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POSITIVE SOLUTIONS FOR FRACTIONAL LAPLACIAN SYSTEM INVOLVING CONCAVE-CONVEX NONLINEARITIES AND SIGN-CHANGING WEIGHT FUNCTIONS

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Abstract. In this paper, we consider a fractional Laplacian system (1.1) with both concave-convex nonlinearities and sign-changing weight functions in bounded domains. With the help of the Nehari manifold, we prove that the system has at least two positive solutions when the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^n$.
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1. Introduction

In this paper, we study the following system involving fractional Laplacian:

$$\begin{cases} (-\Delta)^s u = \lambda f(x)|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta} h(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^s v = \mu g(x)|v|^{q-2}v + \frac{2\beta}{\alpha+\beta} h(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain of $\mathbb{R}^n$, $s \in (0,1)$, $n > 2s$, $1 < q < 2$ and $\alpha > 1$, $\beta > 1$ satisfy

$$2 < \alpha + \beta < 2^\star = \frac{2n}{n-2s}.$$

The pair of parameters $(\lambda, \mu) \in \mathbb{R}^n \setminus \{(0,0)\}$ and the weight functions f, g, h satisfy the following conditions;

(A) $f, g \in L^{p^\star}(\Omega)$ where

$$p^\star = \frac{\alpha+\beta}{\alpha+\beta-q} \text{ and } f^+ = \max\{\pm f, 0\} \neq 0 \text{ or } g^+ = \max\{\pm g, 0\} \neq 0,$$

(B) $h \in C(\overline{\Omega})$ with $\|h\|_\infty = 1$.

In the past decades, the Laplacian equation or system has been widely investigated and a lot of work has been done for ground state solutions, multiple positive solutions, sign-changing solutions and so on (see [34, 33, 14, 16, 17, 15, 18] and references therein).

Recently, a great attention has been focused on the study of equations or systems involving fractional Laplacian with nonlinear terms, both for their interesting theoretical structure and their concrete applications(see [4, 19, 31, 9, 36, 8, 37, 14, 16, 17] and references therein). This type of operator arises in a quite natural way in many different contexts, such as, the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and many others(see[21, 32, 38] and references therein).

Compared to the Laplacian problems, the fractional Laplacian problems is nonlocal and more challenging. In 2007, L. Caffarelli and L. Silvestre [10] studied an extension problem related to the fractional Laplacian in $\mathbb{R}^n$, which can transform the nonlocal problem into a local problem in $\mathbb{R}^{n+1}_+$. This method can be extended to bounded regions and is extensively used in recent articles. For example, the following fractional Laplacian equation

$$\begin{cases} (-\Delta)^s u = F(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied by many authors under various hypotheses on the nonlinearity f. When $F(u) = \lambda u^q + u^p$ with $0 < q < 1 < p < \frac{n+2s}{n-2s}$ and $\lambda \geq 0$, C. Brändle, E. Colorado, A. de Pablo and U. Sánchez [6] showed that there exists a finite parameter $\Lambda > 0$ such that for $0 < \lambda < \Lambda$ there exist at least two solutions, for $\lambda = \Lambda$ there exists at least one solution and for $\lambda > \Lambda$ there is no solution. Moreover, for $s \geq 1/2$ they prove a universal L^∞ bound for every solution of the problem, independently of λ. Furthermore, when $F(u) = \lambda u^q + u^{\frac{n+2s}{n-2s}}$, B. Barrios, E. Colorado, A. de Pablo and U. Sánchez [4] showed that the existence and multiplicity of solutions under suitable conditions of s and q. When $F(u) = |u|^{2^*-2}u + f(x)$, E. Colorado, A. de Pablo and U. Sánchez [19] showed that the existence and the multiplicity of solutions were proved under appropriate conditions on the size of f

It is also natural to study the coupled system of equations. For the following fractional Laplacian system

$$\begin{cases} (-\Delta)^s u = F(u,v) & \text{in } \mathbb{R}^n, \\ (-\Delta)^s v = G(u,v) & \text{in } \mathbb{R}^n, \\ u, v \in D_s(\mathbb{R}^n), \end{cases}$$

When

$$F(u,v) = \mu_1 |u|^{2^*-2}u + \frac{\alpha\gamma}{2^*}|u|^{\alpha-2}u|v|^\beta, \ G(u,v) = \mu_2 |v|^{2^*-2}v + \frac{\beta\gamma}{2^*}|u|^\alpha |v|^{\beta-2}v,$$

M.D. Zhen, J.C. He and H.Y. Xu [42] showed that the existence and nonexistence of positive least energy solution of the system under proper conditions of $\alpha, \beta, \gamma, N, s$. Z. Guo, S. Luo and W. Zou [26] showed the existence of positive least energy solution, which is radially symmetric with respect to some point in $\mathbb{R}^n$ and decays at infinity with certain rate. For other related articles (see [27, 42, 43] and references therein).

We should point out that in all these works, they only consider equation or system without sign-changing weight functions, in the case of Laplacian system, the problem has been done by T.F. Wu in [41]. For system (1.1), when $f(x) = g(x) = h(x) = 1$ X. He, M. Squassina and W. Zou [27] showed that the system admits at least two positive solutions under proper conditions of λ and μ. When the fractional Laplacian operator is replaced by fractional p-Laplacian operator and $f(x) = g(x) = h(x) = 1$ W.J. Chen, S.B. Deng [13] showed the similar results for system (1.1).

The purpose of this paper is to study system (1.1) in the case of $2 < \alpha+\beta < 2^\star$, by variational methods and a Nehari manifold decomposition, we prove that the system admits at least two positive solutions when the pair of parameters (λ, μ) belongs to certain subset of $\mathbb{R}^2$. We note that the fractional Laplacian operator $(-\Delta)^s$ is defined through the spectral decomposition using the powers of the eigenvalues of the positive Laplace operator $(-\Delta)$ with zero Dirichlet boundary data.

To express the main results, we introduce

$$\Theta = \{z \in \mathbb{R}^2 \setminus \{(0,0)\} \mid 0 < (|\lambda| \|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu| \|g\|_{L^{p^\star}})^{\frac{2}{2-q}} < C(\alpha, \beta, \kappa_s, q, S)\}$$

and

$$C(\alpha, \beta, \kappa_s, q, S) = \left[\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right]^{\frac{2}{\alpha+\beta-2}} \left((\frac{\kappa_s S}{2})^{-\frac{q}{2}} \frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{-\frac{2}{2-q}}.$$

$$\Psi = \{z \in \mathbb{R}^2 \setminus \{(0,0)\} \mid 0 < (|\lambda| \|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu| \|g\|_{L^{p^\star}})^{\frac{2}{2-q}} < D(\alpha, \beta, \kappa_s, q, S)\}$$

and

$$D(\alpha, \beta, \kappa_s, q, S) = \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \left[\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right]^{\frac{2}{\alpha+\beta-2}} \left(\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}} \frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{-\frac{2}{2-q}},$$

where κ_s is a normalization constant and S is the best Sobolev constants that will be introduced later.

Our main results are:

Theorem 1.1. *Suppose that the weight functions f, g, h be satisfied with the conditions (A) and (B), for each $(\lambda, \mu) \in \Theta$, then system* (1.1) *has at least one positive solution in $H_0^s(\Omega) \times H_0^s(\Omega)$.*

Theorem 1.2. *Suppose that the weight functions f, g, h be satisfied with the conditions (A) and (B), for each $(\lambda, \mu) \in \Psi$, then system* (1.1) *has at least two positive solution in $H_0^s(\Omega) \times H_0^s(\Omega)$.*

Remark 1.1. The aim of this paper is to generalized the results in [41] for local Laplacian equation to no-local fractional Laplacian case and when $f(x) = g(x) = h(x) = 1$ X. He, M. Squassina and W. Zou [27] showed that the system admits at least two positive solutions under proper conditions of λ and μ.

Remark 1.2. For system (1.1), if the fractional Laplacian operator is replaced by the fractional p-Laplacian operator, by using the similar method as this paper, we can get similar results. That is

(i) Suppose that the weight functions f, g, h be satisfied with the conditions (A) and (B), for each $(\lambda, \mu) \in \Theta_1$, then system (1.1) has at least one nontrivial solution.

(ii) Suppose that the weight functions f, g, h be satisfied with the conditions (A) and (B), for each $(\lambda, \mu) \in \Psi_1$, then system (1.1) has at least two nontrivial solutions.

Where Θ_1 and Ψ_1 are slightly change in Θ and Ψ.

Remark 1.3. Compared with already know results in [13] for fractional p-Laplacian, the authors in [13] consider the case of $f(x) = g(x) = h(x) = 1$ and get similar results as remark 1.2. However, for system (1.1), we can get at least two positive nontrivial solutions, but for the corresponding fractional p-Laplacian system we only obtain nontrivial solutions.

The paper is organized as follows. In section 2, we introduce some preliminaries and functional setting. In section 3, we define the Nehari manifold and give some Lemmas that will be used later. In section 4, we prove the existence of Palais-Smale sequence. In section 5, we give the results of local minimization problem for system (2.1). Finally, the proofs of Theorem 1.1 and Theorem 1.2 are given in section 6.

2. Preliminaries and functional setting

In this section, we introduce some preliminaries that will be used to establish the energy functional for system (1.1). First, we denote the upper half-space in $\mathbb{R}^{n+1}_+$ by

$$\mathbb{R}^{n+1}_+ = \{z = (x, y) = (x_1, \cdots, x_n, y) \in \mathbb{R}^{n+1} : y > 0\}.$$

Let $\Omega \in \mathbb{R}^n$ be a small bounded domain. Denote $\mathcal{C}_\Omega = \Omega \times (0, +\infty) \in \mathbb{R}^{n+1}_+$ and its boundary by $\partial_L \mathcal{C}_\Omega = \partial\Omega \times (0, \infty)$. The powers $(-\Delta)^s$ of the positive Laplace operator $(-\Delta)$, in a bounded domain Ω with zero Dirichlet date are defined via its spectral decomposition, namely

$$(-\Delta)^s = \sum_{j=1}^{\infty} a_j \rho_j^s \varphi_j(x),$$

where (ρ_j, φ_j) is the sequence of eigenvalues and eigenfunctions of the operator $-\Delta$ in Ω under zero Dirichlet boundary date and a_j are the coefficients of u for the base $\{\varphi_j\}_{j=1}^{\infty}$ in $L^2(\Omega)$. In fact, the fractional Laplacian $(-\Delta)^s$ is well defined in the space of functions

$$H_0^s(\Omega) = \left\{ u = \sum_{j=1}^{\infty} a_j \varphi_j \in L^2(\Omega) : \|u\|_{H_0^s} = \left(\sum_{j=1}^{\infty} a_j^2 \rho_j^s \right)^{\frac{1}{2}} < \infty \right\},$$

and $\|u\|_{H_0^s(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}$. The dual space $H^{-s}(\Omega)$ is defined in the standard way, as well as the inverse operator $(-\Delta)^{-s}$.

Definition 2.1. We say that $(u,v) \in H_0^s(\Omega) \times H_0^s(\Omega)$ is a solution of (1.1) if the identity

$$\int_\Omega \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi_1 + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \varphi_2\right) dx - \int_\Omega \left(\lambda f |u|^{q-2} u \varphi_1 + \mu g |v|^{q-2} v \varphi_2\right) dx$$
$$- \frac{2\alpha}{\alpha+\beta} \int_\Omega h |u|^{\alpha-2} u |v|^\beta \varphi_1 dx - \frac{2\beta}{\alpha+\beta} \int_\Omega h |u|^\alpha |v|^{\beta-2} v \varphi_2 dx = 0$$

holds for all $(\varphi_1, \varphi_2) \in H_0^s(\Omega) \times H_0^s(\Omega)$.

Note that, the energy functional associated with (1.1) is given by

$$J_{\lambda,\mu}(u,v) := \frac{1}{2} \int_\Omega \left(|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2\right) dx - \frac{1}{q} \int_\Omega \left(\lambda f |u|^q + \mu g |v|^q\right) dx$$
$$- \frac{2}{\alpha+\beta} \int_\Omega h |u|^\alpha |v|^\beta dx.$$

The functional is well defined in $H_0^s(\Omega) \times H_0^s(\Omega)$ and the critical points of the functional $J_{\lambda,\mu}$ correspond to solutions of (1.1). Motivated by the works of Caffarelli and Silvestre [10], to deal with the nonlocal problem (1.1), we can study a corresponding extension problem, which allows us to investigate problem (1.1) via classic variational methods.

We define the extension operator and fractional Laplacian for functions in

$$H_0^s(\Omega) \times H_0^s(\Omega).$$

Definition 2.2. For a function $u \in H_0^s(\Omega)$, we denote its s-harmony extension $w = E_s(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution of the problem

$$\begin{cases} div(y^{1-2s} \nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ w = u & \text{on } \Omega \times (0) \end{cases}$$

and

$$(-\Delta)^s u(x) = -\kappa_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x,y),$$

where $\kappa_s = 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)}$ is a normalization constant.

The extension function $w(x,y)$ belongs to the space

$$X_0^s(\mathcal{C}_\Omega) := \overline{C_0^\infty\left(\Omega \times [0,+\infty)\right)}^{\|\cdot\|_{X_0^s(\mathcal{C}_\Omega)}}$$

with the norm

$$\|z\|_{\mathcal{C}_\Omega} := \left(\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla z|^2 dx dy\right)^{\frac{1}{2}}.$$

With the normalization constant κ_s we have that the extension operator is an isometry between $H_0^s(\Omega)$ and $X_0^s(\mathcal{C}_\Omega$, namely

$$\|u\|_{H_0^s(\Omega)} = \|E_s(u)\|_{X_0^s(\mathcal{C}_\Omega}, \ \forall\ u \in H_0^s(\Omega).$$

With this extension we can reformulate (1.1) as the following local problem

$$\begin{cases} -div(y^{1-2s}\nabla w_1) = 0,\ -div(y^{1-2s}\nabla w_2) = 0 & \text{in } \mathcal{C}_\Omega \\ w_1 = w_2 = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ w_1 = u, w_2 = v & \text{on } \Omega \times (0) \\ \frac{\partial w_1}{\partial \nu^s} = \lambda f(x)|w_1|^{q-2}w_1 + \frac{2\alpha}{\alpha+\beta}h(x)|w_1|^{\alpha-2}w_1|w_2|^\beta & \text{on } \Omega \times (0) \\ \frac{\partial w_2}{\partial \nu^s} = \mu g(x)|w_2|^{q-2}w_2 + \frac{2\beta}{\alpha+\beta}h(x)|w_1|^{\alpha}|w_2|^{\beta-2}w_2 & \text{on } \Omega \times (0), \end{cases} \tag{2.1}$$

where

$$\frac{\partial w_i}{\partial \nu^s} = -\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w_i}{\partial y}(x,y), \quad i = 1,2$$

and $w_1, w_2 \in X_0^s(\mathcal{C}_\Omega$ are the s-harmony extension of $u, v \in H_0^s(\Omega)$. Let

$$E_0^s(\mathcal{C}_\Omega) := X_0^s(\mathcal{C}_\Omega) \times X_0^s(\mathcal{C}_\Omega).$$

An energy solution to problem (2.1) is a function $(w_1, w_2) \in E_0^s(\mathcal{C}_\Omega)$ satisfying

$$\begin{aligned} &\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\nabla w_1 \cdot \nabla\varphi_1 dxdy + \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\nabla w_2 \cdot \nabla\varphi_2 dxdy \\ &= \lambda \int_\Omega f(x)|w_1|^{q-2}w_1\varphi_1 dx + \frac{2\alpha}{\alpha+\beta}\int_\Omega h(x)|w_1|^{\alpha-2}w_1|w_2|^\beta\varphi_1 dx \\ &+ \mu \int_\Omega g(x)|w_2|^{q-2}w_2\varphi_2 dx + \frac{2\beta}{\alpha+\beta}\int_\Omega h(x)|w_1|^{\alpha}|w_2|^{\beta-2}w_2\varphi_2 dx. \end{aligned}$$

for all $(\varphi_1, \varphi_2) \in E_0^s(\mathcal{C}_\Omega)$.
If $(w_1, w_2) \in E_0^s(\mathcal{C}_\Omega)$ satisfies (2.1), then $(u, v) = (w_1(\cdot, 0), w_2(\cdot, 0))$ defined in the sense of traces, belongs to the space $H_0^s(\Omega) \times H_0^s(\Omega)$ and it is a solution of the problem (1.1). The energy functional associated with (2.1) is given by

$$\begin{aligned} I_{\lambda,\mu}(w) := I_{\lambda,\mu}(w_1, w_2) = \frac{\kappa_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dxdy \\ - \frac{1}{q}\int_\Omega (\lambda f(x)|w_1|^q + \mu g(x)|w_2|^q)\, dx - \frac{2}{\alpha+\beta}\int_\Omega h(x)|w_1|^\alpha |w_2|^\beta dx. \end{aligned}$$

Critical points of $I_{\lambda,\mu}$ in $E_0^s(\mathcal{C}_\Omega)$ correspond to critical points of

$$J_{\lambda,\mu} : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}.$$

Lemma 2.3. [6] *For any $1 \le r \le 2^\star$ and any $z \in X_0^s(\mathcal{C}_\Omega)$, it holds*

$$\left(\int_\Omega |u(x)|^r dx\right)^{\frac{2}{r}} \le C\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z(x,y)|^2 dxdy, \quad u := \mathrm{T}_r z,$$

for some positive constant $C = C(r, s, n, \Omega)$. Furthermore, the space $X_0^s(\mathcal{C}_\Omega)$ is compactly embedded into $L^r(\Omega)$ for every $r < 2^\star$.

Let S be the best Sobolev constant for the embedding of $X_0^s(\mathcal{C}_\Omega)$ in $L^{\alpha+\beta}(\Omega)$ defined by

$$S = \inf_{z \in X_0^s(\mathcal{C}_\Omega)\setminus\{0\}} \frac{\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z(x,y)|^2 dxdy}{\left(\int_\Omega |z(x)|^{\alpha+\beta}dx\right)^{\frac{2}{\alpha+\beta}}}.$$

In the end of this section, we recall some notations that will be used in the sequel.
• $L^p(\Omega),\ 1 \le p \le \infty$ denotes Lebesgue space with norm $\|\cdot\|_p$ and

$$E = X_0^s(\mathcal{C}_\Omega) \times X_0^s(\mathcal{C}_\Omega)$$

is equipped with the norm $\|z\|^2 = \|(w_1, w_2)\|^2 = \|w_1\|^2_{X_0^s(\mathcal{C}_\Omega)} + \|w_2\|^2_{X_0^s(\mathcal{C}_\Omega)}$.
• The dual space of a Banach space E will be denoted by E^{-1}. We set

$$tz = t(w_1, w_2) = (tw_1, tw_2)$$

for all $z \in E$ and $t \in \mathbb{R}$, $z = (w_1, w_2)$ is said to be positive if $w_1(x, y) > 0,\ w_2(x, y) > 0$ in $\mathcal{C}(\Omega)$ and to be non-negative if $w_1(x, y) \ge 0,\ w_2(x, y) \ge 0$ in $\mathcal{C}(\Omega)$.
• $B(0; r)$ is the ball at the origin with radius r. $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to +\infty$.
• C, C_i, c will denote various positive constants which may vary from line to line.

3. The Nehari manifold

We consider the Nehari minimization problem: for $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$,

$$\theta_{\lambda,\mu} = \inf\{I_{\lambda,\mu}(z) \mid z \in N_{\lambda,\mu}\}$$

where $N_{\lambda,\mu} := \{z \in E \setminus \{0\} \mid \langle I'_{\lambda,\mu}(z), z\rangle = 0\}$ and

$$\langle I'_{\lambda,\mu}(z), z\rangle = \|z\|^2 - \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx - 2\int_\Omega h|w_1|^\alpha |w_2|^\beta dx. \tag{3.1}$$

Note that $N_{\lambda,\mu}$ contains every nonzero solution of problem (2.1).
Define

$$\langle \Phi_{\lambda,\mu}(z), z\rangle = \langle I'_{\lambda,\mu}(z), z\rangle.$$

Then

$$\langle \Phi'_{\lambda,\mu}(z), z\rangle = 2\|z\|^2 - q\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx - 2(\alpha + \beta)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx.$$

Moreover, if $\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx \ne 0$ and $z \in N_{\lambda,\mu}$, we have

$$\langle \Phi'_{\lambda,\mu}(z), z\rangle = (2 - q)\|z\|^2 - 2(\alpha + \beta - q)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx. \tag{3.2}$$

Similarly to the method used in [40], we split $N_{\lambda,\mu}$ into three parts.

$$N^+_{\lambda,\mu} = \{z \in N_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(z), z\rangle > 0\};$$

$$N^0_{\lambda,\mu} = \{z \in N_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(z), z\rangle = 0\};$$

$$N^-_{\lambda,\mu} = \{z \in N_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(z), z\rangle < 0\}.$$

Then, we have the following result.

Lemma 3.1. *For each $(\lambda, \mu) \in \Theta$, we have $N^0_{\lambda,\mu} = \emptyset$.*

Proof. We consider the following two cases
Case 1: $z \in N_{\lambda,\mu}$ and $\int_\Omega h|w_1|^\alpha |w_2|^\beta dx \le 0$, we have

$$\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx = \|z\|^2 - 2\int_\Omega h|w_1|^\alpha |w_2|^\beta dx > 0.$$

Thus $\langle \Phi'_{\lambda,\mu}(z), z\rangle = (2-q)\|z\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx > 0$ and so $z \notin N^0_{\lambda,\mu}$.
Case 2: $z \in N_{\lambda,\mu}$ and $\int_\Omega h|w_1|^\alpha |w_2|^\beta dx > 0$.
Suppose that $N^0_{\lambda,\mu} \ne \emptyset$ for all $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then for each $z \in N^0_{\lambda,\mu}$, we have

$$\langle \Phi'_{\lambda,\mu}(z), z\rangle = (2-q)\|z\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx = 0. \tag{3.3}$$

Thus

$$\|z\|^2 = \frac{2(\alpha+\beta-q)}{2-q}\int_\Omega h|w_1|^\alpha |w_2|^\beta dx$$

and

$$\begin{aligned}\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx &= \|z\|^2 - 2\int_\Omega h|w_1|^\alpha |w_2|^\beta dx \\ &= \frac{2(\alpha+\beta-2)}{2-q}\int_\Omega h|w_1|^\alpha |w_2|^\beta dx > 0.\end{aligned}$$

By the Hölder inequality, Sobolev inequality and 2-p inequality, we have

$$\|z\| \ge \left[\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right]^{\frac{1}{\alpha+\beta-2}} \tag{3.4}$$

and

$$\begin{aligned}\frac{\alpha+\beta-2}{\alpha+\beta-q}\|z\|^2 &= \|z\|^2 - 2\int_\Omega h|w_1|^\alpha |w_2|^\beta dx = \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx \\ &\le |\lambda|\|f\|_{L^{p^\star}}\||w_1\|^q_{L^{\alpha+\beta}} + |\mu|\|g\|_{L^{p^\star}}\||w_2\|^q_{L^{\alpha+\beta}} \\ &\le [(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}]^{\frac{2-q}{2}}(\frac{\kappa_s S}{2})^{-\frac{q}{2}}\|z\|^q.\end{aligned}$$

This implies

$$\|z\| \le \left(\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{\frac{1}{2-q}}\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{1}{2}}. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\begin{aligned}&\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right] \\ &\ge \left[\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right]^{\frac{2}{\alpha+\beta-2}}\left(\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{-\frac{2}{2-q}},\end{aligned}$$

contradicting with the assumption. □

Lemma 3.1 suggests that for each $(\lambda,\mu)\in\Theta$, we can write $N_{\lambda,\mu}=N^+_{\lambda,\mu}\cup N^-_{\lambda,\mu}$. Next, we define

$$\theta^+_{\lambda,\mu}=\inf_{z\in N^+_{\lambda,\mu}} I_{\lambda,\mu}(z) \ \ and \ \ \theta^-_{\lambda,\mu}=\inf_{z\in N^-_{\lambda,\mu}} I_{\lambda,\mu}(z).$$

The following lemma shows that the minimizer on $N_{\lambda,\mu}$ is critical point for $I_{\lambda,\mu}$

Lemma 3.2. *For each $(\lambda,\mu)\in\Theta$, let z_0 be a local minimizer for $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$, then $I'_{\lambda,\mu}(z_0)=0$ in E^{-1}.*

Proof. Since z_0 is a local minimizer for $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$, that is z_0 is a solution of the optimization problem

$$\min\{I_{\lambda,\mu}(z)\mid \Phi_{\lambda,\mu}(z)=0\}.$$

Then, by the theory of Lagrange multipliers, there exists a constant $L\in\mathbb{R}$ such that

$$\langle I'_{\lambda,\mu}(z_0),z_0\rangle=L\langle \Phi'_{\lambda,\mu}(z_0),z_0\rangle.$$

Since $z_0\notin N^0_{\lambda,\mu}$, we have $\langle \Phi'_{\lambda,\mu}(z_0),z_0\rangle\neq 0$, thus $L=0$, this completes the proof. $\square$

Moreover, we have the following properties about the Nehari manifold $N_{\lambda,\mu}$.

Lemma 3.3. *We have*
(i) If $z\in N^+_{\lambda,\mu}$, then $\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx>0$;
(ii) If $z\in N^-_{\lambda,\mu}$, then $\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0$.

Proof. (i) We consider the following two cases.
Case 1: If $\int_\Omega h|w_1|^\alpha|w_2|^\beta dx\le 0$, we have

$$\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx=\|z\|^2-2\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0.$$

Case 2: If $\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0$, since

$$\|z\|^2-\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx-2\int_\Omega h|w_1|^\alpha|w_2|^\beta dx=0$$

and

$$\langle \Phi'_{\lambda,\mu}(z),z\rangle=2\|z\|^2-q\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx-2(\alpha+\beta)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0,$$

it follows that

$$(2-q)\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx-2(\alpha+\beta-2)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0,$$

which implies

$$\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx>\frac{2(\alpha+\beta-2)}{2-q}\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0.$$

(ii) We consider the following two cases.
Case 1: If $\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx=0$, we have

$$2\int_\Omega h|w_1|^\alpha|w_2|^\beta dx=\|z\|^2>0.$$

Case 2: If $\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx \neq 0$, we have

$$(2-q)\|z\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx = \langle \Phi'_{\lambda,\mu}(z), z\rangle < 0.$$

Thus $\int_\Omega h|w_1|^\alpha|w_2|^\beta dx > 0$. $\square$

Lemma 3.4. *The following facts hold*

(i) If $(\lambda,\mu)\in\Theta$, then we have $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$;

(ii If $(\lambda,\mu)\in\Psi$, then we have $\theta^-_{\lambda,\mu} > c_0$ for some positive constant c_0 depending on λ,μ,q,S,κ_s;

(iii) The energy functional $I_{\lambda,\mu}$ is bounded below and coercive on $N_{\lambda,\mu}$.

Proof. (i) Let $z\in N^+_{\lambda,\mu}$, by (3.2), we have

$$\frac{2-q}{2(\alpha+\beta-q)}\|z\|^2 > \int_\Omega h|w_1|^\alpha|w_2|^\beta dx.$$

Hence

$$\begin{aligned} I_{\lambda,\mu}(z) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|z\|^2 + 2\left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx \\ &\le \left[\left(\frac{1}{2}-\frac{1}{q}\right)+\left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\frac{2-q}{\alpha+\beta-q}\right]\|z\|^2 \\ &\le \frac{(q-2)(\alpha+\beta-2)}{2q(\alpha+\beta)}\|z\|^2 < 0. \end{aligned}$$

Therefore, by the definition of $\theta_{\lambda,\mu},\ \theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu}\le\theta^+_{\lambda,\mu}<0$.

(ii) Let $z\in N^-_{\lambda,\mu}$, by (3.2), we have

$$\frac{2-q}{2(\alpha+\beta-q)}\|z\|^2 < \int_\Omega h|w_1|^\alpha|w_2|^\beta dx.$$

By the Hölder inequality and the Sobolev embedding theorem, we have

$$\int_\Omega h|w_1|^\alpha|w_2|^\beta dx \le (\kappa_s S)^{-\frac{\alpha+\beta}{2}}\|z\|^{\alpha+\beta}.$$

Hence,

$$\|z\| > \left(\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right)^{\frac{1}{\alpha+\beta-2}} \quad \textit{for all } z\in N^-_{\lambda,\mu}. \tag{3.6}$$

By (3.6), we have

$$I_{\lambda,\mu}(z) = \frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z\|^2 - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx$$

$$\ge \|z\|^q\left[\frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z\|^{2-q} - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}}+(|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\right]$$

$$> \left\{-\frac{\alpha+\beta-q}{q(\alpha+\beta)}\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}}+(|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\right.$$

$$+\frac{\alpha+\beta-2}{2(\alpha+\beta)}\left(\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right)^{\frac{2-q}{\alpha+\beta-2}}\Bigg\}\times\left(\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right)^{\frac{q}{\alpha+\beta-2}}.$$

Thus, if $(\lambda,\mu)\in\Psi$, then

$$I_{\lambda,\mu} > c_0,\ for\ all\ z\in N^-_{\lambda,\mu},$$

for some positive constant $c_0 = c_0(\lambda,\mu,S,q,\kappa_s)$.
(iii) Let $z\in N_{\lambda,\mu}$, by (3.1), Hölder inequality and Sobolev inequality, we have

$$I_{\lambda,\mu}(z)=\frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z\|^2-\frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx$$

$$\geq\frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z\|^2-\frac{\alpha+\beta-q}{q(\alpha+\beta)}\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}}+(|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\|z\|^q.$$

Since $1<q<2$, then the energy functional $I_{\lambda,\mu}$ is bounded below and coercive on $N_{\lambda,\mu}$. □

For each $z\in N^-_{\lambda,\mu}$, we write

$$t_{max}=\left(\frac{(2-q)\|z\|^2}{2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx}\right)^{\frac{1}{\alpha+\beta-2}}>0.$$

Then the following Lemma holds.

Lemma 3.5. *For each $(\lambda,\mu)\in\Theta$ and $z\in N^-_{\lambda,\mu}$, we have*
(i) *If*

$$\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx\leq 0,$$

then there exists a unique $t^-=t^-(z)>0$ such that $t^-(z)\in N^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^-(z))=\max_{t>0}I_{\lambda,\mu}(tz);$$

(ii) *If*

$$\int_\Omega(\lambda f|w_1|^q+\mu g|w_2|^q)\,dx>0,$$

then there exist unique $0<t^+<t_{max}<t^-$, such that $t^+(z)\in N^+_{\lambda,\mu},\ t^-z\in N^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^+(z))=\min_{0<t<t_{max}}I_{\lambda,\mu}(tz),\ I_{\lambda,\mu}(t^-(z))=\max_{t\geq 0}I_{\lambda,\mu}(tz).$$

Proof. Fix $z\in N^-_{\lambda,\mu}$, by Lemma 3.3, we have $\int_\Omega h|w_1|^\alpha|w_2|^\beta dx>0$. Let

$$m(t)=t^{2-q}\|z\|^2-2t^{\alpha+\beta-q}\int_\Omega h|w_1|^\alpha|w_2|^\beta dx,\ for\ t\geq 0.$$

Clearly, $m(0)=0$, $m(t)\to-\infty$ as $t\to\infty$. Since

$$m'(t)=(2-q)t^{1-q}\|z\|^2-2(\alpha+\beta-q)t^{\alpha+\beta-q-1}\int_\Omega h|w_1|^\alpha|w_2|^\beta dx,$$

we have that $m(t)$ is increasing for $t \in [0, t_{max})$, decreasing for $t \in (t_{max}, +\infty)$ and achieves its maximum at t_{max}. Moreover,

$$m(t_{max}) = \left(\frac{(2-q)\|z\|^2}{2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx} \right)^{\frac{2-q}{\alpha+\beta-2}} \|z\|^2$$

$$- 2\left(\frac{(2-q)\|z\|^2}{2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx} \right)^{\frac{\alpha+\beta-q}{\alpha+\beta-2}} \int_\Omega h|w_1|^\alpha |w_2|^\beta dx$$

$$= \|z\|^q \left[\left(\frac{2-q}{2(\alpha+\beta-q)} \right)^{\frac{2-q}{\alpha+\beta-2}} - 2\left(\frac{2-q}{2(\alpha+\beta-q)} \right)^{\frac{\alpha+\beta-q}{\alpha+\beta-2}} \right] \left(\frac{\|z\|^{\alpha+\beta}}{\int_\Omega h|w_1|^\alpha |w_2|^\beta dx} \right)^{\frac{2-q}{\alpha+\beta-2}}$$

$$\geq \|z\|^q \left(\frac{\alpha+\beta-2}{\alpha+\beta-q} \right) \left((\kappa_s S)^{\frac{\alpha+\beta}{2}} \frac{2-q}{2(\alpha+\beta-q)} \right)^{\frac{2-q}{\alpha+\beta-2}}.$$

That is

$$m(t_{max}) \geq \|z\|^q \left(\frac{\alpha+\beta-2}{\alpha+\beta-q} \right) \left((\kappa_s S)^{\frac{\alpha+\beta}{2}} \frac{2-q}{2(\alpha+\beta-q)} \right)^{\frac{2-q}{\alpha+\beta-2}}. \tag{3.7}$$

(i) If

$$\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx \leq 0,$$

by the property of $m(t)$, there is a unique $t^- > t_{max}$ such that

$$m(t^-) = \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx \text{ and } m'(t^-) < 0.$$

Since

$$\langle \Phi'_{\lambda,\mu}(t^- z), t^- z\rangle = (2-q)(t^-)^2\|z\|^2 - 2(\alpha+\beta-q)(t^-)^{\alpha+\beta} \int_\Omega h|w_1|^\alpha |w_2|^\beta dx \tag{3.8}$$

$$= (t^-)^{1+q} \left[(2-q)(t^-)^{1-q}\|z\|^2 - 2(\alpha+\beta-q)(t^-)^{\alpha+\beta-q-1} \int_\Omega h|w_1|^\alpha |w_2|^\beta dx \right]$$

$$= (t^-)^{1+q} m'(t^-) < 0$$

and

$$\langle I'_{\lambda,\mu}(t^- z), t^- z\rangle$$

$$= (t^-)^2\|z\|^2 - (t^-)^q \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx - 2(t^-)^{\alpha+\beta} \int_\Omega h|w_1|^\alpha |w_2|^\beta dx$$

$$= (t^-)^q \left[(t^-)^{2-q}\|z\|^2 - \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx - 2(t^-)^{\alpha+\beta-q} \int_\Omega h|w_1|^\alpha |w_2|^\beta dx \right]$$

$$= (t^-)^q \left[m(t^-) - \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx \right] = 0.$$

Thus $t^-(z) \in N^-_{\lambda,\mu}$.

For $t > t_{max}$, by (3.8), we know

$$(2-q)t^2\|z\|^2 - 2(\alpha+\beta-q)t^{\alpha+\beta} \int_\Omega h|w_1|^\alpha |w_2|^\beta dx < 0.$$

When $tz \in N_{\lambda,\mu}$, we have

$$\|z\|^2 - t^{q-2}\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx - 2t^{\alpha+\beta-2}\int_\Omega h|w_1|^\alpha|w_2|^\beta dx = 0$$

and

$$\begin{aligned}\frac{d^2}{dt^2}I_{\lambda,\mu}(tz) = \|z\|^2 &- (q-1)t^{q-2}\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx\\ &- 2(\alpha+\beta-1)t^{\alpha+\beta-2}\int_\Omega h|w_1|^\alpha|w_2|^\beta dx.\end{aligned}$$

Consequently,

$$\frac{d^2}{dt^2}I_{\lambda,\mu}(tz) = t^{q-1}m'(t) < 0.$$

Since

$$\frac{d}{dt}I_{\lambda,\mu}(tz) = t\|z\|^2 - t^{q-1}\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx - 2t^{\alpha+\beta-1}\int_\Omega h|w_1|^\alpha|w_2|^\beta dx,$$

we have $\frac{d}{dt}I_{\lambda,\mu}(tz) = 0$ for $t = t^-$. Thus, $I_{\lambda,\mu}(t^-(z)) = \max_{t>0} I_{\lambda,\mu}(tz)$.

(ii) If

$$\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx > 0.$$

Since

$$\begin{aligned}m(0) = 0 &< \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx\\ &\le \left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\|z\|^q\\ &\le \|z\|^q\left(\frac{\alpha+\beta-2}{\alpha+\beta-q}\right)\left((\kappa_s S)^{\frac{\alpha+\beta}{2}}\frac{2-q}{2(\alpha+\beta-q)}\right)^{\frac{2-q}{\alpha+\beta-2}}\\ &\le m(t_{max})\ for\ (\lambda,\mu)\in\Theta,\end{aligned}$$

therefore, there are unique t^+ and t^- such that $0 < t^+ < t_{max} < t^-$,

$$m(t^+) = \int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\,dx = m(t^-)$$

and

$$m'(t^+) > 0 > m'(t^-).$$

By the same arguments as (i), we have

$$t^+z \in N^+_{\lambda,\mu},\ t^-z \in N^-_{\lambda,\mu},\ I_{\lambda,\mu}(t^-z) \ge I_{\lambda,\mu}(tz) \ge I_{\lambda,\mu}(t^+z)$$

for each $t \in [t^+, t^-]$ and $I_{\lambda,\mu}(t^+z) \le I_{\lambda,\mu}(tz)$ for each $t \in [0, t^+]$.
That is

$$I_{\lambda,\mu}(t^+(z)) = \min_{0<t<t_{max}} I_{\lambda,\mu}(tz),\quad I_{\lambda,\mu}(t^-(z)) = \max_{t\ge 0} I_{\lambda,\mu}(tz). \qquad \square$$

4. Existence of Palais-Smale sequence

Definition 4.1. We say that $z_n \in E$ is a $(PS)_c$ sequence in E for $I_{\lambda\mu}$, if

$$I_{\lambda\mu}(z_n) = c + o_n(1)$$

and $I'_{\lambda\mu}(z_n) = o_n(1)$ strongly in E^{-1} as $n \to \infty$. If any $(PS)_c$ sequence in E for $I_{\lambda\mu}$ admits a convergent subsequence, we say that $I_{\lambda\mu}$ satisfies the $(PS)_c$ condition.

First, we will use the idea of [27] to get the following results.

Lemma 4.2. *Let $(\lambda, \mu) \in \Theta$, then for each $z \in N_{\lambda,\mu}$, there exists $r > 0$ and a differentiable function $\xi : B(0, r) \subset E \to \mathbb{R}^+$ such that $\xi(0) = 1$ and $\xi(v)(z-v) \in N_{\lambda,\mu}$ for every $v \in B(0, r)$. Let*

$$\mathrm{T}_1 := 2\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}(\nabla w_1 \nabla v_1 + \nabla w_2 \nabla v_2)dxdy,$$
$$\mathrm{T}_2 := q \int_\Omega \left(\lambda f|w_1|^{q-2}w_1 v_1 + \mu g|w_2|^{q-2}w_2 v_2\right) dx,$$
$$\mathrm{T}_3 := 2 \int_\Omega \left(\alpha h|w_1|^{\alpha-2}w_1 v_1|w_2|^\beta + \beta h|w_1|^\alpha|w_2|^{\beta-2}w_2 v_2\right) dx,$$

then,

$$\langle \xi'(0), v\rangle = \frac{\mathrm{T}_2 + \mathrm{T}_3 - \mathrm{T}_1}{(2-q)\|z\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx} \tag{4.1}$$

holds for all $v \in E$.

Proof. For $z = (w_1, w_2) \in N_{\lambda,\mu}$, define a function $F : \mathbb{R} \times E \to \mathbb{R}$ by

$$\begin{aligned} F_z(\xi, p) &:= \langle I'_{\lambda,\mu}(\xi(z-p)), \xi(z-p)\rangle \\ &= \xi^2\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}(|\nabla(w_1-p_1)|^2 + |\nabla(w_2-p_2)|^2)dxdy \\ &- \xi^q \int_\Omega (\lambda f|w_1-p_1|^q + \mu g|w_2-p_2|^q)\, dx - 2\xi^{\alpha+\beta} \int_\Omega h|w_1-p_1|^\alpha|w_2-p_2|^\beta dx. \end{aligned}$$

Then, $F_z(1, 0) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$ and by lemma 3.1, we have $N^0_{\lambda,\mu} = \emptyset$.
That is

$$\begin{aligned} \frac{dF_z(1,0)}{d\xi} &= 2\|z\|^2 - q\int_\Omega (\lambda f|w_1|^q + \mu g|w_2|^q)\, dx - 2(\alpha+\beta)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx \\ &= (2-q)\|z\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha|w_2|^\beta dx \neq 0. \end{aligned}$$

According to the implicit function theorem, there exist $r > 0$ and a differentiable function $\xi : B(0, r) \subset E \to \mathbb{R}^+$ such that $\xi(0) = 1$ and (4.1) holds. Moreover, $F_z(\xi(v), v) = 0$ holds for all $v \in B(0, r)$ is equivalent to

$$\langle I'_{\lambda,\mu}(\xi(v)(z-v)), \xi(v)(z-v)\rangle = 0$$

for all $v \in B(0, r)$. That is $\xi(v)(z - v) \in N_{\lambda,\mu}$. □

Lemma 4.3. *Let $(\lambda,\mu) \in \Theta$, then for each $z \in N^-_{\lambda,\mu}$, there exists $r > 0$ and a differentiable function $\xi^- : B(0,r) \subset E \to \mathbb{R}^+$ such that*

$$\xi^-(0) = 1 \text{ and } \xi^-(v)(z - v) \in N_{\lambda,\mu}$$

for every $v \in B(0,r)$ and formula (4.1) holds.

Proof. Similar to the argument in Lemma 4.2, there exists $r > 0$ and a differentiable function $\xi^- : B(0,r) \subset E \to \mathbb{R}^+$ such that $\xi^-(0) = 1$ and $\xi^-(v)(z-v) \in N_{\lambda,\mu}$ for every $v \in B(0,r)$ and formula (4.1) holds. Since

$$\langle \Phi'_{\lambda,\mu}(z), z\rangle = (2-q)\|z\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_1|^\alpha |w_2|^\beta dx < 0,$$

by the continuity of function $\Phi'_{\lambda,\mu}$ and ξ^-, we have

$$\begin{aligned}\langle \Phi'_{\lambda,\mu}(\xi^-(v)(z-v)), \xi^-(v)(z-v)\rangle &= (2-q)\|\xi^-(v)(z-v)\|^2 \\ -2(\alpha+\beta-q)\int_\Omega h|(\xi^-(v)(z-v))_1|^\alpha &|(\xi^-(v)(z-v))_2|^\beta dx < 0.\end{aligned}$$

This implies that $\xi^-(v)(z-v) \in N^-_{\lambda,\mu}$. □

Lemma 4.4. *The following facts hold:*

(i) If $(\lambda,\mu) \in \Theta$, then there is a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{z_n\} \subset N_{\lambda,\mu}$ for $I_{\lambda,\mu}$;

(ii) If $(\lambda,\mu) \in \Psi$, then there is a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{z_n\} \subset N^-_{\lambda,\mu}$ for $I_{\lambda,\mu}$.

Proof. (i) By Lemma 3.4(iii) and Ekeland Variational Principle, there exists a minimizing sequence $\{z_n\} \subset N_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n) < \theta_{\lambda,\mu} + \frac{1}{n}, \tag{4.2}$$

$$I_{\lambda,\mu}(z_n) < I_{\lambda,\mu}(w) + \frac{1}{n}\|w - z_n\|, \ \forall\, w \in N_{\lambda,\mu}.$$

By taking n large, from Lemma 3.4(i), we have $\theta_{\lambda,\mu} < 0$, thus

$$\begin{aligned} I_{\lambda,\mu}(z_n) &= \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z_n\|^2 - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)\int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\,dx \\ &< \theta_{\lambda,\mu} + \frac{1}{n} < \frac{\theta_{\lambda,\mu}}{2}. \end{aligned} \tag{4.3}$$

This implies

$$\begin{aligned} -\frac{q(\alpha+\beta)}{2(\alpha+\beta-q)}\theta_{\lambda,\mu} &< \int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\,dx \\ &\le \left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}} \left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}} \|z_n\|^q. \end{aligned} \tag{4.4}$$

Consequently, $z_n \neq 0$ and putting together (4.3), (4.4) and the Hölder inequality, we obtain

$$\|z_n\| > \left[-\frac{q(\alpha+\beta)}{2(\alpha+\beta-q)}\theta_{\lambda,\mu}\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{q-2}{2}}\left(\frac{\kappa_s S}{2}\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}.$$

and

$$\|z_n\| < \left[\frac{2(\alpha+\beta-q)}{q(\alpha+\beta-2)}\left[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\right]^{\frac{1}{2-q}}. \quad (4.5)$$

Now, we will show that

$$\|I'_{\lambda,\mu}(z_n)\|_{E^{-1}} \to 0 \quad as \quad n \to +\infty.$$

Applying Lemma 4.2 to z_n, we can obtain the function $\xi_n : B(0, r_n) \subset E \to \mathbb{R}^+$ such that $\xi_n(0) = 1$ and $\xi_n(v)(z_n - v) \in N_{\lambda,\mu}$ for every $v \in B(0, r_n)$. Taking $0 < \rho < r_n$, let $w \in E$ with $w \neq 0$ and put $v^\star = \frac{\rho w}{\|w\|}$. We set $v_\rho = \xi_n(v^\star)(z_n - v^\star)$, then $v_\rho \in N_{\lambda,\mu}$. By (4.2), we have

$$I_{\lambda,\mu}(v_\rho) - I_{\lambda,\mu}(z_n) \geq -\frac{1}{n}\|v_\rho - z_n\|.$$

By the Mean Value Theorem, we get

$$\langle I'_{\lambda,\mu}(z_n), v_\rho - z_n\rangle + o(\|v_\rho - z_n\|) \geq -\frac{1}{n}\|v_\rho - z_n\|.$$

Thus, we have

$$\langle I'_{\lambda,\mu}(z_n), -v^\star\rangle + (\xi_n(v^\star) - 1)\langle I'_{\lambda,\mu}(z_n), z_n - v^\star\rangle \geq -\frac{1}{n}\|v_\rho - z_n\| + o(\|v_\rho - z_n\|). \quad (4.6)$$

From $\xi_n(v^\star)(z_n - v^\star) \in N_{\lambda,\mu}$ and (4.6), we obtain

$$-\rho\left\langle I'_{\lambda,\mu}(z_n), \frac{w}{\|w\|}\right\rangle + (\xi_n(v^\star) - 1)\langle I'_{\lambda,\mu}(z_n) - I'_{\lambda,\mu}(v_\rho), z_n - v^\star\rangle$$
$$\geq -\frac{1}{n}\|v_\rho - z_n\| + o(\|v_\rho - z_n\|).$$

So, we get

$$\left\langle I'_{\lambda,\mu}(z_n), \frac{w}{\|w\|}\right\rangle \leq \frac{\|v_\rho - z_n\|}{n\rho} + \frac{o(\|v_\rho - z_n\|)}{\rho} + \frac{(\xi_n(v^\star) - 1)}{\rho}\langle I'_{\lambda,\mu}(z_n) - I'_{\lambda,\mu}(v_\rho), z_n - v^\star\rangle. \quad (4.7)$$

Since

$$\|v_\rho - z_n\| \leq \rho|\xi_n(v^\star)| + |\xi_n(v^\star) - 1|\|z_n\|$$

and

$$\lim_{\rho\to 0}\frac{|\xi_n(v^\star) - 1|}{\rho} \leq \|\xi'_n(0)\|.$$

If we let $\rho \to 0$ in (4.7) for fixed $n \in \mathbb{N}$, then by (4.5) we can find a constant $C > 0$, independent of ρ such that

$$\langle I'_{\lambda,\mu}(z_n), \frac{w}{\|w\|}\rangle \leq \frac{C}{n}\left(1 + \|\xi'_n(0)\|\right).$$

Thus, we are done once we show that $\|\xi_n'(0)\|$ is uniformly bounded. By (4.1), (4.5) and Hölder inequality, we have

$$|\langle \xi_n'(0), v\rangle| \leq \frac{C_1\|v\|}{\left|(2-q)\|z_n\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx\right|},$$

for some $C_1 > 0$. We only need to show that

$$\left|(2-q)\|z_n\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx\right| \geq C_2,$$

for some $C_2 > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence z_n such that

$$(2-q)\|z_n\|^2 - 2(\alpha+\beta-q)\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx = o_n(1). \tag{4.8}$$

By (4.8) and the fact that $z_n \in N_{\lambda,\mu}$, we have

$$\|z_n\|^2 = \frac{2(\alpha+\beta-q)}{2-q}\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx + o_n(1)$$

and

$$\|z_n\|^2 = \frac{\alpha+\beta-q}{\alpha+\beta-2}\int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\, dx + o_n(1).$$

By the Hölder and Sobolev inequalities, when n large enough, we have

$$\|z_n\| \geq [\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}]^{\frac{1}{\alpha+\beta-2}} \tag{4.9}$$

and

$$\begin{aligned}
\frac{\alpha+\beta-2}{\alpha+\beta-q}\|z_n\|^2 &= \int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\, dx \\
&\leq |\lambda|\|f\|_{L^{p^\star}}\|\|w_{1,n}\|^q_{L^{\alpha+\beta}} + |\mu|\|g\|_{L^{p^\star}}\|\|w_{2,n}\|^q_{L^{\alpha+\beta}} \\
&\leq [(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}]^{\frac{2-q}{2}}(\frac{\kappa_s S}{2})^{-\frac{q}{2}}\|z_n\|^q.
\end{aligned}$$

This implies

$$\|z_n\| \leq \left(\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{\frac{1}{2-q}}[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}]^{\frac{1}{2}}. \tag{4.10}$$

By (4.9) and (4.10), we have

$$\begin{aligned}
&[(|\lambda|\|f\|_{L^{p^\star}})^{\frac{2}{2-q}} + (|\mu|\|g\|_{L^{p^\star}})^{\frac{2}{2-q}}] \\
&\geq \left[\frac{2-q}{2(\alpha+\beta-q)}(\kappa_s S)^{\frac{\alpha+\beta}{2}}\right]^{\frac{2}{\alpha+\beta-2}}\left(\left(\frac{\kappa_s S}{2}\right)^{-\frac{q}{2}}\frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{-\frac{2}{2-q}},
\end{aligned}$$

contradicting the assumption, that is

$$\left\langle I'_{\lambda,\mu}(z_n), \frac{w}{\|w\|}\right\rangle \leq \frac{C}{n}.$$

This completes the proof of (1).
Similarly, by Lemma 4.3, we can prove (ii), we omit the details here. $\square$

Lemma 4.5. *If $\{z_n\} \subset E$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$, then $\{z_n\}$ is bounded in E.*

Proof. Let $z_n = (w_{1,n}, w_{2,n}) \subset E$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$, suppose by contradiction that $\|z_n\| \to +\infty$ as $n \to +\infty$. Let

$$\widetilde{z_n} = (\widetilde{w}_{1,n}, \widetilde{w}_{2,n}) := \frac{z_n}{\|z_n\|} = \left(\frac{w_{1,n}}{\|z_n\|}, \frac{w_{2,n}}{\|z_n\|}\right).$$

We may assume that $\widetilde{z}_n \rightharpoonup \widetilde{z} = (\widetilde{w}_1, \widetilde{w}_2)$ in E. By the compact embedding theorem, we know $\widetilde{w}_{1,n}(\cdot,0) \to \widetilde{w}_1(\cdot,0)$ and $\widetilde{w}_{2,n}(\cdot,0) \to \widetilde{w}_2(\cdot,0)$ strongly in $L^r(\Omega)$ for all $1 \le r < 2^\star$. Thus, by Hölder inequality and Dominated convergence theorem, we have

$$\int_\Omega (\lambda f|\widetilde{w}_{1,n}|^q + \mu g|\widetilde{w}_{2,n}|^q)\,dx = \int_\Omega (\lambda f|\widetilde{w}_1|^q + \mu g|\widetilde{w}_2|^q)\,dx + o_n(1).$$

Since $\{z_n\}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ and $\|z_n\| \to +\infty$, we have

$$\begin{aligned} &\frac{\kappa_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \\ &- \frac{\|z_n\|^{q-2}}{q}\int_\Omega (\lambda f(x)|\widetilde{w}_{1,n}|^q + \mu g(x)|\widetilde{w}_{2,n}|^q)\,dx \\ &- \frac{2\|z_n\|^{\alpha+\beta-2}}{\alpha+\beta}\int_\Omega h(x)|\widetilde{w}_{1,n}|^\alpha|\widetilde{w}_{2,n}|^\beta dx = o_n(1) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\kappa_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \\ &- \|z_n\|^{q-2}\int_\Omega (\lambda f(x)|\widetilde{w}_{1,n}|^q + \mu g(x)|\widetilde{w}_{2,n}|^q)\,dx \\ &- 2\|z_n\|^{\alpha+\beta-2}\int_\Omega h(x)|\widetilde{w}_{1,n}|^\alpha|\widetilde{w}_{2,n}|^\beta dx = o_n(1). \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), as $n \to \infty$, we obtain

$$\begin{aligned} &\kappa_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \\ &= \frac{2(\alpha+\beta-q)}{q(\alpha+\beta-2)}\|z_n\|^{q-2}\int_\Omega (\lambda f(x)|\widetilde{w}_{1,n}|^q + \mu g(x)|\widetilde{w}_{2,n}|^q)\,dx + o_n(1). \end{aligned} \tag{4.13}$$

Since $1 < q < 2$ and $\|z_n\| \to +\infty$ as $n \to +\infty$, (4.13) implies

$$\kappa_s\int_{\mathcal{C}_\Omega} y^{1-2s}\left(|\nabla\widetilde{w}_{1,n}|^2 + |\nabla\widetilde{w}_{2,n}|^2\right)dxdy \to 0.$$

Which contradicts the fact that $\|\widetilde{z}_n\| = 1$ for any $n \ge 1$. $\square$

5. Local minimization problem

Now, we establish the existence of a local minimum for $I_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$.

Theorem 5.1. *Let $(\lambda,\mu)\in\Theta$, then $I_{\lambda,\mu}$ has a local minimizer z^+ in $N^+_{\lambda,\mu}$ satisfying*
(*i*) $I_{\lambda,\mu}(z^+)=\theta_{\lambda,\mu}=\theta^+_{\lambda,\mu}$;
(*ii*) *z^+ is a positive solution of* (2.1).

Proof. By (*i*) of Lemma 4.4 there exists a minimizing sequence $\{z_n\}=\{(w_{1,n},w_{2,n})\}$ for $I_{\lambda,\mu}$ in $N_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n)=\theta_{\lambda,\mu}+o_n(1)\ \textit{and}\ I'_{\lambda,\mu}(z_n)=o_n(1)\ \textit{in}\ E^{-1}. \tag{5.1}$$

By Lemma 3.4, Lemma 4.5 and the compact imbedding theorem, we know there is a subsequence, still denoted by $\{z_n\}$ and $z^+=(w_1^+,w_2^+)\in E$ such that

$$\begin{cases} w_{1,n}\rightharpoonup w_1^+, w_{2,n}\rightharpoonup w_2^+, & \textit{weakly} \text{ in } X_0^s(\Omega),\\ w_{1,n}\to w_1^+, w_{2,n}\to w_2^+, & \textit{srongly} \text{ in } L^r(\Omega)\ \textit{for all}\ 1\le r<2^\star. \end{cases}$$

As $n\to\infty$, by Hölder inequality and Dominated convergence theorem, we obtain

$$\int_\Omega (\lambda f|w_{1,n}|^q+\mu g|w_{2,n}|^q)\,dx=\int_\Omega \left(\lambda f|w_1^+|^q+\mu g|w_2^+|^q\right)dx+o_n(1) \tag{5.2}$$

and

$$\int_\Omega h|w_{1,n}|^\alpha|w_{2,n}|^\beta dx=\int_\Omega h|w_1^+|^\alpha|w_2^+|^\beta dx+o_n(1). \tag{5.3}$$

First, we claim that

$$\int_\Omega \left(\lambda f|w_1^+|^q+\mu g|w_2^+|^q\right)dx\neq 0,$$

we argue by contradiction, then we have $\int_\Omega (\lambda f|w_{1,n}|^q+\mu g|w_{2,n}|^q)\,dx\to 0$ as $n\to\infty$. Thus

$$\|z_n\|^2=2\int_\Omega h|w_{1,n}|^\alpha|w_{2,n}|^\beta dx+o_n(1)$$

and

$$\begin{aligned} I_{\lambda,\mu}(z_n)&=\frac{1}{2}\|z_n\|^2-\frac{2}{\alpha+\beta}\int_\Omega h(x)|w_{1,n}|^\alpha|w_{2,n}|^\beta dx+o_n(1)\\ &=\left(\frac{1}{2}-\frac{1}{\alpha+\beta}\right)\|z_n\|^2+o_n(1). \end{aligned}$$

This contradicts $I_{\lambda,\mu}(z_n)\to\theta_{\lambda,\mu}<0$ as $n\to\infty$.
Now, we claim z^+ is a nontrivial solution of (2.1). From (5.1), (5.2) and (5.3), we know z^+ is a weak solution of (2.1). From $z_n\in N_{\lambda,\mu}$, we have

$$I_{\lambda,\mu}(z_n)=\frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z_n\|^2-\frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega (\lambda f|w_{1,n}|^q+\mu g|w_{2,n}|^q)\,dx. \tag{5.4}$$

That is

$$\int_\Omega (\lambda f|w_{1,n}|^q+\mu g|w_{2,n}|^q)\,dx=\frac{q(\alpha+\beta-2)}{2(\alpha+\beta-q)}\|z_n\|^2-\frac{q(\alpha+\beta)}{\alpha+\beta-q}I_{\lambda,\mu}(z_n). \tag{5.5}$$

Let $n \to \infty$ in (5.5), by (5.1), (5.2) and $\theta_{\lambda,\mu} < 0$, we have

$$\int_\Omega \left(\lambda f|w_1^+|^q + \mu g|w_2^+|^q\right) dx \geq -\frac{q(\alpha+\beta)}{\alpha+\beta-q}\theta_{\lambda,\mu} > 0.$$

Therefore, $z^+ \in N_{\lambda,\mu}$ is a nontrival solution of (2.1). Next, we show that $z_n \to z^+$ strongly in E and $I_{\lambda,\mu}(z^+) = \theta_{\lambda,\mu}$. Since $z^+ \in N_{\lambda,\mu}$, then by (5.4), we obtain

$$\begin{aligned} \theta_{\lambda,\mu} \leq I_{\lambda,\mu}(z^+) &= \frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z^+\|^2 - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega \left(\lambda f|w_1^+|^q + \mu g|w_2^+|^q\right) dx \qquad (5.6)\\ &\leq \liminf_{n\to\infty}\left(\frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z_n\|^2 - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega \left(\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q\right) dx\right)\\ &\leq \lim_{n\to\infty}\left(\frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z_n\|^2 - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega \left(\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q\right) dx\right)\\ &\leq \lim_{n\to\infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}. \end{aligned}$$

This implies that $I_{\lambda,\mu}(z^+) = \theta_{\lambda,\mu}$ and $\lim\limits_{n\to\infty}\|z_n\|^2 = \|z^+\|^2$. Hence $z_n \to z^+$ srongly in E.

Finally, we claim that $z^+ \in N^+_{\lambda,\mu}$. Assume by contradiction that $z^+ \in N^-_{\lambda,\mu}$, then by Lemma 3.5, there exist unique t_1^+ and t_1^-, such that $t_1^+(z^+) \in N^+_{\lambda,\mu}$, $t_1^-(z^+) \in N^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt}I_{\lambda,\mu}(t_1^+z^+) = 0 \text{ and } \frac{d^2}{dt^2}I_{\lambda,\mu}(t_1^+z^+) > 0,$$

there exists $t_1^+ < t^\star < t_1^-$ such that $I_{\lambda,\mu}(t_1^+z^+) < I_{\lambda,\mu}(t^\star z^+)$. By Lemma 3.5, we have

$$I_{\lambda,\mu}(t_1^+z^+) < I_{\lambda,\mu}(t^\star z^+) \leq I_{\lambda,\mu}(t_1^-z^+) = I_{\lambda,\mu}(z^+),$$

a contraction. Since $I_{\lambda,\mu}(z^+) = I_{\lambda,\mu}(|w_1^+|, |w_2^+|)$ and $(|w_1^+|, |w_2^+|) \in N_{\lambda,\mu}$, by Lemma 3.2, we may assume that z^+ is a nontrivial nonnegative solution of (2.1). Then by the Strong Maximum Principle [11], we have $w_1^+, w_2^+ > 0$ in $\mathcal{C}(\Omega)$, hence z^+ is positive solution for (2.1). □

Next, we establish the existence of a local minimum for $I_{\lambda,\mu}$ on $N^-_{\lambda,\mu}$.

Theorem 5.2. *Let $(\lambda,\mu) \in \Psi$, then $I_{\lambda,\mu}$ has a local minimizer z^- in $N^-_{\lambda,\mu}$ satisfying*

(i) $I_{\lambda,\mu}(z^-) = \theta^-_{\lambda,\mu}$;

(ii) z^- is a positive solution of (2.1).

Proof. By (ii) of Lemma 4.4 there exists a minimizing sequence $\{z_n\} = \{(w_{1,n}, w_{2,n})\}$ for $I_{\lambda,\mu}$ in $N^-_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n) = \theta^-_{\lambda,\mu} + o_n(1) \textit{ and } I'_{\lambda,\mu}(z_n) = o_n(1) \textit{ in } E^{-1}.$$

By Lemma 3.4 (iii), Lemma 4.5 and the compact imbedding theorem, we know there is a subsequence, still denoted by $\{z_n\}$ and $z^- = (w_1^-, w_2^-) \in N^-_{\lambda,\mu}$ such that

$$\begin{cases} w_{1,n} \rightharpoonup w_1^-, w_{2,n} \rightharpoonup w_2^-, & \textit{weakly in} X_0^s(\Omega),\\ w_{1,n} \to w_1^-, w_{2,n} \to w_2^-, & \textit{srongly in} L^r(\Omega) \textit{ for all } 1 \leq r < 2^\star. \end{cases}$$

As $n \to \infty$, this implies that

$$\int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\, dx = \int_\Omega \left(\lambda f|w_1^-|^q + \mu g|w_2^-|^q\right) dx + o_n(1)$$

and

$$\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx = \int_\Omega h|w_1^-|^\alpha |w_2^-|^\beta dx + o_n(1).$$

First, we claim that

$$\int_\Omega \left(\lambda f|w_1^-|^q + \mu g|w_2^-|^q\right) dx \neq 0,$$

suppose by contradiction, then we have

$$\int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\, dx \to 0 \text{ as } n \to \infty.$$

Thus

$$\|z_n\|^2 = 2\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx + o_n(1)$$

and

$$\begin{aligned} I_{\lambda,\mu}(z_n) &= \frac{1}{2}\|z_n\|^2 - \frac{2}{\alpha+\beta}\int_\Omega h(x)|w_{1,n}|^\alpha |w_{2,n}|^\beta dx + o_n(1) \\ &= \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z_n\|^2 + o_n(1). \end{aligned}$$

This contradicts $I_{\lambda,\mu}(z_n) \to \theta_{\lambda,\mu} < 0$ as $n \to \infty$.
Now, we prove that $z_n \to z^-$ strongly in E. Othercase, we have

$$\begin{aligned} &\|z^-\|^2 - \int_\Omega \left(\lambda f|w_1^-|^q + \mu g|w_2^-|^q\right) dx - 2\int_\Omega h|w_1^-|^\alpha |w_2^-|^\beta dx \\ &\leq \liminf_{n\to\infty} \left(\|z_n\|^2 - \int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\, dx - 2\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx\right) \\ &\leq \lim_{n\to\infty} \left(\|z_n\|^2 - \int_\Omega (\lambda f|w_{1,n}|^q + \mu g|w_{2,n}|^q)\, dx - 2\int_\Omega h|w_{1,n}|^\alpha |w_{2,n}|^\beta dx\right) = 0. \end{aligned}$$

Which contradicts $z^- \in N^-_{\lambda,\mu}$. Hence $z_n \to z^-$ strongly in E. This implies

$$I_{\lambda,\mu}(z_n) \to I_{\lambda,\mu}(z^-) = \theta^-_{\lambda,\mu} \ as\ n \to +\infty.$$

Since $I_{\lambda,\mu}(z_n = I_{\lambda,\mu}(|z^-|)$ and $|z^-| \in N^-_{\lambda,\mu}$, by Lemma 3.2, we have z^- is a solution of problem (2.1), such that $z^- \geq 0$ in $\mathcal{C}(\Omega)$. Finally, by the same arguements as in the proof of Theorem 5.1, we have that z^- is a positive solotion of (2.1). □

6. Proof of Theorem 1.1 and Theorem 1.2

Now, we complete the proof of Theorem 1.1 and Theorem 1.2.

Proof. For $(\lambda,\mu)\in\Theta$, by Theorem 5.1, system (2.1) admits at least one positive solution $z^+\in N^+_{\lambda,\mu}$ such that $z^+>0$ in $\mathcal{C}(\Omega)$. By Theorem 5.1 and Theorem 5.2, we obtain that for $(\lambda,\mu)\in\Psi$, system (2.1) admits at least two positive solution z^+ and z^- such that $z^+\in N^+_{\lambda,\mu}$, $z^-\in N^-_{\lambda,\mu}$ and $z^+>0, z^->0$ in $\mathcal{C}(\Omega)$. Since $N^+_{\lambda,\mu}\cap N^+_{\lambda,\mu}=\emptyset$, then z^+ and z^- are distinct solutions of syetem (2.1). In turn, $(u^\pm(x),v^\pm(x))=(w_1^\pm(x,0),w_2^\pm(x,0))$ are distinct solutions of (1.1). □

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