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APPROXIMATING COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE COSINE FAMILIES IN HILBERT SPACES BY ALGORITHMS WITH ERROR TERMS

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Abstract. In this paper the Mann type iterative scheme with error term to approximate a common fixed point of one-parameter asymptotically nonexpansive cosine family is investigated in Hilbert spaces. By using the theory of cosine families, some strong convergence theorems of the sequences generated by these schemes are established on closed convex subsets and compact convex subsets, respectively. As special cases, strong convergence results for nonexpansive cosine families are also obtained.

Key Words and Phrases: Asymptotically nonexpansive cosine family, common fixed point, Mann type iterative scheme with error term, strong convergence.
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1. Introduction and preliminaries

Throughout this paper we denote by $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{R}$ and $\mathbb{R}^+$ the sets of nonnegative integers, positive integers, real numbers and nonnegative real numbers, respectively. Let H be a real Hilbert space with the origin θ. Let D be a nonempty closed convex subset of H with $\theta \in D$. A operator T on D is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$; and is called asymptotically nonexpansive if there exists $\{\lambda_n\} \subset \mathbb{R}^+$ such that $\lambda_n \to 0$ as $n \to \infty$ and

$$\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|, \text{ for all } x, y \in D \text{ and } n \in \mathbb{N}^+.$$

The class of asymptotically nonexpansive operators was introduced by Goebel and Kirk in [10] as an important generalization of the class of nonexpansive operators. It is well known that the set of fixed points of an asymptotically nonexpansive operator on a nonempty closed convex bounded subset is nonempty (see [10]). For approximating the fixed points of nonexpansive or asymptotically nonexpansive operators, iterative techniques have been studied by a number of authors and many meaningful results were obtained (for example, see [3, 4, 18, 19, 20, 26, 30, 6, 28, 8, 1, 31, 32]). Huang [11] generalized the classical Mann scheme to the one with errors for an asymptotically nonexpansive operator T and studied the following explicit iterative scheme:

$$\begin{cases} x_1 \in D \text{ arbitrary chosen}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n,\ n \in \mathbb{N}^+, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in the interval $[0,1]$ and $\{u_n\}$ is an error sequence in D. Fukhar-ud-din and Khan [9] introduced and studied the following implicit iterative scheme with errors for N asymptotically quasi-nonexpansive operators $\{T_1, T_1, \cdots, T_N\}$:

$$\begin{cases} x_0 \in D \text{ arbitrary chosen}, \\ x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n; \\ k, n \in \mathbb{N}^+,\ n = (k-1)N + i,\ i \in \{1, 2, \cdots, N\}, \end{cases} \tag{1.1}$$

where $\{u_n\}$ is an error sequence in D, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in the interval $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Chang et al. [5] introduced and studied the implicit iteration process with error for a finite family of asymptotically nonexpansive mappings and some extend results were presented. Recently, by modifying Mann's algorithm, some iterative methods have been continuously proposed and analyzed to solve the variational inequality, the equilibrium problem and the minimization problem, for instance, see Cianciaruso et al. [7], Kamraksa and Wangkeere [14], Hussain et al. [13, 12].

Recall that the problem of approximating common fixed points of the one- parameter semigroups $\{T(t) : t \in \mathbb{R}^+\}$ arose from the behavior study of solutions of the first-order differential inclusion. In the past few decades, weak and strong convergence problems concerning the semigroups have been extensively studied (for instance, see [17, 22, 2, 7, 14, 15] and the references therein). It is akin to one-parameter semigroup, Xiao et al [29] considered the problems of approximating common fixed points of the one-parameter nonexpansive cosine family $\{C(t) : t \in \mathbb{R}\}$, which is directly linked to solutions of the abstract second-order differential inclusions.

The main aim of this paper is to continue the study in this direction and consider iterative techniques for the asymptotically nonexpansive cosine family in Hilbert spaces. Motivated by Reich [18], Fukhar-ud-din and Khan [9], Suzuki [23], Kamraksa and Wangkeere [14], we suggest and analyze the following Mann type iterative scheme with error term, which is an implicit iteration modified by (1.1) and defined by

$$\text{(IIS)} \begin{cases} x_0 \in D \text{ chosen arbitrarily}, \\ x_n = (1 - a_n - b_n)x_{n-1} + a_n C(t_n)x_n + b_n u_n,\ n \in \mathbb{N}^+, \end{cases}$$

where $\{u_n\}$ is a sequence in D, $\{a_n\}$, $\{b_n\}$ and $\{t_n\}$ are sequences in $\mathbb{R}^+$ such that $a_n + b_n \leq 1$. By using the theory of cosine families, we establish some strong convergence theorems of the sequences generated by (IIS) on closed convex subsets and compact convex subsets, respectively. As special cases, we obtain strong convergence results for nonexpansive cosine families.

The following definitions and results will be needed in the sequel. Let $\{C(t) : t \in \mathbb{R}\}$ be a family of operators on D. By $\int_a^b C(t)dt$ we denote the Bochner integral $\int_a^b C(t)xdt$ for all $x \in D$ in the compact interval $[a, b]$.

Definition 1.1. ([29]) A one-parameter family $\{C(t) : t \in \mathbb{R}\}$ of operators on D is said to be a strongly continuous cosine family if the following conditions are satisfied:
(C-1) $C(0)x = x$, for all $x \in D$ and $C(t)\theta = \theta$ for all $t \in \mathbb{R}$;
(C-2) $C(t + r) + C(t - r) = 2C(t)C(r)$, for all $t, r \in \mathbb{R}$;
(C-3) $C(t + r) - C(t - r) = \pm \int_{t-r}^{t+r} d\tau \int_0^{\tau} C(\lambda)d\lambda$, for all $t, r \in \mathbb{R}$;
(C-4) $\{C(t)\}$ is strongly continuous on D, i.e., for each $x \in D$, the operator $C(\cdot)x$ from $\mathbb{R}$ into D is continuous.

If $\{C(t) : t \in \mathbb{R}\}$ is a cosine family, then $\{S(t) : t \in \mathbb{R}\}$ is the associated sine family defined by

$$S(t) = \int_0^t C(\tau)d\tau, \ t \in \mathbb{R}.$$

A cosine family $\{C(t) : t \in \mathbb{R}\}$ is said to have a common fixed point if there exists $p \in D$ such that $C(t)p = p$ for all $t \in \mathbb{R}$. The common fixed point set of $\{C(t) : t \in \mathbb{R}\}$ is denoted by $F(C)$, i.e.,

$$F(C) = \bigcap_{t \in \mathbb{R}} F(C(t)).$$

From (C-1) we see that $F(C) \neq \emptyset$.

Definition 1.2. A cosine family $\{C(t) : t \in \mathbb{R}\}$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|C(t)x - C(t)y\| \leq L\|x - y\|, \text{ for all } x, y \in D \text{ and } t \in \mathbb{R}^+.$$

If $L = 1$, then $\{C(t) : t \in \mathbb{R}\}$ is said to be nonexpansive. A cosine family $\{C(t) : t \in \mathbb{R}\}$ is said to be locally asymptotically nonexpansive on $[0, \gamma]$ (where $\gamma > 0$) if there exists a bounded measurable function $\lambda : [0, \gamma] \to [0, +\infty)$ such that $\limsup_{t \to 0+} \lambda(t) = 0$ and

$$\|C(t)x - C(t)y\| \leq [1 + \lambda(t)]\|x - y\|$$

for all $x, y \in D$ and $t \in [0, \gamma]$ (cf. [17, 21, 27]).

It is clear that a locally asymptotically nonexpansive cosine family is locally L-Lipschitzian, where $L = L(\gamma) = 1 + \sup_{t \in [0,\gamma]} \lambda(t)$.

Lemma 1.1. ([29]) *Let $\{C(t) : t \in \mathbb{R}\}$ be a cosine family on D. Then the following assertions are true.*
(1) *For each $x \in D$, the operator $S(\cdot)x$ from $\mathbb{R}$ into D is continuous.*
(2) *$C(t) = C(-t)$, $S(t) = -S(-t)$, for all $t \in \mathbb{R}$.*
(3) *$S(t + r) + S(t - r) = 2S(t)C(r)$, for all $t, r \in \mathbb{R}$.*
(4) *$C(t) - C(r) = \pm 2S\left(\frac{t + r}{2}\right) S\left(\frac{t - r}{2}\right)$, for all $t, r \in \mathbb{R}$.*
(5) *$C(t), C(r), S(t)$ and $S(r)$ are commutative, for all $t, r \in \mathbb{R}$.*

Lemma 1.2. ([16]) *Let H be a real Hilbert space, $x, y \in H$ and $a \in [0,1]$. Then*

$$\|(1-a)x + ay\|^2 = (1-a)\|x\|^2 + a\|y\|^2 - a(1-a)\|x-y\|^2.$$

Using Lemma 1.2, we can prove the the following identity.

Lemma 1.3. *Let H be a real Hilbert space, $x, y, z \in H$, $a, b, c \in [0,1]$ with $a+b+c = 1$. Then*

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2.$$

Lemma 1.4. ([29]) *Let $\{t_n\} \subset \mathbb{R}^+$ be a sequence such that*

$$\liminf_{n\to\infty} t_n = 0 \text{ and } 0 < \limsup_{n\to\infty} t_n < +\infty.$$

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\lim_{n\to\infty} g(t_n) = 0$. Suppose that either

$$\liminf_{n\to\infty}(t_{n+1} - t_n) = 0 \text{ or } \liminf_{n\to\infty}(t_n - t_{n+1}) = 0.$$

Then for each $i \in \mathbb{N}$ there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} t_{n_k} = 0 \text{ and } \lim_{k\to\infty} \frac{g(t_{n_k})}{(t_{n_k})^i} = 0.$$

Lemma 1.5. ([25]) *Let $\{r_n\}, \{s_n\}, \{\mu_n\}$ be three nonnegative real sequences satisfying*

$$r_{n+1} \le (1+\mu_n)r_n + s_n, \text{ for all } n \in \mathbb{N}^+; \ \sum_{n=1}^{\infty} s_n < +\infty, \ \sum_{n=1}^{\infty} \mu_n < +\infty.$$

Then $\lim_{n\to\infty} r_n$ exists.

2. Properties concerning Lipschitzian cosine family

Lemma 2.1. *Let D be a nonempty closed convex subset of a Hilbert space H. Let $\{C(t)\}$ be a strongly continuous cosine family of operators on D. If $\{C(t) : t \in \mathbb{R}\}$ is L-Lipschitzian, then $\|S(t)x\| \le L|t|\|x\|$ and*

$$\|S(kt)x\| \le k \max\{L, 1\}\|S(t)x\|, \text{ for all } t \in \mathbb{R}, k \in \mathbb{N}^+ \text{ and } x \in D. \tag{2.1}$$

Proof. Assume that $L \ge 1$, without loss of generality. For $t > 0$ we have

$$\begin{aligned} \|S(t)x\| &= \left\| \int_0^t [C(\tau)x - C(\tau)\theta] d\tau \right\| \le \int_0^t \|C(\tau)x - C(\tau)\theta\| d\tau \\ &\le \int_0^t L\|x\| d\tau = Lt\|x\|. \end{aligned}$$

Since $S(-t) = -S(t)$ and $S(0) = 0$, we have $\|S(t)x\| \le L|t|\|x\|$ for $t \in \mathbb{R}$.

The inequality (2.1) is obvious for $k = 1$. For $s, t \in \mathbb{R}$, from Lemma 1.1 (2) (3) (5) we have

$$S(2t) = 2S(t)C(t) \tag{2.2}$$

and

$$S(t) - S(r) = 2C\left(\frac{t+r}{2}\right) S\left(\frac{t-r}{2}\right). \tag{2.3}$$

Thus, using (2.2) and (C-1) we have

$$\|S(2t)x\| = 2\|C(t)S(t)x - C(t)\theta\| \leq 2L\|S(t)x\|, \tag{2.4}$$

which shows that (2.1) holds for $k = 2$. Let $k \in \mathbb{N}^+$ and $k > 2$. Using (2.3) and (C-1) we have

$$\begin{aligned} \|S(kt)x - S[(k-2)t]x\| &= 2\|C[(k-1)t]S(t)x - C[(k-1)t]\theta\| \\ &\leq 2L\|S(t)x\|. \end{aligned} \tag{2.5}$$

Since $\|S(\gamma t)x\| \leq \gamma L\|S(t)x\|$ for $\gamma = 1$ or $\gamma = 2$, using (2.5) we have

$$\begin{aligned} \|S(kt)x\| &\leq \|S(kt)x - S(\gamma t)x\| + \|S(\gamma t)x\| \\ &\leq \|S(\gamma t)x\| + \sum_{i=1}^{\frac{k-\gamma}{2}} \|S[(k+2-2i)t]x - S[(k-2i)t]\| \\ &\leq \gamma L\|S(t)x\| + \frac{k-\gamma}{2} 2L\|S(t)x\| = kL\|S(t)x\|, \end{aligned}$$

which is the desired inequality. □

Inspired by the methods of Suzuki in [22, 23, 24] and Zhu et al in [33], we give the following assertion.

Lemma 2.2. *Let D be a nonempty closed convex subset of a Hilbert space H and $\{C(t)\}$ a strongly continuous cosine family of L-Lipschitzian operators on D and $L_0 = \max\{L, 1\}$. Let $\{x_n\}$ be a sequence in D, and $\{t_n\}$ be a real sequence satisfying $0 < t_n < t$ for all $n \in \mathbb{N}^+$. Then*
(1) *the following inequality holds*

$$\|x_n - C(t)x_n\| \leq \frac{L_0 t^2}{t_n^2} \|x_n - C(t_n)x_n\| + 2L_0 t \sup_{0 \leq r \leq t_n} \left\| S\left(\frac{r}{2}\right) x_n \right\|.$$

(2) $\lim_{n\to\infty} \sup_{0\leq r\leq t_n} \left\|S\left(\frac{r}{2}\right) x_n\right\| = 0$ *if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} t_n = 0$.
Proof. For each $t \in \mathbb{R}$ and $x \in D$,

$$\begin{aligned} \|S(t)x\| &= \left\| \int_0^t [C(\tau)x - C(\tau)\theta] d\tau \right\| \leq \int_0^t \|C(\tau)x - C(\tau)\theta\| d\tau \\ &\leq L \int_0^t \|x\| d\tau = |t|L\|x\|. \end{aligned} \tag{2.6}$$

For each $x \in D$ and $r \in \mathbb{R}$, from (C-1) and Lemma 1.1 (4) we have

$$C(r)x - x = C(r)x - C(0)x = \pm 2S\left(\frac{r}{2}\right) S\left(\frac{r}{2}\right) x. \tag{2.7}$$

(1) For a real number $t > 0$, we denote by $[t]$ the maximum integer not exceeding t. Setting $[\frac{t}{t_n}] = m_n$ and using (2.6), (2.7) and Lemmas 1.1 and 2.1 we have

$$\begin{aligned}
\|x_n - C(t)x_n\| &\leq \sum_{k=0}^{m_n-1} \|C(kt_n)x_n - C((k+1)t_n)x_n\| + \|C(m_nt_n)x_n - C(t)x_n\| \\
&\leq \sum_{k=0}^{m_n-1} 2\left\|S\left(\frac{(2k+1)t_n}{2}\right)S\left(\frac{t_n}{2}\right)x_n\right\| \\
&\quad +2\left\|S\left(\frac{m_nt_n+t}{2}\right)S\left(\frac{m_nt_n-t}{2}\right)x_n\right\| \\
&\leq L_0\sum_{k=0}^{m_n-1} 2(2k+1)\left\|S\left(\frac{t_n}{2}\right)S\left(\frac{t_n}{2}\right)x_n\right\| \\
&\quad +2L_0t\sup_{0\leq r\leq t_n}\left\|S\left(\frac{r}{2}\right)x_n\right\| \\
&= 2L_0m_n^2\left\|S\left(\frac{t_n}{2}\right)S\left(\frac{t_n}{2}\right)x_n\right\| + 2L_0t\sup_{0\leq r\leq t_n}\left\|S\left(\frac{r}{2}\right)x_n\right\| \\
&\leq \frac{2L_0t^2}{t_n^2}\left\|S\left(\frac{t_n}{2}\right)S\left(\frac{t_n}{2}\right)x_n\right\| + 2L_0t\sup_{0\leq r\leq t_n}\left\|S\left(\frac{r}{2}\right)x_n\right\| \\
&= \frac{L_0t^2}{t_n^2}\|x_n - C(t_n)x_n\| + 2L_0t\sup_{0\leq r\leq t_n}\left\|S\left(\frac{r}{2}\right)x_n\right\|.
\end{aligned}$$

(2) Since $\{x_n\}$ is bounded, there exists $\rho > 0$ such that $\|x_n\| \leq \rho$ for all $n \in \mathbb{N}^+$. Thus, using (2.6) we have

$$\lim_{n\to\infty}\sup_{0\leq r\leq t_n}\left\|S\left(\frac{r}{2}\right)x_n\right\| \leq \lim_{n\to\infty}\sup_{0\leq r\leq t_n}\frac{rL}{2}\|x_n\| \leq \lim_{n\to\infty}\frac{L\rho t_n}{2} = 0.$$

This completes the proof. □

3. Strong convergence

Theorem 3.1. *Let D be a nonempty compact convex subset of a Hilbert space H and $\theta \in D$. Let $\{C(t)\}$ be a strongly continuous and locally asymptotically nonexpansive cosine family of operators on D and $[0,\gamma]$. Let $\{a_n\}, \{b_n\} \subset [0,1]$ and $\{t_n\} \subset [0,\gamma]$ be sequences satisfying*

(i) $\lim\limits_{n\to\infty} a_n = 0$, $\sum\limits_{n=1}^{\infty} |a_{n+1} - a_n| < +\infty$, *and* $\sum\limits_{n=1}^{\infty} a_n = +\infty$;

(ii) $\sum\limits_{n=1}^{\infty} a_n\lambda(t_n) < +\infty$, $\sum\limits_{n=1}^{\infty} b_n < +\infty$;

(iii) $\liminf\limits_{n\to\infty} t_n = 0$, $0 < \limsup\limits_{n\to\infty} t_n < +\infty$ *and* $\lim\limits_{n\to\infty}(t_{n+1} - t_n) = 0$.

Let $\{u_n\}$ be a sequence in D. Then the sequence $\{x_n\}$ defined by (IIS) *converges strongly to an element of $F(C)$ if and only if $\lim\limits_{n\to\infty}\|C(t_n)x_n - x_n\|$ exists.*

Proof. Let $L = 1 + \sup_{t\in[0,\gamma]} \lambda(t)$. Since D is compact, D is bounded. So there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in D$.

"Necessity". Let $x_n \to q \in F(C)$. Then from

$$\|C(t_n)x_n - x_n\| = \|[C(t_n)x_n - q] - (x_n - q)\| \leq (L+1)\|x_n - q\|$$

we see that $\lim_{n\to\infty} \|C(t_n)x_n - x_n\| = 0$. The necessity is proved.

"Sufficiency". Suppose that $\lim_{n\to\infty} \|C(t_n)x_n - x_n\|$ exists. First, we show that the sequence $\{x_n\}$ generated by (IIS) is well defined. For any $n \in \mathbb{N}^+$, assume that x_{n-1} is defined. We define a operator T_n as follows:

$$T_n x = (1 - a_n - b_n)x_{n-1} + a_n C(t_n)x + b_n u_n, \text{ for all } x \in D.$$

Since $\{C(t)\}$ is locally asymptotically nonexpansive, we have

$$\|T_n x - T_n y\| = a_n\|C(t_n)x - C(t_n)y\| \leq a_n(1 + \lambda(t_n))\|x - y\|$$

for all $x, y \in D$. Since we can suppose $0 \leq a_n(1+\lambda(t_n)) < 1$ by (i), T_n is a contraction with the contractive coefficient $a_n(1 + \lambda(t_n))$. From the Banach fixed point theorem we see that T_n has a unique fixed point, denoted as x_n, which means that $\{x_n\}$ is well defined.

Next, for each $p \in F(C)$ we show that $\lim_{n\to\infty} \|x_n - p\|$ exists. In fact, from (IIS) it follows that

$$\begin{aligned} \|x_n - p\| &= \|(1 - a_n - b_n)(x_{n-1} - p) + a_n[C(t_n)x_n - p] + b_n(u_n - p)\| \\ &\leq (1 - a_n - b_n)\|x_{n-1} - p\| + a_n\|C(t_n)x_n - p\| + b_n\|u_n - p\| \\ &\leq (1 - a_n)\|x_{n-1} - p\| + a_n(1 + \lambda(t_n))\|x_n - p\| \\ &\quad + b_n\|u_n - p\|. \end{aligned} \tag{3.1}$$

Since we can suppose $1 - a_n(1 + \lambda(t_n)) > 0$ by (i), from (3.1) we have

$$\|x_n - p\| \leq \left[1 + \frac{a_n\lambda(t_n)}{1 - a_n(1 + \lambda(t_n))}\right] \|x_{n-1} - p\| + \frac{2Mb_n}{1 - a_n(1 + \lambda(t_n))}. \tag{3.2}$$

Set $\mu_n := \frac{a_n\lambda(t_n)}{1-a_n(1+\lambda(t_n))}$ and $\nu_n := \frac{2Mb_n}{1-a_n(1+\lambda(t_n))}$. From (ii) it follows that

$$\begin{aligned} \lim_{n\to\infty} \frac{\mu_n}{a_n\lambda(t_n)} &= \lim_{n\to\infty} \frac{1}{1 - a_n(1 + \lambda(t_n))} = 1; \text{ and} \\ \lim_{n\to\infty} \frac{\nu_n}{b_n} &= \lim_{n\to\infty} \frac{2M}{1 - a_n(1 + \lambda(t_n))} = 2M. \end{aligned} \tag{3.3}$$

Thus, from (3.3) and conditions (i) and (ii) we infer that

$$\sum_{n=1}^{\infty} \mu_n < +\infty \text{ and } \sum_{n=1}^{\infty} \nu_n < +\infty.$$

Hence, using Lemma 1.5, from (3.2) we deduce that $\lim_{n\to\infty} \|x_n - p\|$ exists.
Next, we show that

$$\lim_{n\to\infty} \|C(t_n)x_n - x_n\| = 0. \tag{3.4}$$

For fixed $p \in F(C)$, using Lemma 1.3 we have

$$\begin{aligned}
\|x_n - p\|^2 &= \|(1-a_n-b_n)(x_{n-1}-p) + a_n(C(t_n)x_n - p) + b_n(u_n - p)\|^2 \\
&\leq (1-a_n-b_n)\|x_{n-1}-p\|^2 + a_n\|C(t_n)x_n - p\|^2 \\
&\quad + b_n\|u_n - p\|^2 - (1-a_n-b_n)a_n\|x_{n-1} - C(t_n)x_n\|^2 \\
&\leq (1-a_n)\|x_{n-1}-p\|^2 + a_n(1+\lambda(t_n))^2\|x_n - p\|^2 \\
&\quad + b_n\|u_n - p\|^2 - (1-a_n-b_n)a_n\|x_{n-1} - C(t_n)x_n\|^2. \qquad (3.5)
\end{aligned}$$

From conditions (i) and (ii) it follows that $\lim_{n\to\infty}(a_n + b_n) = 0$. Then there exists $n_0 \in \mathbb{N}^+$ such that $1 - a_n - b_n \geq 2^{-1}$ for all $n \geq n_0$. Hence from (3.5) we have

$$\begin{aligned}
2^{-1}a_n\|x_{n-1} - C(t_n)x_n\|^2 &\leq (1-a_n)(\|x_{n-1}-p\|^2 - \|x_n - p\|^2) \\
&\quad + a_n[(1+\lambda(t_n))^2 - 1]\|x_n - p\|^2 + b_n\|u_n - p\|^2 \\
&\leq \big[(1-a_{n-1})\|x_{n-1}-p\|^2 - (1-a_n)\|x_n - p\|^2\big] \\
&\quad + 2M^2|a_n - a_{n-1}| + 2M^2(L+1)a_n\lambda(t_n) \\
&\quad + 2M^2 b_n. \qquad (3.6)
\end{aligned}$$

For any $m \geq n_0$, (3.6) yields

$$\begin{aligned}
&\sum_{n=n_0}^{m} 2^{-1}a_n\|x_{n-1} - C(t_n)x_n\|^2 \\
\leq\ &\big[(1-a_{n_0-1})\|x_{n_0-1}-p\|^2 - (1-a_m)\|x_m - p\|^2\big] \\
&+ 2M^2\sum_{n=n_0}^{m}|a_n - a_{n-1}| + 2M^2(L+1)\sum_{n=n_0}^{m}a_n\lambda(t_n) \\
&+ 2M^2\sum_{n=n_0}^{m} b_n. \qquad (3.7)
\end{aligned}$$

Let $m \to \infty$, by conditions (i) and (ii), from (3.7) we have

$$\sum_{n=n_0}^{\infty} 2^{-1}a_n\|x_{n-1} - C(t_n)x_n\|^2 < +\infty. \qquad (3.8)$$

But $\sum_{n=n_0}^{\infty} a_n = +\infty$, (3.8) means that

$$\liminf_{n\to\infty}\|x_{n-1} - C(t_n)x_n\| = 0. \qquad (3.9)$$

Since D is bounded, from (IIS) we observe that

$$\begin{aligned}
\|x_n - C(t_n)x_n\| &\leq (1-a_n-b_n)\|x_{n-1} - C(t_n)x_n\| + b_n\|u_n - C(t_n)x_n\| \\
&\leq \|x_{n-1} - C(t_n)x_n\| + 2Mb_n. \qquad (3.10)
\end{aligned}$$

Combining with (3.9) and (3.10) we obtain

$$\liminf_{n\to\infty}\|x_n - C(t_n)x_n\| = 0. \qquad (3.11)$$

Since $\lim_{n\to\infty}\|C(t_n)x_n - x_n\|$ exists, from (3.11) we see that (3.4) holds.

Finally we show that $\{x_n\}$ converges strongly to an element of $F(C)$. Using Lemma 1.4, from (3.4) and condition (iii) we see that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} t_{n_k} = 0 \text{ and } \lim_{k\to\infty} \frac{\|x_{n_k} - C(t_{n_k})x_{n_k}\|}{t_{n_k}^2} = 0. \tag{3.12}$$

Since D is compact, there exists a subsequence of $\{x_{n_k}\}$ (for simplicity we still denote it by $\{x_{n_k}\}$) which converges strongly to $q \in D$. Suppose that $t > 0$ is arbitrary. Since $\lim_{k\to\infty} t_{n_k} = 0$, we can assume $t_{n_k} < t$ for all $k \in \mathbb{N}^+$, without loss of generality. Using Lemma 2.2, we have

$$\|x_{n_k} - C(t)x_{n_k}\| \le \frac{L_0 t^2}{t_{n_k}^2} \|x_{n_k} - C(t_{n_k})x_{n_k}\| + 2L_0 t \sup_{0\le r\le t_{n_k}} \left\| S\left(\frac{r}{2}\right) x_{n_k} \right\|, \tag{3.13}$$

where $L_0 = \max\{1, 1 + \sup_{r\in[0,t]} \lambda(r)\}$. It follows from (3.12) and (3.13) that

$$\lim_{k\to\infty} \|x_{n_k} - C(t)x_{n_k}\| = 0.$$

Observing that

$$\begin{aligned} \|q - C(t)q\| &\le \|q - x_{n_k}\| + \|x_{n_k} - C(t)x_{n_k}\| + \|C(t)x_{n_k} - C(t)q\| \\ &\le \|x_{n_k} - C(t)x_{n_k}\| + (2 + \lambda(t))\|q - x_{n_k}\|, \end{aligned}$$

we have $q = C(t)q$. This implies that $q \in F(C)$ due to $C(t) = C(-t)$. Since $q \in F(C)$, from (3.1) we see that $\lim_{n\to\infty} \|x_n - q\|$ exists. Hence, $\lim_{n\to\infty} \|x_n - q\| = \lim_{k\to\infty} \|x_{n_k} - q\| = 0$. This completes the proof. □

Theorem 3.2. *Let D be a nonempty closed convex subset of a Hilbert space H and $\theta \in D$. Let $\{C(t)\}$ be a strongly continuous and locally asymptotically nonexpansive cosine family of operators on D and $[0, \gamma]$. Let $\{a_n\}, \{b_n\} \subset [0, 1]$ and $\{t_n\} \subset [0, \gamma]$ be sequences satisfying*

(i) $\sum_{n=1}^{\infty} a_n < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$;

(ii) $\liminf_{n\to\infty} t_n = 0$, $0 < \limsup_{n\to\infty} t_n < +\infty$ *and* $\lim_{n\to\infty} (t_{n+1} - t_n) = 0$.

Let $\{u_n\}$ be a bounded sequence in D. Then the sequence $\{x_n\}$ defined by (IIS) *converges strongly to an element of $F(C)$ if and only if $\lim_{n\to\infty} \|C(t_n)x_n - x_n\| = 0$.*

Proof. Let $L = 1 + \sup_{t\in[0,\gamma]} \lambda(t)$. By duplicating the proof of Theorem 3.1, we can show that "Necessity" holds, and from conditions (i) we see that the sequence $\{x_n\}$ generated by (IIS) is well defined. We only prove "Sufficiency". Suppose that $\lim_{n\to\infty} \|C(t_n)x_n - x_n\| = 0$, i.e., (3.4) holds.

First, for each $p \in F(C)$ we show that $\lim_{n\to\infty} \|x_n - p\|$ exists. Since $\{u_n\}$ is bounded, we set $M_0 = \sup_{n\ge1} \|u_n - p\|$. Condition (i) implies that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$. Thus, from (3.1) we have

$$\|x_n - p\| \le \left[1 + \frac{a_n L}{1 - a_n L}\right] \|x_{n-1} - p\| + \frac{M_0 b_n}{1 - a_n L}. \tag{3.14}$$

From condition (i) we infer that

$$\sum_{n=1}^{\infty} \frac{a_n L}{1 - a_n L} < +\infty \text{ and } \sum_{n=1}^{\infty} \frac{M_0 b_n}{1 - a_n L} < +\infty.$$

Hence, using Lemma 1.5, from (3.14) we deduce that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded. In view of

$$\|C(t_n)x_n\| = \|C(t_n)x_n - C(t_n)\theta\| \le (1 + \lambda(t_n))\|x_n\| \le L\|x_n\|,$$

$\{C(t_n)x_n\}$ is also bounded. Set

$$M = \sup_{n\ge 1}\{\|C(t_n)x_n - u_n\|, \|C(t_n)x_n - x_{n-1}\|, \|u_n - x_{n-1}\|\}.$$

Next, we show that $\{x_n\}$ converges strongly to a point in D. For every $n, k \in \mathbb{N}^+$, by (IIS) we have

$$\begin{aligned} \|x_{n+k} - x_n\| &\le \sum_{j=n}^{n+k} \|x_j - x_{j-1}\| \le \sum_{j=n}^{n+k} \|a_j(C(t_j)x_j - x_{j-1}) + b_j(u_j - x_{j-1})\| \\ &\le M \sum_{j=n}^{n+k} (a_j + b_j). \end{aligned} \tag{3.15}$$

Since $\sum_{n=1}^{\infty}(a_n + b_n) < +\infty$, from (3.15) we infer that $\{x_n\}$ is a Cauchy sequence. Since D is closed, $\{x_n\}$ converges strongly to a point q in D.

Finally we show that $q \in F(C)$. Using Lemma 1.4, from (3.4) and condition (ii) we can obtain (3.12). Thus (3.13) holds by using Lemma 2.2, and so we have $\lim_{k\to\infty} \|x_{n_k} - C(t)x_{n_k}\| = 0$ for $t > 0$. Hence

$$\|q - C(-t)q\| = \|q - C(t)q\| = \lim_{k\to\infty} \|x_{n_k} - C(t)x_{n_k}\| = 0,$$

and so $q \in F(C)$. This completes the proof. □

Remark 3.1. In Theorems 3.1 and 3.2, by taking $\lambda(t_n) \equiv 0$ we can obtain the corresponding convergence theorems for nonexpansive cosine families; by taking $b_n \equiv 0$ we can obtain some convergence results of iterative schemes without error terms. Our results contain the case of nonexpansive cosine family and the case of iterative scheme without error term as their special cases. In the case of nonexpansive cosine family, our results are different from the ones in [29]. For example, as corollaries of Theorem 3.1, we have the following result immediately:

Corollary 3.3. *Let D be a nonempty compact convex subset of a Hilbert space H and $\theta \in D$. Let $\{C(t)\}$ be a strongly continuous and nonexpansive cosine family of operators on D. Let $\{a_n\}, \{b_n\} \subset [0,1]$ and $\{t_n\} \subset \mathbb{R}^+$ be sequences satisfying*

(i) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < +\infty$, *and* $\sum_{n=1}^{\infty} a_n = +\infty$;

(ii) $\sum_{n=1}^{\infty} b_n < +\infty$;

(iii) $\liminf_{n\to\infty} t_n = 0$, $0 < \limsup_{n\to\infty} t_n < +\infty$ *and* $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$.

Let $\{u_n\}$ *be a sequence in* D. *Then the sequence* $\{x_n\}$ *defined by* (IIS) *converges strongly to an element of* $F(C)$ *if and only if* $\lim_{n\to\infty} \|C(t_n)x_n - x_n\|$ *exists.*

Example 3.1. Let H_0, H_1, H_2 be closed linear subspaces of H and $H = H_0 \oplus H_1 \oplus H_2$. Let P_1 and P_2 be two orthogonal projection operators from H onto H_1 and H_2, respectively. Let

$$C(t) = (\cosh t)P_1 \oplus (\cos t)P_2.$$

Then it is easy to verify that $\{C(t)\}$ is a strongly continuous cosine family of (linear) operators on $H_1 \oplus H_2$ satisfying

$$\|C(t)x - C(t)y\| \leq [1 + \lambda(t)]\|x - y\| \text{ for all } x, y \in H_1 \oplus H_2,$$

where $\lambda(t) = \cosh t - 1$. Let D be a nonempty closed convex subset of $H_1 \oplus H_2$ with $\theta \in D$. Let $\{u_n\}$ be a bounded sequence in D. Let $a_n = b_n = \frac{1}{n^2}$. Define the sequence $\{t_n\}$ by

$$t_n = \begin{cases} \frac{i}{2k-1}, & \text{if } n = (k-1)(2k-1) + i,\ i = 1, 2, \cdots, 2k-1, \\ \\ \frac{2k+1-j}{2k}, & \text{if } n = k(2k-1) + j,\ j = 1, 2, \cdots, 2k. \end{cases}$$

Namely, the sequence $\{t_n\}$ is as follows:

$$1; \quad 1, \quad \tfrac{1}{2}; \quad \tfrac{1}{3}, \quad \tfrac{2}{3}, \quad 1; \quad 1, \quad \tfrac{3}{4}, \quad \tfrac{2}{4}, \quad \tfrac{1}{4}; \quad \tfrac{1}{5}, \quad \cdots;$$

$$\tfrac{1}{2k-1}, \quad \tfrac{2}{2k-1}, \quad \cdots, \quad \tfrac{2k-2}{2k-1}, \quad 1; \quad 1, \quad \tfrac{2k-1}{2k}, \quad \tfrac{2k-2}{2k}, \quad \cdots, \quad \tfrac{2}{2k}, \quad \tfrac{1}{2k}; \quad \cdots.$$

It is plain that $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Therefore, conditions (i) and (ii) in Theorem 4.2 are satisfied. Note that

$$\lim_{t\to 0+} \frac{(\cosh t) - 1}{t^2} = \frac{1}{2}.$$

If D is bounded and $\lim_{n\to\infty} \|C(t_n)x_n - x_n\| = 0$, then we obtain an iterative scheme defined by (IIS) which converges strongly to a common fixed point of $\{C(t)\}$.

Example 3.2. Let $C(t)$ be a strongly continuous cosine family of (linear) operators defined in Example 3.1. Let H, H_0, H_1, H_2 and D be as taken in Example 3.1. In particular, we take $H = \mathbb{R}^2$ as an Euclidian space,

$$H_0 = \{(0,0)\},\ H_1 = \{(\delta_1, 0) : \delta_1 \in \mathbb{R}\},\ H_2 = \{(0, \delta_2) : \delta_2 \in \mathbb{R}\},$$
$$D = \{(\delta_1, \delta_2) : \delta_1, \delta_2 \in [-2, 2]\}.$$

Then D be a nonempty compact convex subset of H with $\theta \in D$. Let $\{u_n\}$ be a sequence in D. Let

$$b_n = \frac{1}{n^2},\ a_n = \frac{1}{n}.$$

Then it is evident that

$$\sum_{n=1}^{\infty} b_n < +\infty,\ \lim_{n\to\infty} a_n = 0,\ \sum_{n=1}^{\infty} |a_{n+1} - a_n| < +\infty, \text{ and } \sum_{n=1}^{\infty} a_n = +\infty.$$

Define the sequence $\{t_n\}$ by

$$t_n = \begin{cases} \left(\frac{i}{2k-1}\right)^k, & \text{if } n = (k-1)(2k-1)+i,\ i = 1, 2, \cdots, 2k-1, \\ \left(\frac{2k+1-j}{2k}\right)^k, & \text{if } n = k(2k-1)+j,\ j = 1, 2, \cdots, 2k. \end{cases}$$

Then it is evident

$$\liminf_{n\to\infty} t_n = 0,\ 0 < \limsup_{n\to\infty} t_n < +\infty \text{ and } \lim_{n\to\infty}(t_{n+1} - t_n) = 0.$$

Note that $\lambda(t) = \cosh t - 1 \le t^2 \le t$ for $t \in [0, 1]$. We have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \lambda(t_n) &\le \sum_{n=1}^{\infty} a_n t_n \\ &= \sum_{k=1}^{\infty} \left[\sum_{i=1}^{2k-1} \frac{\left(\frac{i}{2k-1}\right)^k}{(k-1)(2k-1)+i} + \sum_{j=1}^{2k} \frac{\left(\frac{2k+1-j}{2k}\right)^k}{k(2k-1)+j} \right] \\ &\le \sum_{k=1}^{\infty} \left[\sum_{i=1}^{2k-1} \frac{\left(\frac{i}{2k-1}\right)^k}{(k-1)(2k-1)+1} + \sum_{j=1}^{2k} \frac{\left(\frac{2k+1-j}{2k}\right)^k}{k(2k-1)+1} \right] \end{aligned}$$

Set

$$v_k = \sum_{i=1}^{2k-1} \frac{\left(\frac{i}{2k-1}\right)^k}{(k-1)(2k-1)+1} + \sum_{j=1}^{2k} \frac{\left(\frac{2k+1-j}{2k}\right)^k}{k(2k-1)+1}.$$

Since

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{2k-1} i^k}{(k+1)^{-1}(2k-1)^{k+1}} = 1$$

and

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{2k} (2k+1-j)^k}{(k+1)^{-1}(2k)^{k+1}} = \lim_{k\to\infty} \frac{\sum_{j=1}^{2k} j^k}{(k+1)^{-1}(2k)^{k+1}} = 1,$$

we have

$$\lim_{k\to\infty} \frac{v_k}{k^{-2}} = \lim_{k\to\infty} \left[\frac{k^2(k+1)^{-1}(2k-1)}{(k-1)(2k-1)+1} + \frac{k^2(k+1)^{-1}(2k)}{k(2k-1)+1} \right] = 2.$$

This implies that $\sum_{n=1}^{\infty} a_n \lambda(t_n) < +\infty$. Therefore, conditions (i)-(iii) in Theorem 3.1 are satisfied. From Theorem 3.1 we can obtain an iterative scheme defined by (IIS) which converges strongly to a common fixed point of $\{C(t)\}$.

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References

[1] S.A. Ahmed, A.A. El-Sayed, A.K.A. Bukhari, V.Ć. Rajić, *Fixed points by some iterative algorithms in Banach and Hilbert spaces with some applications*, J. Comput. Anal. Appl., **20**(2016), 707-717.

[2] A. Aleyner, Y. Censor, *Best approximation to common fixed points of a semigroup of nonexpansive operators*, J. Nonlinear Convex Anal., **6**(2005), 137-151.

[3] J. Borwein, S. Reich, I. Shafrir, *Krasnoselskii-Mann iterations in normed spaces*, Canad. Math. Bull., **35**(1992), 21-28.

[4] R.E. Bruck, T. Kuczumow, S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloquium Math., **65**(1993), 169-179.

[5] S.S. Chang, K.K. Tan, H.W.J. Lee, C. K. Chan, *On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **313**(2006), 273-283.

[6] Y.J. Cho, X.L. Qin, *Convergence of a general iterative method for nonexpansive mappings in Hilbert spaces*, J. Comput. Appl. Math., **228**(2009), 458-65.

[7] F. Cianciaruso, G. Marino, L. Muglia, *Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces*, J. Optim. Theory Appl., **146**(2010), 491-509.

[8] W.Q. Deng, *Strong convergence of Mann's type iteration method for an infinite family of generalized asymptotically nonexpansive nonself mappings in Hilbert spaces*, Optim. Letters, **8**(2014), 533-42.

[9] H. Fukhar-ud-din, A.R. Khan, *Convergence of implicit iterates with errors for mappings with unbounded domain in Banach spaces*, Inter. J. Math. Math. Sci., **10**(2005), 1643-1653.

[10] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.

[11] Z. Huang, *Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl., **37**(1999), 1-7.

[12] N. Hussain, G. Marino, L. Muglia, A.A.N. Abdou, *On the strong convergence of a general-type Krasnosel'skii Mann's algorithm depending on the coefficients*, J. Fixed Point Theory Appl., **18**(2016), 1-25.

[13] N. Hussain, G. Marino, L. Muglia, B.A.S. Alamr, *On some Mann's type iterative algorithms*, Fixed Point Theory Appl., **17**(2015), 1-16.

[14] U. Kamraksa, R. Wangkeere, *Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces*, J. Global Optim., **51**(2011), 689-714.

[15] U. Kohlenbach, A. Koutsoukou-Argyraki, *Effective asymptotic regularity for one-parameter nonexpansive semigroups*, J. Math. Anal. Appl., **433**(2016), 1883-1903.

[16] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl., **329**(2007), 336-346.

[17] H. Piri, P. Kumam, *Strong convergence theorems for fixed points of asymptotically nonexpansive semigroups in Banach spaces*, Fixed Point Theory Appl., **225**(2014), 1-15.

[18] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979), 274-276.

[19] J. Schu, *Weak and Strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43**(1991), 153-159.

[20] J. Schu, *Approximation of fixed points of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **112**(1991), 143-151.

[21] Y. Shehu, *An iterative method for nonexpansive semigroups, variational inclusions and generalized equilibrium problems*, Math. Comput. Modelling, **55**(2012), 1301-1314.

[22] T. Suzuki, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert space*, Proc. Amer. Math. Soc., **131**(2002), 2133-2136.

[23] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227-239.

[24] T. Suzuki, *Browder's type convergence theorems for one parameter semi-groups of nonexpansive mappings in Banach spaces*, Israel J. Math., **157**(2007), 239-257.
[25] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., **178**(1993), 301-308.
[26] K.K. Tan, X.H. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **122**(1994), 733-739.
[27] N.T.T. Thuy, *Hybrid Mann-Halpern iteration methods for finding fixed points involving asymptotically nonexpansive mappings and semigroups*, Vietnam. J. Math., **42**(2014), 219-232.
[28] J.-Z. Xiao, J. Sun, X. Huang, *Approximating common fixed points of asymptotically quasi-nonexpansive mappings by a k+1-step iterative scheme with error terms*, J. Comput. Appl. Math., **233**(2010), 2062-2070.
[29] J.Z. Xiao, J. Yan, X.H. Zhu, *Explicit, implicit and viscosity iterations for nonexpansive cosine families in Hilbert spaces*, Acta Math. Sci. (Ser. A), **34**(2014), 1518-1531.
[30] B.L. Xu, M.A. Noor, *Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **267**(2002), 444-453.
[31] A.J. Zaslavski, *Approximate Solutions of Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, 2016.
[32] A.J. Zaslavski, *Algorithms for Solving Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, 2018.
[33] X.H. Zhu, J.Z. Xiao, *Minimum selections and fixed points of set-valued operators in Banach spaces with some uniform convexity*, Appl. Math. Comput., **217**(2011), 6004-6010.

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