priori estimate

$$\rho(x_n, \xi) \leq \frac{k^n}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0)).$$

The proof that $\rho(y_n, \xi) \leq \frac{k^n}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0))$ is done in a similar fashion. Therefore

$$\max\{\rho(x_n, \xi), \rho(y_n, \xi)\} \leq \frac{k^n}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0)).$$

(III) From the inequality (6.8) applied for $l = k+1$ we get

$$\begin{aligned}\rho(x_n, x_{n+m}) &\leq \sum_{j=0}^{m-1} \rho(x_{n+j}, x_{n+j+1}) \leq \sum_{j=0}^{m-1} k^{j+1}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n)) \\ &\leq \frac{k}{1-k}(1-k^{m+1})(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n)).\end{aligned}$$

Letting $m \to \infty$ we obtain the a posteriori estimate

$$\rho(x_n, \xi) \leq \frac{k}{1-k}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n)).$$

The proof that $\rho(y_n, \xi) \leq \frac{k}{1-k}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n))$ is done in a similar fashion and thus

$$\max\{\rho(x_n, \xi), \rho(y_n, \xi)\} \leq \frac{k}{1-k}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n)).$$

(IV) Considering that the pair (ξ, η) is a coupled fixed point for (F, f) and (4.6) we have the inequalities

$$\begin{aligned}S_{15} &= \rho(x_{2n}, \xi) + \rho(y_{2n}, \eta) \\ &= \rho(G(x_{2n-1}, y_{2n-1}), F(\xi, \eta)) + \rho(g(x_{2n-1}, y_{2n-1}), f(\xi, \eta)) \\ &\leq \alpha\rho(x_{2n-1}, \xi) + \beta\rho(y_{2n-1}, \eta) + \gamma\rho(x_{2n-1}, \xi) + \delta\rho(y_{2n-1}, \eta) \\ &= (\alpha+\gamma)\rho(x_{2n-1}, \xi) + (\beta+\delta)\rho(y_{2n-1}, \eta) \leq k(\rho(x_{2n-1}, \xi) + \rho(y_{2n-1}, \eta)).\end{aligned}$$

By similar arguments we get

$$\begin{aligned}\rho(x_{2n+1}, \xi) + \rho(y_{2n+1}, \eta) &= \rho(F(x_{2n}, y_{2n}), G(\xi, \eta)) + \rho(f(x_{2n}, y_{2n}), g(\xi, \eta)) \\ &\leq \alpha\rho(x_{2n}, \xi) + \beta\rho(y_{2n}, \eta) + \gamma\rho(x_{2n}, \xi) + \delta\rho(y_{2n}, \eta) \\ &= (\alpha+\gamma)\rho(x_{2n}, \xi) + (\beta+\delta)\rho(y_{2n}, \eta) \\ &\leq k(\rho(x_{2n}, \xi) + \rho(y_{2n}, \eta)).\end{aligned}$$

Consequently $\rho(x_n, \xi) + \rho(y_n, \eta) \leq k(\rho(x_{n-1}, \xi) + \rho(y_{n-1}, \eta))$. □

7. Applications

If put $A_x = A_y = A$, $B_x = B_y = B$, $f(x,y) = F(y,x)$, $g(x,y) = G(y,x)$, $z = y$, $w = x$, $t = v$, $s = u$, $\gamma = \beta$ and $\delta = \alpha$, then we get the results from ([21], Theorem 2 and Theorem 3) as corollaries of Theorem 4.1 and Theorem 4.2.

Following [20] we may assume that $(\mathbb{R}, |\cdot|)$ is a uniformly convex and $\delta_{(\mathbb{R},|\cdot|)}(\varepsilon) = \frac{\varepsilon}{2}$. In this case the inverse function δ_X^{-1} exists ant is equal to 2ε. Thus we get the following corollary of Theorem 4.1:

Corollary 7.1. *Let A_x, A_y, B_x and B_y be nonempty convex subsets of a $(\mathbb{R}, |\cdot|)$, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let the ordered pair $((F, f), (G, g))$ be a cyclic contraction. Then (F, f) has a unique coupled best proximity point $(\xi, \eta) \in A_x \times A_y$ and (G, g) has a unique coupled best proximity point $(\zeta, \varsigma) \in B_x \times B_y$, (i.e. $|\xi - F(\xi, \eta)| = d_x$, $|\eta - f(\xi, \eta)| = d_y$ and $|\zeta - G(\zeta, \varsigma)| = d_x$, $|\varsigma - g(\zeta, \varsigma)| = d_y$). Moreover, $\zeta = F(\xi, \eta)$, $\varsigma = f(\xi, \eta)$, $\xi = G(\zeta, \varsigma)$ and $\eta = g(\zeta, \varsigma)$. For any arbitrary point $(x, y) \in A \times A$ there hold*

$$\lim_{n\to\infty} x_{2n} = \xi, \ \lim_{n\to\infty} y_{2n} = \eta, \ \lim_{n\to\infty} x_{2n+1} = \zeta, \ \lim_{n\to\infty} y_{2n+1} = \varsigma$$

and

$$|\xi-\zeta|+|\eta-\varsigma|=d_x+d_y.$$

Moreover there hold

$$\begin{array}{c} G(F(\xi,\eta),f(\xi,\eta))=\xi,\ g(F(\xi,\eta),f(\xi,\eta))=\eta, \\ F(G(\zeta,\varsigma),g(\zeta,\varsigma))=\zeta,\ f(G(\zeta,\varsigma),g(\zeta,\varsigma))=\varsigma. \end{array} \tag{7.1}$$

(i) *a priori error estimates hold*

$$|\xi-x_{2m}|\le 2P_{0,1}(x,y)\frac{W_{0,1}(x,y)}{d_x}\cdot\frac{k^{2m}}{1-k^2}; \tag{7.2}$$

$$|\eta-y_{2m}|\le 2P_{0,1}(x,y)\frac{W_{0,1}(x,y)}{d_y}\cdot\frac{k^{2m}}{1-k^2}; \tag{7.3}$$

(ii) *a posteriori error estimates hold*

$$|\xi-x_{2n}|\le 2P_{2n,2n-1}(x,y)\frac{W_{2n,2n-1}(x,y)}{d_x}\cdot\frac{k}{1-k^2}; \tag{7.4}$$

$$|\eta-y_{2n}|\le 2P_{2n,2n-1}(x,y)\frac{W_{2n,2n-1}(x,y)}{d_y}\cdot\frac{k}{1-k^2}. \tag{7.5}$$

We will illustrate Corollary 7.1 by solving the next system:

Example 7.2. Let us consider the system of nonlinear equations:

$$\left\{\begin{array}{rcrcl} 36x & + & e^y & = & e+68 \\ 4\arctan\left(\frac{x}{2}\right) & + & 18y & = & \pi+18. \end{array}\right. \tag{7.6}$$

Solution. Let us consider the functions

$$F(x,y)=-\frac{x}{8}-\frac{e^y}{32}+\frac{e-60}{32},\quad G(x,y)=-\frac{x}{8}-\frac{e^y}{32}-\frac{e-60}{32},$$

$$f(x,y)=-\frac{\arctan\left(\frac{x}{2}\right)}{4}-\frac{y}{8}+\frac{\pi-14}{16},\quad g(x,y)=-\frac{\arctan\left(\frac{x}{2}\right)}{4}-\frac{y}{8}-\frac{\pi-14}{16}.$$

It is easy to check that $F:[2,+\infty)\times[1,1.5]\to(-\infty,-2]$, $f:[2,+\infty)\times[1,1.5]\to[-1.5,-1]$, $G:(-\infty,-2]\times[-1.5,-1]\to[2,+\infty)$, $g:(-\infty,-2]\times 5-1.5,-1]\to[1,1.5]$ and the system

$$\left\{\begin{array}{rcl} x-F(x,y) & = & 4 \\ y-f(x,y) & = & 2 \end{array}\right. \tag{7.7}$$

is equivalent to (7.6).
Using the inequalities $|e^y-e^v|\le e^{1.5}|y-v|$ for $y,v\in[1,1.5]$ and

$$\left|\arctan\left(\frac{z}{2}\right)-\arctan\left(\frac{t}{2}\right)\right|\le\frac{|z-t|}{4}$$

for $z,t\in[2,+\infty)$ it is easy to obtain that

$$\begin{array}{rcl} S_{17} & = & |F(x,y)-G(u,v)|+|f(z,w)-g(t,s)| \\ & \le & \frac{1}{8}|x-u|+\frac{e^{1.5}}{32}|y-v|+\frac{60-e}{16}+\frac{1}{16}|z-t|+\frac{1}{8}|w-s|+\frac{\pi-14}{8}. \end{array}$$

We will check that Corollary 7.1 holds for $\alpha = 1/8$, $\beta = e^{1.5}/32$, $\gamma = 1/16$, $\delta = 1/8$, $d_x = 4$ and $d_y = 2$. From the equality

$$\left|\frac{e-60}{16}\right| + \left|\frac{\pi-14}{16}\right| - 4\left(1-\left(\frac{1}{8}+\frac{1}{16}\right)\right) - 2\left(\frac{7}{8}-\frac{e^{1.5}}{32}\right)$$

we get

$$\begin{aligned} S_{18} &= |F(x,y)-G(u,v)| + |f(z,w)-g(t,s)| \\ &\le \frac{1}{8}|x-u| + \frac{e^{1.5}}{32}|y-v| + \frac{1}{16}|z-t| + \frac{1}{8}|w-s| \\ &+ \left(1-\left(\frac{1}{8}+\frac{1}{16}\right)\right)d_x + \left(\frac{7}{8}-\frac{e^{1.5}}{32}\right)d_y. \end{aligned}$$

Therefore the ordered pair $((F,f),(G,g))$ is a cyclic contraction with constants $\frac{1}{8}$, $\frac{e^{1.5}}{32}$, $\frac{1}{16}$, $\frac{1}{8}$ and the unique solution of (7.6) is $(2,1)$. $\square$

Table 1. Number $2m$ of iterations needed by the a priori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	4	6	8	10	12	14

Table 2. Number $2m$ of iterations needed by the a posteriori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	4	8	12	14	16	20

If we try to solve system (7.6) with the help of Maple 2016.0, we get as an answer

$$x = 2\tan(RootOf(72\tan(\_Z) + e^{-\frac{2}{9}\_Z+\frac{1}{18}\pi+1} - e - 72))$$

$$y = -\frac{2}{9}RootOf(72\tan(\_Z) + e^{-\frac{2}{9}\_Z+\frac{1}{18}\pi+1} - e - 72) + \frac{\pi}{18} + 1.$$

If we try numerically approximate the solutions of the system (7.6) with the help of Maple 2016.0, we get as an answer $\{x = 2.000000000, y = .9999999998\}$.

8. Conclusion

It is interesting to apply the technique from the article for tripled fixed points and tripled best proximity points [9, 10, 11, 5, 12], as well as for quadruple fixed points and quadruple best proximity points [23]. It will be also interesting if similar results could be obtained for coupled, tripled or quadruple fixed points in partially ordered metric spaces or best proximity points in partially ordered uniformly convex Banach spaces [15, 26]. An open question can be to generalize the ideas from [7, 19] about coupled fixed or best proximity points.

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