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COINCIDENCE POINTS FOR SET-VALUED MAPPINGS WITH DIRECTIONAL REGULARITY

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Abstract. This paper is devoted to investigate the interrelations between directional metric regularity and coincidence points for set-valued mappings. Under the assumption of directional metric regularity and directional Aubin continuity, new coincidence point theorems were established through iteration procedures for both local and global cases. As an application, the (global) directional Aubin continuity for the solution mapping of partial-parametrized variational system was established.
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1. Introduction

The concept of metric regularity (closely tied with the covering property, Aubin property) traces back to the classical Banach open mapping principle for linear continuous operators. In the past few decades, metric regularity has been widely recognized as a fundamental property in various aspects of optimization, and especially it's valuable in convergence analysis of algorithms for solving optimization problems and beyond. For a comprehensive understanding of the developments, historical remarks together with applications of mapping regularities, see [15, 23].

Let X and Y be normed vector spaces. We recall that a closed multifunction $F : X \rightrightarrows Y$ is said to be metrically regular around

$$(\bar{x}, \bar{y}) \in \operatorname{gph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

with constant $\kappa > 0$, if there are $\delta > 0$ and some neighborhoods $U \subset X$ and $V \subset Y$ of $\bar{x}$ and $\bar{y}$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \quad \forall (x, y) \in U \times V \text{ with } d(y, F(x)) < \delta, \tag{1.1}$$

where $d(y, F(x))$ denotes the distance of point y to the set $F(x)$. It is well-known that the metric regularity is equivalent to the property of local covering, for more details, see [15, 23].

For a pair of set-valued mappings $F, G : X \rightrightarrows Y$, $x \in X$ is said to be the coincidence point of (F, G), if $F(x) \cap G(x) \neq \emptyset$. In the papers [3, 4], A.V. Arutyunov established the theory of coincidence point for covering mappings and Lipschitzian mappings. Later these results were extended to local case in [10]. There exists a tight connection between the property of covering/metric regularity and the theory of coincidence points for set-valued mappings due to the fact that both theories share common Lipschitz-type properties (see, [1, 3, 4, 12, 14, 20, 22]). Earliest discussions on the relationship between the aforementioned theories can be traced back to the proof of the Lyusternik-Graves theorem by virtue of the set-valued contraction mapping principle (see [19, 24]). Based on the approach and theory proposed by A.V. Arutynov, many scholars studied the theory of coincidence point and its related applications. For instance, in [7, 8], the authors studied the theory of coincidence points in partially ordered spaces. Under regularity assumptions, the author in [20, 22] considered a so called "double fixed point" and the authors in [14] established both local and global versions of fixed point theorems for $F^{-1}G$. The aforementioned metric fixed point theories can be viewed as modifications of the results from [3] on coincidence points of a pair of mappings (F, G). The theory of coincidence point and its related metric fixed point theory was applied to various problems, see [2, 9, 11, 13, 21] and the references therein.

Recently, the classical notions of regularity which imply the action of mappings around the reference points in all directions have been extended to the case where the relations defining these properties hold only on some directions by many authors, for more details (see [6, 5, 17, 18, 26], and the comments therein). Motivated by the fact that the classical regularity properties do not distinguish between minima and maxima, Durea, Pantiruc and Strugariu [17] introduced a new directional metric regularity property for set-valued mappings acting between normed vector spaces using directional minimal time function instead of the distance of point to set in (1.1). It then naturally brings us the idea to propose a question that if we relax the hypotheses in coincidence point theories to directional cases, would the corresponding results still be true?

The main objective in this paper is to explore the connection between the aforementioned directional metric regularity and coincidence point theories for set-valued mappings. Under the assumptions of $F : X \rightrightarrows Y$ being directionally metrically regular and $G : X \rightrightarrows Y$ being directional Aubin continuous, the existence of the coincidence points of set-valued mappings (F, G) and (F^{-1}, G^{-1}) were considered and the "directional distance" with the corresponding directions from given sets to the set of coincidence points of set-valued mappings (F, G) and (F^{-1}, G^{-1}), respectively, were all estimated.

The rest of the paper is organized as follows. Section 2 contains definitions of the basic properties under consideration and some preliminary results. In sections 3, we investigate in detail the interrelation between directional regularity properties and coincidence point theories for both local and global cases. A sufficient condition for directional Aubin continuity of the solution mapping of partial-parameterized variational system was also established.

2. Notations and preliminary results

This section presents basic definitions and preliminaries widely used in what follows. Let X be a normed vector space, the symbol S_X stands for the unit sphere of X while $B_\alpha(x)$ indicates the closed ball of radius $\alpha > 0$ centered at $x \in X$ in the space X. For a set-valued mapping $F : X \rightrightarrows Y$, its domain and graph are defined as $\mathrm{dom}(F) := \{x \in X : F(x) \neq \emptyset\}$ and $\mathrm{gph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$, respectively. The symbol $F^{-1} : Y \rightrightarrows X$ stands for the inverse mapping of F with $F^{-1}(y) = \{x \in X : y \in F(x)\}$.

Let $A \subset X$ be a nonempty set. The cone generated by A is designated by cone$A$. A is said to be locally closed (resp., complete) at $\bar{x} \in A$ when there exists $r \in (0, \infty)$ such that $A \cap B_r(\bar{x})$ is closed (resp., complete). Recall that for a point $u \in X \backslash \{0\}$, the minimal time function is defined as follows [16, 25]: for any $x \in X$,

$$T_u(x, A) := \inf\{t \geq 0 : x + tu \in A\}.$$

It is easy to see that $T_u(x, A) = \lambda T_{\lambda u}(x, A)$, for any $\lambda > 0$, and hence one can consider that $u \in S_X$. To study the directional regularity for multifunctions, the authors in [17] introduced the following definition of directional minimal time function:

Definition 2.1. Consider $M \subset S_X$ and nonempty sets $A, B \subset X$. Then the function

$$\begin{aligned} T_M(x, A) : &= \inf\{t \geq 0 : \exists u \in M \; s.t. \; x + tu \in A\} \\ &= \inf\{t \geq 0 : (x + tM) \cap A \neq \emptyset\} \;\forall x \in X \end{aligned} \tag{2.2}$$

is called the directional minimal time function with respect to M.
We put $T_M(x, A) = \infty$ if $(x+tM) \cap A = \emptyset$ for every $t \geq 0$. We consider the directional excess from A to B with respect to M as

$$e_M(A, B) := \sup_{x \in A} T_M(x, B).$$

For the purpose of covering all the situations, the following conventions are adopted:

$$T_M(x, \emptyset) := \infty, \forall x \in X, \;\; e_M(\emptyset, B) := 0, \forall B \subset X.$$

Note that $e_M(A, \emptyset) = \infty$ for every $A \subset X \backslash \{\emptyset\}$. For convenience, we denote in what follows $T_M(x, \{u\})$ by $T_M(x, u)$. It is easy to see that

$$T_M(x, u) = T_{-M}(u, x) = T_{-M}(-x, -u) \text{ and } T_M(x, u) = \|x - u\|$$

whenever $u \in x + \mathrm{cone}M$. It also has been established that if cone$M$ is convex, then T_M has the properties of a generalized extended-valued quasi-metric:

(i) $T_M(x, u) = 0$ if and only if $x = u$,

(ii) $T_M(x, u) \leq T_M(x, v) + T_M(v, u) \;\forall x, u, v \in X$.

Moreover, for any set $A \subset X$, one has $T_{S_X}(\cdot, A) = d(\cdot, A)$. In addition, for any nonempty set $M \subset S_X$ and $x \in X$, $T_M(x, A) < \infty$ implies that $A \neq \emptyset$ and $x \in A - \mathrm{cone}M$, for more discussion on the directional minimal time function, see [17] and the references therein.

We denote the set of coincidence points of a pair of set-valued mappings $F, G : X \rightrightarrows Y$ by

$$\mathrm{Coin}(F, G) := \{x \in X : F(x) \cap G(x) \neq \emptyset\}.$$

Next we recall the major notions of directional metric regularity and Lipschitzian properties of set-valued mappings in our study, which are introduced in [17].

Definition 2.2. Let $F : X \rightrightarrows Y$ be a multifunction, $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, $\emptyset \neq M \subset S_X$ and $\emptyset \neq N \subset S_Y$.

(i) F is said to be directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to M and N with constant $\kappa > 0$, if there are $\delta > 0$ and some neighborhoods $U \subset X$ and $V \subset Y$ of $\bar{x}$ and $\bar{y}$ such that

$$T_M(x, F^{-1}(y)) \leq \kappa T_N(y, F(x)) \quad \forall (x, y) \in U \times V \text{ with } T_N(y, F(x)) < \delta. \tag{2.3}$$

We say that F is globally directionally metrically regular with respect to M and N with constant $\kappa > 0$, if

$$T_M(x, F^{-1}(y)) \leq \kappa T_N(y, F(x)) \ \forall (x, y) \in X \times Y. \tag{2.4}$$

(ii) F is said to be directionally Aubin continuous around $(\bar{x}, \bar{y})$ with respect to M and N with constant $\mu > 0$, if there are neighborhoods $U \subset X$ and $V \subset Y$ of $\bar{x}$ and $\bar{y}$ such that

$$e_N(F(x) \cap V, F(x')) \leq \mu T_M(x', x) \quad \forall x, x' \in U. \tag{2.5}$$

We say that F is globally directionally Aubin continuous with respect to M and N with constant $\mu > 0$, if (2.5) holds for $U = X$ and $V = Y$.

Observe that if we take $M = S_X$ and $N = S_Y$, then the aforementioned concepts reduce to the usual metric regularity and Aubin property around the reference points. The relation between directional metric regularity and directional Aubin continuity is similar to the equivalence between metric regularity of mappings and Aubin continuity of their inverses (see [17, Proposition 2.4]). For global notions we also have the same conclusion as follows and for completeness we provide the proof as well.

Proposition 2.3. *Let $F : X \rightrightarrows Y$ be an arbitrary mapping, $\emptyset \neq M \subset S_X$, $\emptyset \neq N \subset S_Y$ and $\kappa > 0$. Then F is globally directionally metrically regular with respect to M and N with constant κ if and only if the inverse mapping F^{-1} is globally directionally Aubin continuous with respect to $-N$ and M with constant κ.*

Proof. Suppose that F is globally directionally metrically regular with respect to M and N with constant κ, i.e. (2.4) holds and we are going to show that

$$e_M(F^{-1}(y), F^{-1}(y')) \leq \kappa T_{-N}(y, y') \ \forall y, y' \in Y. \tag{2.6}$$

Pick $y, y' \in Y$ and $x \in F^{-1}(y)$. If there is no such x or $T_{-N}(y, y') = \infty$, then (2.6) holds automatically. Otherwise $T_{-N}(y, y') < \infty$, which implies that $y' \in y - \mathrm{cone}N$. According to inequality (2.4), one has that

$$T_M(x, F^{-1}(y')) \leq \kappa T_N(y', F(x)) \leq \kappa T_N(y', y) = \kappa T_{-N}(y, y'),$$

which shows that inequality (2.6) also holds. Therefore, F^{-1} is directionally Aubin continuous with respect to $-N$ and M with constant κ.

Conversely we assume that (2.6) holds and aim to show the validity of (2.4). To this end, we pick any $x \in X$ and $y \in Y$. If $T_N(y, F(x)) = \infty$, then (2.4) holds automatically. So we assume that $T_N(y, F(x)) < \infty$. Then, for any $\varepsilon > 0$, there exists $v \in F(x) \cap (y + \text{cone}N)$ such that $T_N(y, v) < T_N(y, F(x)) + \varepsilon$. Thus, $x \in F^{-1}(v)$ and from (2.6), we have

$$\begin{aligned} T_M(x, F^{-1}(y)) &\leq e_M(F^{-1}(v), F^{-1}(y)) \\ &\leq \kappa T_{-N}(v, y) = \kappa T_N(y, v) \\ &< \kappa T_N(y, F(x)) + \kappa\varepsilon. \end{aligned}$$

As $\varepsilon \to 0$, the above inequality ensures that (2.4) is true. The proof is complete.

3. Main results

We first give a coincidence point theorem regarding two set-valued mappings under the assumption of directional metric regularity and directional Aubin continuity, the proof of which is based on the inverse mapping iteration.

Theorem 3.1. *Let X, Y be normed vector spaces, $M \subset S_X$, $N \subset S_Y$ be nonempty sets such that* cone$M$ *and* cone$N$ *be closed and convex and $\alpha, \beta, \delta, \kappa, \mu$ be positive constants such that $\kappa\mu < 1$. Consider any two set-valued mappings $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ and points $(\bar{x}, \bar{y}) \in \text{gph}(F)$ and $(\tilde{x}, \tilde{y}) \in \text{gph}(G)$ with $\|\bar{x} - \tilde{x}\| < \alpha$ and $\|\bar{y} - \tilde{y}\| < \beta$ such that the following conditions hold:*

(i) either one of the sets $\text{gph}(F) \cap (B_\alpha(\bar{x}) \times B_\beta(\bar{y}))$ *and* $\text{gph}(G) \cap (B_\alpha(\tilde{x}) \times B_\beta(\tilde{y}))$ *is complete while the other is closed;*

(ii) F is directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to M and $-N$ for constants κ, δ and neighborhoods $B_\alpha(\bar{x})$ and $B_\beta(\bar{y})$, that is,

$$T_M(x, F^{-1}(y)) \leq \kappa T_{-N}(y, F(x)) \ \forall (x, y) \in B_\alpha(\bar{x}) \times B_\beta(\bar{y}) \text{ with } T_{-N}(y, F(x)) < \delta; \tag{3.7}$$

(iii) G is directionally Aubin continuous around $(\tilde{x}, \tilde{y})$ with respect to M and N for constant μ and neighborhoods $B_\alpha(\tilde{x})$ and $B_\beta(\tilde{y})$, that is,

$$e_N(G(x') \cap B_\beta(\tilde{y}), G(x)) \leq \mu T_M(x', x) \ \forall x, x' \in B_X(\tilde{x}, \alpha). \tag{3.8}$$

Let a, b and r be any positive reals satisfying

$$r < \delta, \frac{\kappa r}{1 - \kappa\mu} + a < \alpha, \frac{\mu\kappa r}{1 - \kappa\mu} + b < \beta. \tag{3.9}$$

Then for any $x \in B_a(\bar{x}) \cap B_a(\tilde{x})$,

$$T_M(x, \text{Coin}(F, G)) \leq \frac{\kappa}{1 - \kappa\mu} \inf_{y \in G(x) \cap V(x)} T_{-N}(y, F(x)), \tag{3.10}$$

where $V(x) := \{y \in B_b(\bar{y}) \cap B_b(\tilde{y}) : T_{-N}(y, F(x)) < r\}$, and for any $y \in G(x) \cap V(x)$,

$$T_N(y, \text{Coin}(F^{-1}, G^{-1})) \leq \frac{\kappa\mu}{1 - \kappa\mu} T_{-N}(y, F(x)) \tag{3.11}$$

Proof. According the assumptions on constants a, b, r in (3.9), we pick sufficiently small $\varepsilon \in (0,1)$ such that

$$\begin{aligned} &\kappa(\mu+\varepsilon)+\varepsilon<1,\\ &r<\delta, (\mu+\varepsilon)(\kappa r+\varepsilon)<\delta,\\ &\frac{\kappa r+\varepsilon}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}+a<\alpha,\\ &\frac{(\mu+\varepsilon)(\kappa r+\varepsilon)}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}+b<\beta, \end{aligned} \tag{3.12}$$

and aim to show the validity of (3.10) and (3.11). Pick any $x \in B_a(\bar{x}) \cap B_a(\tilde{x})$ and let it be fixed. If there is no such x or $G(x) \cap V(x) = \emptyset$, then there is nothing more to prove. Otherwise, for any $y \in G(x) \cap V(x)$, one has that $\|\bar{y}-y\| \leq b$, $\|\tilde{y}-y\| \leq b$ and $T_{-N}(y, F(x)) < r$. Note that $r < \delta$, it follows from assumption (3.7) that, there exists $x_1 \in F^{-1}(y)$ with $x_1 \in x + \text{cone}M$ such that

$$T_M(x, x_1) < T_M(x, F^{-1}(y)) + \varepsilon \leq \kappa T_{-N}(y, F(x)) + \varepsilon \leq \kappa r + \varepsilon. \tag{3.13}$$

Furthermore,

$$\begin{aligned} \max\{\|\bar{x}-x_1\|, \|\tilde{x}-x_1\|\} &\leq \max\{\|\bar{x}-x\|, \|\tilde{x}-x\|\} + \|x-x_1\| \\ &\leq T_M(x, x_1) + a \leq \kappa r + \varepsilon + a < \alpha. \end{aligned} \tag{3.14}$$

If $x_1 = x$, then $x \in \text{Coin}(F, G)$ and $y \in \text{Coin}(F^{-1}, G^{-1})$, and hence (3.10) and (3.11) hold automatically. Next we assume that $x_1 \neq x$. Then $0 < T_X(x, x_1) < \infty$ and according to (3.8) and the definition of $V(x)$, there exists $y_1 \in G(x_1) \cap (y + \text{cone}N)$ such that

$$\begin{aligned} T_N(y, y_1) &< T_N(y, G(x_1)) + \varepsilon T_M(x, x_1)\\ &\leq e_N(G(x) \cap B_\beta(\tilde{y}), G(x_1)) + \varepsilon T_M(x, x_1)\\ &\leq (\mu+\varepsilon) T_M(x, x_1). \end{aligned} \tag{3.15}$$

It then follows that

$$\max\{\|\bar{y}-y_1\|, \|\tilde{y}-y_1\|\} \leq b + \|y-y_1\| = b + T_N(y, y_1) < (\mu+\varepsilon)(\kappa r+\varepsilon) + b < \beta. \tag{3.16}$$

We proceed our proof with the approach of induction. To this end, we construct sequences of points $x_k \in B_\alpha(\bar{x}) \cap B_\alpha(\tilde{x})$ and $y_k \in B_\beta(\bar{y}) \cap B_\beta(\tilde{y})$, with $x_0 = x$ and $y_0 = y$ such that, for $k = 0, 1, 2, \ldots$,

$$x_{k+1} \in F^{-1}(y_k) \cap (x_k + \text{cone}M) \text{ and } y_{k+1} \in G(x_{k+1}) \cap (y_k + \text{cone}N) \tag{3.17}$$

with

$$T_M(x_k, x_{k+1}) \leq (\kappa(\mu+\varepsilon)+\varepsilon)^k T_M(x_0, x_1) \text{ and } T_N(y_k, y_{k+1}) \leq (\mu+\varepsilon) T_M(x_k, x_{k+1}). \tag{3.18}$$

By (3.15), we see that x_1 and y_1 satisfy (3.17) and (3.18) for $k = 0$. Suppose that for some $n \geq 1$ we have generated $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ satisfying (3.17) and (3.18) and we are going to show that there exist $x_{n+1} \in B_\alpha(\bar{x}) \cap B_\alpha(\tilde{x})$ and $y_{n+1} \in B_\beta(\bar{x}) \cap B_\beta(\tilde{x})$ such that the induction formulation can proceed. If $x_n = x_{n-1}$, we set $x_{n+1} = x_n$ and $y_{n+1} = y_n$. Then $x_{n-1} \in \text{Coin}(F, G)$ and $y_n \in \text{Coin}(F^{-1}, G^{-1})$. Note that $x_{k+1} \in x_k + \text{cone}M$ and $y_{k+1} \in y_k + \text{cone}N$ for all $k = 0, 1, \ldots, n-1$ and cone$M$, cone$N$ are convex sets, one has $x_n \in x_0+\text{cone}M$ and $y_n \in y_0+\text{cone}N$. Hence, it follows from (3.13) and (3.18) that

$$\begin{aligned} T_M(x_0, \text{Coin}(F,G)) \le T_M(x_0, x_{n-1}) &\le \sum_{i=1}^{n-1} T_M(x_{i-1}, x_i) \\ &\le \frac{T_M(x_0,x_1)}{1-(\kappa(\mu+\varepsilon)+\varepsilon)} \le \frac{\kappa T_{-N}(y, F(x))+\varepsilon}{1-(\kappa(\mu+\varepsilon)+\varepsilon)} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} T_M(y_0, \text{Coin}(F^{-1}, G^{-1})) \le T_N(y_0, y_n) &\le (\mu+\varepsilon)\sum_{i=1}^{n} T_M(x_{i-1}, x_i) \\ &\le \frac{(\mu+\varepsilon)(\kappa T_{-N}(y, F(x))+\varepsilon)}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}. \end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that (3.10) and (3.11) holds.
Next we assume that $x_n \neq x_{n-1}$. Since $x_n \in F^{-1}(y_{n-1}) \cap B_\alpha(\bar{x}) \cap B_\alpha(\tilde{x})$ and $y_n \in y_{n-1} + \text{cone}N$, it follows from (3.12), (3.13) and (3.18) that

$$\begin{aligned} T_{-N}(y_n, F(x_n)) \le T_N(y_{n-1}, y_n) &\le (\mu+\varepsilon) T_M(x_{n-1}, x_n) \\ &\le (\mu+\varepsilon) T_M(x_0, x_1) < (\mu+\varepsilon)(\kappa r+\varepsilon) < \delta. \end{aligned}$$

Hence, according to (3.7) there exists $x_{n+1} \in F^{-1}(y_n) \cap (x_n + \text{cone}M)$ such that

$$\begin{aligned} T_M(x_n, x_{n+1}) &\le T_M(x_n, F^{-1}(y_n)) + \varepsilon T_M(x_{n-1}, x_n) \\ &\le \kappa T_{-N}(y_n, F(x_n)) + \varepsilon T_M(x_{n-1}, x_n) \\ &\le \kappa T_N(y_{n-1}, y_n) + \varepsilon T_M(x_{n-1}, x_n). \end{aligned}$$

Then, by invoking the induction hypothesis (3.18), we have

$$T_N(y_{n-1}, y_n) \le (\mu+\varepsilon) T_M(x_{n-1}, x_n).$$

And therefore,

$$T_M(x_n, x_{n+1}) \le (\kappa(\mu+\varepsilon)+\varepsilon) T_M(x_{n-1}, x_n) \le (\kappa(\mu+\varepsilon)+\varepsilon)^n T_M(x_0, x_1).$$

By repeating the arguments above, if $x_{n+1} = x_n$, we obtain the validity of (3.10) and (3.11) and then set $y_{n+1} = y_n$. If $x_{n+1} \neq x_n$, note that $y_n \in G(x_n) \cap B_\beta(\tilde{y})$, then by (3.8) there exists $y_{n+1} \in G(x_{n+1}) \cap (y_n + \text{cone}N)$ such that

$$T_N(y_n, y_{n+1}) \le e_N(G(x_n) \cap B_\beta(\tilde{y}), G(x_{n+1})) + \varepsilon T_M(x_n, x_{n+1}) \le (\mu+\varepsilon) T_M(x_n, x_{n+1}).$$

Note that $x_0 \in B_a(\bar{x}) \cap B_a(\tilde{x})$ and $y_0 \in B_b(\bar{y}) \cap B_b(\tilde{y})$, next we are going to show that $x_{n+1} \in B_\alpha(\bar{x}) \cap B_\alpha(\tilde{x})$ and $y_{n+1} \in B_\beta(\bar{y}) \cap B_\beta(\tilde{y})$. Since $x_n \in x_0 + \text{cone}M$ and $y_n \in y_0 + \text{cone}N$, one has that $x_{n+1} \in x_0 + \text{cone}M$ and $y_{n+1} \in y_0 + \text{cone}M$. Utilizing (3.17) and (3.18), we have

$$\begin{aligned} T_M(x_0, x_{n+1}) &\le \sum_{j=0}^{n} T_M(x_j, x_{j+1}) \le \sum_{j=0}^{n} (\kappa(\mu_x+\varepsilon)+\varepsilon)^j T_M(x_0, x_1) \\ &\le \frac{T_M(x_0, x_1)}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}, \end{aligned}$$

and therefore, through (3.12) and (3.13),

$$\begin{aligned}\max\{\|\bar{x}-x_{n+1}\|,\|\tilde{x}-x_{n+1}\|\} &\le \max\{\|\bar{x}-x_0\|,\|\tilde{x}-x_0\|\}+\|x_0-x_{n+1}\|\\ &\le a+T_M(x_0,x_{n+1})<\frac{\kappa r+\varepsilon}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}+a<\alpha.\end{aligned}$$

Furthermore, by (3.18),

$$T_N(y_0,y_{n+1})\le\sum_{j=0}^{n}T_N(y_j,y_{j+1})\le\sum_{j=0}^{n}(\mu+\varepsilon)T_M(x_j,x_{j+1})\le\frac{(\mu+\varepsilon)T_M(x_0,x_1)}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}.$$

Hence, by (3.12) and (3.13),

$$\begin{aligned}\max\{\|\bar{y}-y_{n+1}\|,\|\tilde{y}-y_{n+1}\|\} &\le \max\{\|\bar{y}-y_0\|,\|\tilde{y}-y_0\|\}+\|y_0-y_{n+1}\|\\ &\le b+T_M(y_0,y_{n+1})<\frac{(\mu+\varepsilon)(\kappa r+\varepsilon)}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}+b<\beta.\end{aligned}$$

This completes the induction step, which guarantees that (3.17) and (3.18) hold for all k.

We have already shown that if $x_{k+1}=x_k$ for some k, then (3.10) and (3.11) hold true. Suppose now that $x_{k+1}\neq x_k$ for all k. Note that $x_{k+1}\in x_k+\mathrm{cone}M$ and $y_{k+1}\in y_k+\mathrm{cone}N$, one has $T_M(x_k,x_{k+1})=\|x_k-x_{k+1}\|$, $T_N(y_k,y_{k+1})=\|y_k-y_{k+1}\|$, where $x_k\in x_0+\mathrm{cone}M$ and $y_k\in y_0+\mathrm{cone}N$ for all k (thanks to assumption of $\mathrm{cone}M,\mathrm{cone}N$ being convex). By virtue of the inequalities for x_k and y_k in (3.18), we see that for any natural m and n,

$$\begin{aligned}\|x_n-x_{n+m}\| &\le \sum_{k=n}^{n+m-1}\|x_k-x_{k+1}\|=\sum_{k=n}^{n+m-1}T_M(x_k,x_{k+1})\\ &\le \sum_{k=n}^{n+m-1}(\kappa(\mu+\varepsilon)+\varepsilon)^kT_M(x_0,x_1)\le\frac{(\kappa(\mu+\varepsilon)+\varepsilon)^n}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}T_M(x_0,x_1)\end{aligned}$$

and

$$\begin{aligned}\|y_n-y_{n+m}\| &\le \sum_{k=n}^{n+m-1}\|y_k-y_{k+1}\|=\sum_{k=n}^{n+m-1}T_N(y_k,y_{k+1})\\ &\le \sum_{k=n}^{n+m-1}(\mu+\varepsilon)T_M(x_k,x_{k+1})\le\frac{(\mu+\varepsilon)(\kappa(\mu+\varepsilon)+\varepsilon)^n}{1-(\kappa(\mu+\varepsilon)+\varepsilon)}T_M(x_0,x_1).\end{aligned}$$

The above inequalities ensure that both $\{x_k\}$ and $\{y_k\}$ are Cauchy sequences. Note that

$$(x_k,y_k)\in((B_\alpha(\bar{x})\cap B_\alpha(\tilde{x}))\times(B_\beta(\bar{y})\cap B_\beta(\tilde{y}))\cap((x_0+\mathrm{cone}M)\times(y_0+\mathrm{cone}N)),$$

$(x_k,y_{k-1})\in\mathrm{gph}(F),(x_k,y_k)\in\mathrm{gph}(G)$ and $\mathrm{cone}M,\mathrm{cone}N$ are closed, then according to assumption (i), we may conclude that for the case of

$$\mathrm{gph}(F)\cap((B_\alpha(\bar{x})\cap B_\alpha(\tilde{x}))\times(B_\beta(\bar{y})\cap B_\beta(\tilde{y}))$$

being complete and $\mathrm{gph}(G)((B_\alpha(\bar{x}) \cap B_\alpha(\tilde{x})) \times (B_\beta(\bar{y}) \cap B_\beta(\tilde{y}))$ being closed, there exists $(\hat{x}, \hat{y}) \in \mathrm{gph}(F)$ such that $(x_k, y_{k-1}) \to (\hat{x}, \hat{y})$, and then

$$(\hat{x}, \hat{y}) \in \mathrm{gph}(G) \cap ((x_0 + \mathrm{cone}M) \times (y_0 + \mathrm{cone}N));$$

while for the other case we could obtain similar results. This shows that

$$\hat{x} \in \mathrm{Coin}(F, G) \cap (x_0 + \mathrm{cone}M) \text{ and } \hat{y} \in \mathrm{Coin}(F^{-1}, G^{-1}) \cap (y_0 + \mathrm{cone}N).$$

Utilizing (3.13) and (3.18), we finally obtain that

$$\begin{aligned} T_M(x_0, \mathrm{Coin}(F, G)) &\leq T_M(x_0, \hat{x}) = \|x_0 - \hat{x}\| = \lim_{k\to\infty} \|x_0 - x_k\| \\ &\leq \sum_{k=0}^{\infty} \|x_k - x_{k+1}\| = \sum_{k=0}^{\infty} T_M(x_k, x_{k+1}) \\ &\leq \frac{T_M(x_0, x_1)}{1 - (\kappa(\mu + \varepsilon) + \varepsilon)} \leq \frac{\kappa T_N(y, F(x)) + \varepsilon}{1 - (\kappa(\mu + \varepsilon) + \varepsilon)} \end{aligned}$$

and

$$\begin{aligned} T_N(y_0, \mathrm{Coin}(F^{-1}, G^{-1})) &\leq T_N(y_0, \hat{y}) \leq \sum_{k=0}^{\infty} T_N(y_k, y_{k+1}) \\ &\leq \frac{(\mu + \varepsilon)(\kappa T_N(y, F(x)) + \varepsilon)}{1 - (\kappa(\mu + \varepsilon) + \varepsilon)}. \end{aligned}$$

Taking the limit of $\varepsilon \to 0$, we obtain the validity of inequalities (3.10) and (3.11). The proof is complete.

It is worth to note that if there exists x such that the right side of (3.10) does not equal infinity, then $\mathrm{Coin}(F, G) \neq \emptyset$. In [10, Theorem 3.1], the authors derive a uniform estimate for the distance to the set of coincidence points of two set-valued mappings F and G under the assumptions that F is local covering and G is local Aubin continuous in metric spaces. In Theorem 3.1, our assumption is placed on directional metric regularity and directional Aubin continuity in normed vector spaces, which leads to the estimate of “directional distance” to the set of coincidence points, see (3.10) and (3.11). Therefore, our results is the development of the results from [10].

Example 3.2. Consider $X = Y = \mathbb{R}$, $M = \{1\}, N = \{1\}$, and let

$$F(x) := \begin{cases} \{x\}, & if\, x \geq 0, \\ \emptyset, & if\, x < 0 \end{cases} \text{ and } G(x) := \begin{cases} \{\frac{x}{2}\}, & if\, x \geq 0, \\ \emptyset, & if\, x < 0. \end{cases}$$

First, we claim that F is directionally metrically regular around $(0, 0)$ with respect to M and $-N$ for constant 1. Indeed, it suffices to show that

$$T_M(x, F^{-1}(y)) \leq T_{-N}(y, F(x)) \; \forall (x, y) \in \mathbb{R}^2. \tag{3.20}$$

Since $F(x) = \emptyset$ for $x < 0$, we have $T_{-N}(y, F(x)) = \infty$ for any $y \in \mathbb{R}$, which implies that (3.20) holds; When $x \geq 0$ and $y < 0$, it is clear that $y \notin F(x) - \mathrm{cone}(-N)$ which indicates that $T_{-N}(y, F(x)) = \infty$ and hence (3.20) holds; In the case of $x \geq 0$ and $y \geq 0$, we have $T_M(x, F^{-1}(y)) = T_M(x, y) = T_{-N}(y, x) = T_{-N}(y, F(x))$ which also guaranties the validity of (3.20). It's then easy to observe that G is directionally Aubin continuous around $(0, 0)$ with respect to M and N for constant $\frac{1}{2}$. Also it is clear that F and G satisfy all other conditions in Theorem 3.1 and we can check that $\mathrm{Coin}(F,G)=\{0\}$. However, F is not metrically regular around $(0,0)$ and G is not Aubin continuous around $(0,0)$.

When the assumption of directional metric regularity and directional Aubin continuity in Theorem 3.1 is generalized to the global case, we obtain the following coincidence point theorem of two set-valued mappings, the proof of which follows from the lines in Theorem 3.1 but needs a modification.

Theorem 3.3. *Let X,Y be normed vector spaces, $M\subset S_X$, $N\subset S_Y$ be nonempty sets such that* $\mathrm{cone}M$ *and* $\mathrm{cone}N$ *be closed and convex and κ,μ be positive constants such that $\kappa\mu<1$. Consider any two set-valued mappings $F:X\rightrightarrows Y$ and $G:X\rightrightarrows Y$ such that the following conditions hold:*

(i) either one of the sets $\mathrm{gph}(F)$ *and* $\mathrm{gph}(G)$ *is complete while the other is closed;*

(ii) F is globally directionally metrically regular with respect to M and $-N$ with constant κ;

(iii) G is globally directionally Aubin continuous with respect to M and N with constant μ.

Then for any $x\in X$,

$$T_M(x,\mathrm{Coin}(F,G))\le\frac{\kappa}{1-\kappa\mu}\inf_{y\in G(x)}T_{-N}(y,F(x)),\tag{3.21}$$

and for any $y\in G(x)$,

$$T_N(y,\mathrm{Coin}(F^{-1},G^{-1}))\le\frac{\kappa\mu}{1-\kappa\mu}T_{-N}(y,F(x)).\tag{3.22}$$

Proof. By the assumptions on F and G, we have (2.4) holds and

$$e_N(G(x'),G(x))\le\mu T_M(x',x)\ \forall x,x'\in X.\tag{3.23}$$

Pick any $\varepsilon>0$ such that $\kappa(\mu+\varepsilon)+\varepsilon<1$. Fix any $x\in X$ and pick some $y\in G(x)$ with $T_{-N}(y,F(x))<\infty$. If there is not such y, then the right side of (3.21) and (3.22) is ∞ and the proof is trivial. Otherwise, it follows from (2.4) that there exists $x_1\in F^{-1}(y)$ with $x_1\in x+\mathrm{cone}M$ such that

$$T_M(x,x_1)<T_M(x,F^{-1}(y))+\varepsilon\le\kappa T_{-N}(y,F(x))+\varepsilon.$$

If $x_1=x$, then $x\in\mathrm{Coin}(F,G)$ and $y\in\mathrm{Coin}(F^{-1},G^{-1})$. Hence (3.21) and (3.22) hold automatically. Let $x_1\ne x_0$, then $0<T_X(x,x_1)<\infty$. It follows from (3.23) that there exists $y_1\in G(x_1)\cap(y+\mathrm{cone}N)$ such that

$$\begin{aligned}T_N(y,y_1)\ &<\ T_N(y,G(x_1))+\varepsilon T_M(x,x_1)\le e_N(G(x),G(x_1))+\varepsilon T_M(x,x_1)\\ &\le\ (\mu+\varepsilon)T_M(x,x_1).\end{aligned}\tag{3.24}$$

Note that $y\in F(x_1)\cap(y_1-\mathrm{cone}N)$, then inequalities (2.4) and (3.24) imply that, there exists $x_2\in F^{-1}(y_1)$ with $x_2\in x_1+\mathrm{cone}M$ such that

$$\begin{aligned}T_M(x_1,x_2)&<T_M(x_1,F^{-1}(y_1))+\varepsilon T_M(x,x_1)\le\kappa T_{-N}(y_1,F(x_1))+\varepsilon T_M(x,x_1)\\ &\le\kappa T_{-N}(y_1,y)+\varepsilon T_M(x,x_1)=\kappa T_N(y,y_1)+\varepsilon T_M(x,x_1)\\ &\le(\kappa(\mu+\varepsilon)+\varepsilon)T_M(x,x_1).\end{aligned}$$

Employing the approach of induction similar to that in Theorem 3.1, we can construct Cauchy sequences $(x_k, y_k) \in X \times Y$ with $x_0 = x$ and $y_0 = y$, which converge to some

$$(\hat{x}, \hat{y}) \in \mathrm{gph}(F) \cap \mathrm{gph}(G) \cap ((x_0 + \mathrm{cone}M) \times (y_0 + \mathrm{cone}N)).$$

The rest of the proof follows similarly and we omit it. The proof is completed.

Remark 3.4. It is worth to mention that Theorem 3.3 is the development of [3, Theorem 2] which focuses on the covering property in metric space. When $M = S_X$, $N = S_Y$ Theorem 3.3 is a supplement of [3, Theorem 2] in normed vector space. From the the proof of the above theorem, we have $\mathrm{Coin}(F, G) \neq \emptyset$, provided that there exists $x \in X$ and $y \in G(x)$ such that $(y - \mathrm{cone}N) \cap F(x) \neq \emptyset$. When $M = S_X$ and $N = S_Y$, our result in Theorem 3.3 covers [1, Theorem 6].

The following result extended the global Lyusternik-Graves Theorem to the case with directions.

Corollary 3.5. *Let X, Y be normed vector spaces, $M \subset S_X$, $N \subset S_Y$ be nonempty sets such that* cone$M$ *and* cone$N$ *be closed and convex and κ, μ be positive constants such that $\kappa\mu < 1$. Consider any two set-valued mappings $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ such that condition (i) and (iii) in Theorem 3.3 hold and F is globally directionally metrically regular with respect to M and N with constant κ. Then $F + G$ is globally directionally metrically regular with respect to M and N with constant $\frac{\kappa}{1-\kappa\mu}$.*

Proof. We only have to show that

$$T_M(x, (F+G)^{-1}(z)) \leq \frac{\kappa}{1-\kappa\mu} T_N(z, (F+G)(x)) \ \forall (x, z) \in X \times Y. \tag{3.25}$$

Fix any $(x, z) \in X \times Y$. If $T_N(z, (F + G)(x)) = \infty$, then (3.25) holds automatically. We assume next that $T_N(z, (F + G)(x)) < \infty$ which implies that

$$(F + G)(x) \cap (z + \mathrm{cone}N) \neq \emptyset.$$

Then, for any $\varepsilon > 0$, there exist $u_1 \in F(x)$ and $u_2 \in G(x)$ with $u_1 + u_2 \in z + \mathrm{cone}N$, such that

$$\begin{aligned} T_N(z, (F+G)(x)) + \varepsilon > T_N(z, u_1 + u_2) &= T_N(z - u_2, u_1) \\ &\geq T_N(z - u_2, F(x)) \geq \inf_{w \in z - G(x)} T_N(w, F(x)). \end{aligned}$$

By the assumption, it is easy to see that $z - G(\cdot)$ is globally directionally Aubin continuous with respect to M and $-N$ with constant μ. Note that

$$(F + G)^{-1}(z) = \mathrm{Coin}(F, z - G(\cdot))$$

(in fact $x \in (F+G)^{-1}(z) \Leftrightarrow z \in (F+G)(x) \Leftrightarrow F(x) \cap (z - G(x)) \neq \emptyset$), now applying Theorem 3.3 to F and $z - G(\cdot)$ (by $-N$ instead of N), we get

$$T_M(x, (F+G)^{-1}(z)) \leq \frac{\kappa}{1-\kappa\mu} \inf_{w \in z - G(x)} T_N(w, F(x)) < T_N(z, (F+G)(x)) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain that (3.25) holds. The proof is complete.

As an application, we consider the following partial-parametrized version of the variational system:

$$0 \in f(p, x) + F(x) \tag{3.26}$$

where $f : P \times X \to Y$ is a single-valued mapping while $F : X \rightrightarrows Y$ is a set-valued mapping between arbitrary Banach spaces. The solution mapping $S : P \rightrightarrows X$ associated to (3.26) is given by

$$S(p) := \{x \in X : 0 \in f(p, x) + F(x)\}.$$

The following result provides a sufficient condition for directional Aubin continuity of the solution mapping S.

Theorem 3.6. *Let P, X and Y be Banach spaces, $L \subset S_P, M \subset S_X$ and $N \subset S_Y$ be nonempty sets such that* $\text{cone}L, \text{cone}M$ *and* $\text{cone}N$ *be closed and convex and κ, μ, λ be positive constants such that $\kappa\mu < 1$. Consider the parametric generalized equation (3.26) and a pair $(\bar{p}, \bar{x}) \in P \times X$ and $\bar{y} := -f(\bar{p}, \bar{x})$ with $\bar{x} \in S(\bar{p})$. Suppose that the following conditions are satisfied:*

(i) f is continuous around $(\bar{p}, \bar{x})$ and is directionally Lipschitz continuous around $(\bar{x}, \bar{y})$ with respect to L, M and N with constants λ and μ, i.e., there exists $a > 0$, such that

$$T_N(f(p, x), f(p', x')) \leq \lambda T_L(p, p') + \mu T_M(x, x') \ \forall (p, x), (p', x') \in B_a(\bar{p}) \times B_a(\bar{x}), \tag{3.27}$$

(ii) $\text{gph}(F)$ *is locally closed around $(\bar{x}, \bar{y})$ and F is directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to M and $-N$ with constants κ,*

Then S is directionally Aubin continuous around $(\bar{p}, \bar{x})$ with respect to L and M with constant $\frac{\kappa\lambda}{1-\kappa\mu}$.

Proof. According to assumption (ii), there exist positive constants α, β and δ such that $\text{gph}(F) \cap (B_\alpha(\bar{x}) \times B_\beta(\bar{y}))$ is closed and (3.7) holds. Let a, b and r be such that $2\lambda a < r$ and satisfies (3.9) and (3.27) simultaneously. By condition (i), making a smaller if necessary, we may assume that $\text{gph}(f(p, \cdot))$ is closed for all $p \in B_a(\bar{p})$ and

$$\|f(p, x) - f(p, \bar{x})\| < b \text{ and } \|f(\bar{p}, \bar{x}) - f(p, x)\| \leq b \ \forall (p, x) \in B_a(\bar{p}) \times B_a(\bar{x}). \tag{3.28}$$

Then it suffices to show that

$$e_M(S(p) \cap B_a(\bar{x}), S(p')) \leq \frac{\kappa\lambda}{1 - \kappa\mu} T_L(p, p') \ \forall p, p' \in B_a(\bar{p}). \tag{3.29}$$

Pick any $p, p' \in B_a(\bar{p})$ and $x \in S(p) \cap B_a(\bar{x})$. If there is no such x or $T_L(p', p) = \infty$, then (3.29) holds automatically.
Otherwise, we have $T_L(p, p') < \infty$ and $-f(p, x) \in F(x)$. Then $p' \in p + \text{cone}L$.
It follows from (3.27) that

$$T_{-N}(-f(p', x), -f(p, x)) = T_N(f(p, x), f(p', x)) \leq \lambda T_L(p, p'),$$

which implies that$-f(p, x) \in -f(p', x) - \text{cone}N$. Note that $-f(p, x) \in F(x)$, then

$$T_{-N}(-f(p', x), F(x)) \leq T_{-N}(-f(p', x), -f(p, x)) \leq \lambda T_L(p, p') \leq 2\lambda a < r.$$

The above inequality together with (3.28) show that

$$-f(p', x) \in B_b(\bar{y}) \cap B_b(-f(p', \bar{x})) \cap V(x),$$

where $V(x)$ is defined as in Theorem 3.1. Now we can apply Theorem 3.1 for F and $G = -f(p', \cdot)$ with $(\bar{x}, \bar{y})$ and $(\tilde{x}, \tilde{y}) = (\bar{x}, -f(p', \bar{x}))$. It is easy to see that $\mathrm{Coin}(F, G) = S(p')$. Thus, it follows from (3.10) that

$$\begin{aligned} T_M(x, S(p')) &\leq \frac{\kappa}{1-\kappa\mu} T_{-N}((-f(p', x)) \cap V(x), F(x)) \\ &= \frac{\kappa}{1-\kappa\mu} T_{-N}(-f(p', x), F(x)) \\ &\leq \frac{\kappa}{1-\kappa\mu} T_{-N}(-f(p', x), -f(p, x)) \leq \frac{\kappa\lambda}{1-\kappa\mu} T_L(p, p'). \end{aligned}$$

Since x is arbitrarily chosen, we conclude that (3.29) is true. The proof is complete.

When $L = S_P, M = S_X$ and $N = S_Y$, the above Theorem goes back to [2, Theorem 5.1 (ii)]. The next example illustrates that, different from the usual Lipschitz continuity, directionally Lipschitz continuous does not imply the continuity of f, even at $(\bar{p}, \bar{x})$.

Example 3.7. Let $L = \{-1, 1\}, M = \{(1, 0)\}, N = \{1\}$ and $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(p, (x, y)) := \begin{cases} x, & if\, y = 0, \\ 1, & if\, y \neq 0. \end{cases}$$

Then, f is not continuous at $(0, (0, 0))$. In fact, for any $p \in \mathbb{R}$,

$$f\left(p, \left(\frac{1}{k}, \frac{1}{k}\right)\right) = 1 \nrightarrow 0 = f(p, (0, 0)),$$

when $k \to \infty$. Next we show that

$$T_N(f(p, (x, y)), f(q, (u, v))) \leq T_M((x, y), (u, v)) \;\forall p, q \in \mathbb{R} \text{ and } (x, y), (u, v) \in \mathbb{R}^2. \tag{3.30}$$

Pick any $p, q \in \mathbb{R}$ and $(x, y), (u, v) \in \mathbb{R}^2$. If $(u, v) \notin (x, y) + \mathrm{cone}M = (x, y) + \mathbb{R}_+ \times \{0\}$, one has $T_M((x, y), (u, v)) = \infty$ and then (3.30) holds automatically. Next we assume that $(u, v) \in (x, y) + \mathbb{R}_+ \times \{0\}$, i.e. $x \leq u$ and $v = y$. If $v = y = 0$, then $f(p, (x, y)) = x$, $f(q, (u, v)) = u$ and

$$T_N(f(p, (x, y)), f(q, (u, v))) = u - x = T_M((x, y), (u, v)).$$

If $v = y \neq 0$, we have $f(p, (x, y)) = f(q, (u, v)) = 1$, and then

$$T_N(f(p, (x, y)), f(q, (u, v))) = 0 \leq u - x = T_M((x, y), (u, v)).$$

Therefore, we conclude that (3.30) holds, i.e. f is globally Lipschitz continuous with respect to $\{-1, 1\}$, $\{(1, 0)\}$ and $\{1\}$.

As an application of Theorem 3.3, next we provide a sufficient condition for global directional Aubin continuity of the solution mapping S.

Theorem 3.8. *Let P, X and Y be Banach spaces, $L \subset S_P, M \subset S_X$ and $N \subset S_Y$ be nonempty sets such that* $\mathrm{cone}L, \mathrm{cone}M$ *and* $\mathrm{cone}N$ *be closed and convex and κ, μ, λ be positive constants such that $\kappa\mu < 1$. Consider the parametric generalized equation* (3.26). *Suppose that the following conditions are satisfied:*

(i) for any $p \in P$, $\mathrm{gph}(f(p,\cdot))$ is closed and f is globally directionally Lipschitz continuous with respect to L, M and N with constants λ and μ, i.e.,

$$T_N(f(p,x), f(p',x')) \leq \lambda T_L(p,p') + \mu T_M(x,x') \;\forall (p,x),(p',x') \in P \times X,$$

(ii) $\mathrm{gph}(F)$ is closed and F is globally directionally metrically regular with respect to M and $-N$ with constants κ,

Then S is globally directionally Aubin continuous with respect to L and M with constant $\frac{\kappa\lambda}{1-\kappa\mu}$.

Proof. It suffices to show that

$$e_M(S(p), S(p')) \leq \frac{\kappa\lambda}{1-\kappa\mu} T_L(p,p') \;\forall p,p' \in \mathrm{dom}(S). \tag{3.31}$$

Pick any $p, p' \in P$. If $S(p) = \emptyset$ or $T_L(p',p) = \infty$, then (3.31) holds automatically. Otherwise, we assume that $T_L(p,p') < \infty$ and pick any $x \in S(p)$, then we have $-f(p,x) \in F(x)$. It follows from assumption (i) that

$$T_{-N}(-f(p',x), -f(p,x)) = T_N(f(p,x), f(p',x)) \leq \lambda T_L(p,p') < \infty,$$

which implies that$-f(p,x) \in -f(p',x) - \mathrm{cone}N$. Then

$$T_{-N}((-f(p',x)), F(x)) \leq T_{-N}(-f(p',x), -f(p,x)) \leq T_L(p,p'). \tag{3.32}$$

According to the assumptions, we can apply Theorem 3.3 for F and $G = -f(p',\cdot)$. It is easy to see that $\mathrm{Coin}(F,G) = S(p')$. Thus, from (3.21) and (3.32), one has

$$T_M(x, S(p')) \leq \frac{\kappa}{1-\kappa\mu} T_{-N}((-f(p',x)), F(x)) \leq \frac{\kappa\lambda}{1-\kappa\mu} T_L(p,p').$$

Since x is arbitrarily chosen, we conclude that (3.31) holds. The proof is complete.

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