Fixed Point Theory, 22(2021), No. 1, 359-376
DOI: 10.24193/fpt-ro.2021.1.25
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A NOVEL HYBRID METHOD FOR EQUILIBRIUM PROBLEM AND A COUNTABLE FAMILY OF GENERALIZED NONEXPANSIVE-TYPE MAPS WITH APPLICATIONS

MARKJOE OLUNNA UBA[*], EMMANUEL EZZAKA OTUBO[**,+],
MARIA AMARAKRISTI ONYIDO[*,++]

[*]University of Nigeria, Nsukka
E-mail: markjoe.uba@unn.edu.ng

[**]Auburn University, Abuja
[+]Ebonyi State University, Abakaliki
E-mail: eeo0010@auburn.edu

[++]Auburn University, USA
E-mail: mao0021@auburn.edu

Abstract. Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^*. A novel hybrid method for finding a solution of an equilibrium problem and a common element of fixed points for a family of a general class of nonlinear nonexpansive maps is constructed. The sequence of the method is proved to converge strongly to a common element of the family and a solution of the equilibrium problem. Finally, an application of our theorem complements, generalizes and extends some recent important results (*Takahashi et al.*, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 341 (2008), 276-286., *Nakajo and Takahashi*, Strong convergence theorems for nonexpansive mappings and nonexpansive semi-groups, J. Math. Anal. Appl. vol. 279 (2003), 372-379., *Qin and Su*, Strong convergence of monotone hybrid method for fixed point iteration process, J. Syst. Sci. and Complexity **21** (2008) 474-482., *Klin-eam et al.*, Hybrid method for the equilibrium problem and a family of generalized nonexpansive mappings in Banach spaces, J. Nonlinear Sci. Appl. 9 (2016), 4963-4975).
Key Words and Phrases: Equilibrium problem, J_* fixed points, strong convergence.
2020 Mathematics Subject Classification: 47H09, 47H05, 47J25, 47J05, 47H10.

1. Introduction

Let E be a real Banach space and E^* the topological dual space of E. A map $J : E \to 2^{E^*}$ defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x||||x^*||, ||x|| = ||x^*||\}$$

is called the *normalized duality map* on E.
Let E be a real normed space with dual space E^*. A map $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, $\big\langle \eta - \nu, x - y \big\rangle \geq 0 \ \forall \ \eta \in Ax, \ \nu \in Ay$. Consider, for

example, the following: Let $g : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of g, $\partial g : E \to 2^{E^*}$, is defined for each $x \in E$ by

$$\partial g(x) = \big\{x^* \in E^* : g(y) - g(x) \geq \big\langle y - x, x^* \big\rangle \ \forall \ y \in E\big\}.$$

It is easy to check that ∂g is a *monotone map* on E, and that $0 \in \partial g(u)$ *if and only if u is a minimizer of g*. Setting $\partial g \equiv A$, it follows that solving the inclusion

$$0 \in Au, \tag{1.1}$$

in this case, is solving for a minimizer of g.

A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that $\big\langle \eta - \nu, j(x-y)\big\rangle \geq 0 \ \forall \ \eta \in Ax, \ \nu \in Ay$. Accretive operators have been studied extensively by numerous mathematicians (see e.g., Browder [10], Deimling [21], Kato [26], Cioranescu [18], Reich [43], and a host of other authors).

It is well known (see e.g., Zeidler [62]) that many physically significant problems can be modeled in terms of an initial-value problem of the form

$$0 \in \frac{du}{dt} + Au, \quad u(0) = u_0, \tag{1.2}$$

where A is a nonlinear map on an appropriate real Banach space. Typical examples of such evolution equations are found in models involving the heat, wave or Schrödinger equations (see e.g., Browder [10], Zeidler [62]). Observe that in the model (1.2), if the solution u is independent of time (i.e., at the equilibrium state of the system), then $\dfrac{du}{dt} = 0$ and (1.2) reduces to

$$0 \in Au. \tag{1.3}$$

In a case where A is accretive, solutions to (1.3) correspond to the equilibrium state of the system described by (1.2). Solutions of inclusion (1.3) can also represent solutions of partial differential equations (see e.g., Benilan, Crandall and Pazy [6], Khatibzadeh and Moroanu [27], Khatibzadeh and Shokri [28], Showalter [45], Volpert [54] and so on). In studying the inclusion $0 \in Au$, where A is a multi-valued accretive operator on a Hilbert space H, Browder[9] introduced an operator T defined by $T := I - A$ where I is the identity map on H. He called such an operator *pseudo-contractive*. It is clear that solutions of $0 \in Au$, if they exist, correspond to fixed points of T. In past years, methods for approximating solutions of inclusion (1.3) when A is an accretive-type operator have been investigated by numerous mathematicians (see e.g., Agarwal *et al.* [1]; Berinde [7]; Chidume [14]; Censor and Reich [12]; William and Shahzad [56], and the references contained in them).

Moreover, in a case where A is monotone, if $A = \partial g$ which is known to be monotone, solutions to (1.3) correspond to minimizers of some convex functional. It is well known (see e.g., Cioranescu [18]) that most monotone operators on a normed space are subdifferentials of some convex function. Even though the class of monotone-type operators have a wider variety of applications (see e.g., Pascali and Sburian [40], p. 101), unfortunately, developing algorithms for approximating solutions of equations of type (1.3) *when $A : E \to 2^{E^*}$ is of monotone-type* has not been very fruitful. Part of the difficulty seems to be that the technique of converting the inclusion (1.3) into a fixed point problem for $T := I - A : E \to 2^E$ is not applicable since, in this case

when A is monotone, A maps E into 2^{E^*} and the identity map does not make sense. However, a new concept of *fixed points for maps from a real normed space E to its dual space E^** has very recently been introduced and studied (see Chidume and Idu [16], Liu [34], Zegeye [58]). The development of this fixed point theory have provided a technique for studying the inclusion $0 \in Au$ where A maps a space E to 2^{E^*}. In particular, for studying the inclusion $0 \in Au$ where A is the subdifferential of a convex function this fixed point theory for maps from a space E to 2^{E^*} is appropriate.
With this evolving fixed point theory, we study the equilibrium problem. Let C be a closed subset of E such that JC is a closed and convex subset of E^*, where J is the duality mapping on E. Let $f : JC \times JC \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is finding

$$x^* \in C \text{ such that } f(Jx^*, Jy) \geq 0, \forall\, y \in C. \tag{1.4}$$

The solution set of (1.4) is denoted by $EP(f)$. Numerous problems in physics, optimization and economics reduce to finding a solution of (1.4) (see, e.g. Zegeye and Shahzad [61], Combettes and Hirstoaga [20], Flam and Antipin [22], Takahashi and Takahashi [46] and the references in them). In the past two decades, the theory of equilibrium problems has been extensively studied in the literature. In most of the papers published on the equilibrium problems, the existence of their solutions and applications have been studied (see, e.g. Blum and Oettli [8], Combettes and Hirstoaga [20], Flam and Antipin [22], Moudafi and The ŕa [35], Takahashi and Takahashi [46] and the references therein). Moreover, in the past few years, several researchers have started working on the approximate solution of the equilibrium problems and their generalizations (see, e.g. Iiduka and Takahashi [24], Kamimura and Takahashi [25], Zegeye et al. [59], Zegeye and Shahzad [60] and the references therein). Furthermore, Ibaraki and Takahashi [23], and Kohsaka and Takahashi [32] also studied some properties for generalized nonexpansive retractions in Banach spaces. Recently, Takahashi and Zembayashi [48] considered equilibrium problem with a bifunction defined on the dual space of a Banach space and they proved a strong convergence theorem for finding a solution of the equilibrium problem which generalized the result of Combettes and Hirstoaga [20].
Very recently in 2016, Klin-eam et al. [31] presented a new and interesting monotone hybrid iterative method *for equilibrium problem* and *convex feasibility problem* for a family of generalized nonexpansive maps in a Banach space more general than Hilbert spaces. They proved the following Theorem:

Theorem 1.1 Let E be a uniformly smooth and uniformly convex real Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, and $T_n : C \to E$, $n = 1, 2, 3, ...$ be a countable family of generalized nonexpansive maps and τ be a family of closed and generalized nonexpansive maps from C to E such that

$$\bigcap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$$

and $F(\Gamma) \cap EP(f) \neq \emptyset$. Assume that $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ u_n \in C, \ \textit{such that} \ f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall \ y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(\Gamma)\cap EP(f)}x$, where $R_{F(\Gamma)\cap EP(f)}$ is the sunny generalized nonexpansive retraction of E onto $F(\Gamma) \cap EP(f)$.

On the other hand, the problem of finding a point in the intersection of a family of closed and convex subsets of a Banach space is generally referred to as *the convex feasibility problem.* This so called *convex feasibility problem* which appears frequently in various areas of physical sciences has been studied well in the framework of Hilbert spaces and has found applications in areas such as image restoration, computer tomography, radiation therapy treatment planning (see e.g., Combettes [19]). Significant research has also been done on the convex feasibility problem in real Banach spaces more general than Hilbert space (see e.g., Kitahara and Takahashi [30], O'Hare *et al.* [38], Chang *et al.* [13], Qin *et al.* [41], Zhou and Tan [63], Wattanwitoon and Kumam [55], Li and Su [33], Takahashi and Zembayashi [49], Kikkawa and Takahashi [29], Aleyner and Reich [5], Sahu *et al.* [44], Ceng *et al.* [11], and the references contained in them).

In this paper, motivated and inspired by Klin-eam et al. [31], very recent results involving monotone type operators (see e.g. [17, 15, 51, 50, 53, 52, 39]) and onging research in *fixed point theory for maps from a real Banach space E to its dual space E^**, We prove a novel strong convergence theorem for finding a solution of an equilibrium problem and an infinite family of an important general class of nonexpansive maps from a normed space E into its dual space E^*. Finally, our result complements and extends several important results and our method of proof is of independent interest.

2. Preliminaries

In the sequel, we shall need the following definitions and results.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , $\rho_E : [0, \infty) \to [0, \infty)$, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau, \ \tau > 0\right\}.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \ x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1.$$

The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1;\ \epsilon = \|x-y\|\Big\}.$$

The space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. The norm of E is said to be *Fréchet differentiable* if for each

$$x \in S := \{u \in E : \|u\| = 1\},\ \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t},$$

exists and is attained uniformly for $y \in E$. We list some properties of the normalized duality map (defined earlier) which are well known (see e.g., Cioranescu [18]).

- $J(0) = 0$,
- For $x \in E$, Jx is nonempty closed and convex,
- If E is strictly convex, then J is one-to-one, i.e., if $x \neq y$, then $Jx \cap Jy = \emptyset$,
- If E is reflexive, then J is onto,
- If E is smooth, then J is single-valued,
- If E is uniformly smooth, then J is uniformly continuous on bounded subsets of E.

Let E be a smooth real Banach space with dual E^*. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$, is defined by:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \text{for } x,y \in E, \tag{2.1}$$

where J is the normalized duality map. It was introduced by Alber and has been studied by Alber [2], Alber and Guerre-Delabriere [3], Kamimura and Takahashi [25], Reich [43] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x,y) = \|x-y\|^2$ for $x,y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2 \text{ for } x,y \in E. \tag{2.2}$$

Remark 2.1 Following Alber [2] , the generalized projection $\Pi_C : E \to C$ is defined by $\Pi_C(x) = \inf \phi(y,x), \forall x \in E$.

Definition 2.2 Let C be a nonempty closed and convex subset of a real Banach space E and T be a map from C to E. The map T is called *generalized nonexpansive* if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and $\phi(Tx,p) \leq \phi(x,p)$ for all $x \in C, p \in F(T)$. A map R from E onto C is said to be a retraction if $R^2 = R$. The map R is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction R from E onto C. We now list some lemmas which will be used in the sequel.

Lemma 2.3 (see e.g., Alber [2]) Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, the following are equivalent.

(i) C is a sunny generalized nonexpansive retract of E,
(ii) C is a generalized nonexpansive retract of E,

(iii) JC is closed and convex.

Lemma 2.4 (see e.g., Alber [2]) Let C be a nonempty closed and convex subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C.Then, the following hold.
(i) $z = Rx$ iff $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$,
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Lemma 2.5 (see e.g., Xu [57]) Let E be a uniformly convex Banach space. Let $r > 0$. Then, there exists a strictly increasing continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and the following inequality holds:

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||),$$

for all $x, y \in B_r(0)$, where $B_r(0) := \{v \in E : ||v|| \leq r\}$ and $\lambda \in [0, 1]$.

Lemma 2.6 (see e.g., Kamimura and Takahashi [25]) Let E be a smooth and uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.7 (see e.g., Ibaraki and Takahashi [23]) Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E to C is uniquely determined.

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. For solving the equilibrium problem, let us assume that a bifunction $f : JC \times JC \to \mathbb{R}$ satisfies the following conditions:
(A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;
(A2) f is monotone, i.e. $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;
(A3) for all $x^*, y^*, z^* \in JC$, $\limsup_{t\downarrow 0} f(tz^* + (1 - t)x^*, y^*) \leq f(x^*, y^*)$;
(A4) for all $x^* \in JC$, $f(x^*, \cdot)$ is convex and lower semicontinuous;

Lemma 2.8 (see e.g., Blum and Oettli [8]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $r > 0$ and let $x \in E$. Then there exists $z \in C$ such that $f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0$, $\forall\, y \in C$.

Lemma 2.9 (see e.g., Takahashi and Zembayashi [48]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and let $x \in E$, define a mapping $T_r(x) : E \to C$ as follows:

$$T_r(x) = \{z \in C : f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0,\ \forall\, y \in C\}.$$

Then the following hold:
(i) T_r is single valued;
(ii) for all $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle x - y, JT_r x - JT_r y\rangle$;
(iii) $F(T_r) = EP(f)$;
(iv) $JEP(f)$ is closed and convex.

Lemma 2.10 (see e.g., Takahashi and Zembayashi [48]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and $r > 0$. For $x \in E$ and $p \in F(T_r)$,

$$\phi(x, T_r(x)) + \phi(T_r(x), p) \leq \phi(x, p).$$

Lemma 2.11 [Alber [2]] Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Then, the following conclusion hold:

$$\varphi(x, \Pi_C y) + \varphi(\Pi_C y, y) \leq \varphi(x, y), \forall x \in C, y \in E.$$

NST-condition. Let C be a closed subset of a Banach space E. Let $\{T_n\}$ and Γ be two families of generalized nonexpansive maps of C into E such that

$$\bigcap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset,$$

where $F(T_n)$ is the set of fixed points of $\{T_n\}$ and $F(\Gamma)$ is the set of common fixed points of Γ.

Definition 2.12 The sequence $\{T_n\}$ satisfies the NST-condition (see e.g., Nakajo, Shimoji and Takahashi [36]) with Γ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} ||x_n - T_n x_n|| = 0 \Rightarrow \lim_{n\to\infty} ||x_n - T x_n|| = 0, \ for\ all\ T \in \Gamma.$$

Remark 2.13 If $\Gamma = \{T\}$ a singleton, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. If $T_n = T$ for all $n \geq 1$, then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$.

3. Main results

Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^*. Let J be the normalized duality map on E and J_* be the normalized duality map on E^*. Observe that under this setting, J^{-1} exists and $J^{-1} = J_*$. With these notations, we have the following definitions.
Let C be a nonempty subset of a real normed space E with dual space E^*.

Definition 3.1 A map $T : C \to E^*$ is called J_**-closed* if $(J_* oT) : C \to E$ is a closed map, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \to x$ and $(J_* oT)x_n \to y$, then $(J_* oT)x = y$.

Definition 3.2 A point $x^* \in C$ is called a J*-fixed point of* T if $Tx^* = Jx^*$. The set of J-fixed points of T will be denoted by $F_J(T)$.

Definition 3.3 A map $T : C \to E^*$ will be called *generalized* J_**-nonexpansive* if $F_J(T) \neq \emptyset$, and $\phi(p, (J_* oT)x) \leq \phi(p, x)$ for all $x \in C$ and for all $p \in F_J(T)$.

Let C be a closed subset of a real Banach space E. Let $\{T_n\}$ and Γ be two families of generalized J_*-nonexpansive maps of C into E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$, where $F_J(\Gamma)$ denotes the set of common J-fixed points of Γ.

Definition 3.4 A sequence $\{T_n\}$ of maps from C to E^* will be said to *satisfy the NST-condition with* Γ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} ||Jx_n - T_n x_n|| = 0 \Rightarrow \lim_{n\to\infty} ||Jx_n - Tx_n|| = 0, \text{ for every } T \in \Gamma.$$

Lemma 3.5 Let E be a uniformly smooth and uniformly convex real Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized J_*-nonexpansive map from C to E^* such that $F_J(T) \neq \emptyset$, then $F_J(T)$ and $JF_J(T)$ are closed.

Proof. First, we show that $F_J(T)$ is closed. For this, let $\{x_n\} \subset F_J(T)$ with $x_n \to x$. Since T is generalized J_*-nonexpansive, we have that $\phi(x_n, (J_* \circ T)x) \leq \phi(x_n, x)$ $\forall\ n \in \mathbb{N}$. This implies that

$$\phi(x, (J_* \circ T)x) = \lim_{n\to\infty} \phi(x_n, (J_* \circ T)x) \leq \lim_{n\to\infty} \phi(x_n, x) = \phi(x, x) = 0.$$

Therefore, $\phi(x, (J_* \circ T)x) = 0$ and hence $x \in F_J(T)$.
Next, we show that $JF_J(T)$ is closed. Clearly, $JF_J(T) = J_*^{-1}F_J(T)$. It is also well known that if E is a uniformly smooth and uniformly convex real Banach space, J and J_* are uniformly continuous on bounded subsets of E (see e.g., Cioranescu [18] and Chidume [14]). This implies that $JF_J(T)$ is closed as a preimage of a closed set.

Using Lemmas 2.3 and 3.5, we give the following lemma.

Lemma 3.6 Let E be a uniformly smooth and uniformly convex real Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized J_*-nonexpansive map from C to E^* such that $F_J(T) \neq \emptyset$. If $JF_J(T)$ is convex, then $F_J(T)$ is a sunny generalized nonexpansive retract of E.

We now prove the following theorem.

Theorem 3.7 Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $T_n : C \to E^*, n = 1, 2, 3, ...$ be an infinite family of generalized J_*-nonexpansive maps and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$ and $F_J(\Gamma) \cap EP(f) \neq \emptyset$. Assume that $JF_J(\Gamma)$ is convex and $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* o T_n)x_n), \\ u_n \in C, \ \textit{such that } f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall\ y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf\ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(\Gamma)\cap EP(f)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(\Gamma) \cap EP(f)$.

Proof. The proof is given in 6 steps.

Step 1. We show that the sequence $\{x_n\}$ is well defined. We begin by showing that $F_J(\Gamma) \cap EP(f)$ is a sunny generalized retract of E. From the fact that $JF_J(\Gamma)$ is convex together with Lemmas 3.5 and 2.9, we have that $JF_J(\Gamma)$ and $JEP(f)$ are closed and convex. Since E is uniformly convex, J is one-to-one. Thus, we have that, $J(F_J(\Gamma) \cap EP(f)) = JF_J(\Gamma) \cap JEP(f)$ and so $J(F_J(\Gamma) \cap EP(f))$ is closed and convex. From Lemma 2.3, we have $F_J(\Gamma) \cap EP(f)$ is a sunny generalized retract of E. Next, we show by induction that JC_n is closed and convex. Clearly, JC_1 is closed and convex. Also, it is easy to see that $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$0 \leq ||x_n||^2 - ||u_n||^2 - 2\langle z, Jx_n - Ju_n\rangle,$$

which is affine in z and hence JC_n is closed and convex for each $n \geq 1$. Again, from Lemma 2.3, we have that C_n is a sunny generalized nonexpansive retract of E for each $n \geq 1$.
Next, we show $F_J(\Gamma) \cap EP(f) \subset C_n \; \forall n \in \mathbb{N}$. Clearly, for $C_1 = C$, we have $F_J(\Gamma) \cap EP(f) \subset C_1$. Suppose that $F_J(\Gamma) \cap EP(f) \subset C_n$ for some $n \in \mathbb{N}$. Let $u \in F_J(\Gamma) \cap EP(f) \subset C_n$. Setting $u_n = T_{r_n} y_n$ for all $n \in \mathbb{N}$, using the fact that $T_n : C \to E^*, n = 1, 2, 3, ...$ is an infinite family of generalized J_*-nonexpansive maps such that $F_J(\Gamma) \neq \emptyset$, the definition of y_n, Lemmas 2.5 and 2.10, we compute as follows:

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, T_{r_n} y_n) \leq \phi(u, y_n) \qquad (3.1)\\
&= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* o T_n)x_n)\\
&\leq \alpha_n\big[||u||^2 - 2\langle u, Jx_n\rangle + ||x_n||^2\big] + (1-\alpha_n)\big[||u||^2 - 2\langle u, J(J_* o T_n)x_n\rangle\\
&\quad + ||T_n x_n||^2\big] - \alpha_n(1-\alpha_n)g(||Jx_n - J(J_* o T_n x_n)x_n||)\\
&= \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, (J_* o T_n)x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - T_n x_n||),
\end{aligned}$$

which yields

$$\phi(u, u_n) \leq \phi(u, x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - T_n x_n||). \tag{3.2}$$

Hence, $\phi(u, u_n) \leq \phi(u, x_n)$ and we have that $u \in C_{n+1}$.
This implies $F_J(\Gamma) \cap EP(f) \subset C_n \; \forall \; n \in \mathbb{N}$. So that, $\{x_n\}$ is well defined.
Clearly, we have also from (3.1) that

$$\phi(u, y_n) \leq \phi(u, x_n). \tag{3.3}$$

Step 2. We show that $\lim_{n\to\infty} ||x_n - u_n|| = 0$.
From lemma 2.4(ii), and $\{x_n\} = R_{C_n} x$, we have that

$$\phi(x, x_n) = \phi(x, R_{C_n} x) \leq \phi(x, u) - \phi(R_{C_n} x, u) \leq \phi(x, u) \; \forall \; u \in F_J(\Gamma) \cap EP(f) \subset C_n.$$

This implies that $\{\phi(x, x_n)\}$ is bounded. Hence, from equation 2.2, $\{x_n\}$ is bounded. Also, from $C_{n+1} \subset C_n$ and $x_n = R_{C_n} x$, we have

$$\phi(x, x_n) \leq \phi(x, x_{n+1}) \; \forall \; n \in \mathbb{N}.$$

So, $\lim_{n\to\infty} \phi(x, x_n)$ exists. Using Lemma 2.4(ii) and $x_n = R_{C_n} x$, we obtain that for all $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
\phi(x_n, x_m) &= \phi(R_{C_n} x, x_m) \leq \phi(x, x_m) - \phi(x, R_{C_n} x)\\
&= \phi(x, x_m) - \phi(x, x_n) \to 0 \; as \; n \to \infty. \qquad (3.4)
\end{aligned}$$

From Lemma 2.6, we conclude that $||x_n - x_m|| \to 0,\ as\ n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in C, and so, there exists $x^* \in C$ such that $x_n \to x^*$. Using the definitions of C_{n+1} and x_{n+1}, we obtain that $\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. By Lemma 2.6, we have that $||x_n - u_n|| \to 0$ as $n \to \infty$, completing proof of Step 2.

Step 3. $\lim_{n\to\infty} ||Jx_n - Tx_n|| = 0\ \forall\ T \in \Gamma$.

Observe first that since J is uniformly continuous on bounded subsets of E, it follows from Step 2 that $||Ju_n - Jx_n|| \to 0$ as $n \to \infty$. From inequality (3.2) and for some constant $M > 0$, we obtain that:

$$\alpha_n(1-\alpha_n)g(||Jx_n - T_nx_n||) \leq \phi(u, x_n) - \phi(u, u_n) \leq 2||u||.||Ju_n - Jx_n|| + ||u_n - x_n||M.$$

Using $\liminf \alpha_n(1 - \alpha_n) = a > 0$, there exists $n_0 \in \mathbb{N}$:

$$0 < \frac{a}{2} < \alpha_n(1 - \alpha_n) \text{ for all } n \geq n_0.$$

Thus, we have

$$0 \leq \frac{a}{2} g(||Jx_n - T_nx_n||) \leq 2||u||.||Ju_n - Jx_n|| + ||u_n - x_n||M\ \forall\ n \geq n_0.$$

Using step 2, and properties of g, we obtain that $\lim_{n\to\infty} ||Jx_n - T_nx_n|| = 0$. Since $\{T_n\}_{n=1}^{\infty}$ satisfies the NST condition with Γ, we have that

$$\lim_{n\to\infty} ||Jx_n - Tx_n|| = 0\ \forall\ T \in \Gamma,$$

completing proof of Step 3.

Step 4. We prove that $x^* \in F_J(\Gamma)$.

From Step 3, we know that $\lim_{n\to\infty} ||Jx_n - Tx_n|| = 0\ \forall\ T \in \Gamma$. Also, we have proved that $x_n \to x^* \in C$. Assume now that $(J_*oT)x_n \to y^*$. Since T is J_*-closed, we have $y^* = (J_*oT)x^*$. Furthermore, by the uniform continuity of J on bounded subsets of E, we have: $Jx_n \to Jx^*$ and $J(J_*oT)x_n \to Jy^*$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} ||Jx_n - Tx_n|| = \lim_{n\to\infty} ||Jx_n - J(J_*oT)x_n|| = 0,$$

which implies, $||Jx^* - Jy^*|| = ||Jx^* - J(J_*oT)x^*|| = ||Jx^* - Tx^*|| = 0$, and so, $x^* \in F_J(\Gamma)$.

Step 5. We prove that $x^* \in EP(f)$.

We recall that in step 1, we set $u_n = T_{r_n}y_n$ for all $n \in \mathbb{N}$. Also, from step 2, $u_n \to x^*\ as\ n \to \infty$. From lemma 2.10 and inequality (3.1), we have

$$\begin{aligned}
\phi(y_n, u_n) &= \phi(y_n, T_{r_n}y_n) \\
&\leq \phi(y_n, u) - \phi(T_{r_n}y_n, u) \\
&\leq \phi(x_n, u) - \phi(T_{r_n}y_n, u) \\
&= \phi(x_n, u) - \phi(u_n, u)
\end{aligned}$$

Since $\lim_{n\to\infty}(\phi(x_n, u) - \phi(u_n, u)) = 0$, we have that $\lim_{n\to\infty} \phi(y_n, u_n) = 0$. From Lemma 2.6, we have that $\lim_{n\to\infty} ||y_n - u_n|| = 0$. Again, since $r_n \in [a, \infty)$, we have that

$$\lim_{n\to\infty} \frac{||y_n - u_n||}{r_n} = 0. \tag{3.5}$$

From $u_n = T_{r_n} y_n$, we have that

$$f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall\ y \in C.$$

By (A2), we have

$$\frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq -f(Ju_n, Jy) \geq f(Jy, Ju_n), \ \forall\ y \in C. \tag{3.6}$$

Since $f(x, \cdot)$ is convex and lower semicontinuous and $u_n \to x^*$, it follows from equation (3.5) and inequality (3.6) that

$$f(Jy, Jx^*) \leq 0, \ \forall\ y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t^* = tJy + (1-t)Jx^*$. Since, JC is convex, we have that $y_t^* \in JC$ and hence $f(y_t^*, Jx^*) \leq 0$. From (A1),

$$0 = f(y_t^*, y_t^*) \leq tf(y_t^*, Jy) + (1-t)f(y_t^*, Jx^*) \leq tf(y_t^*, Jy), \ \forall\ y \in C.$$

This implies that

$$f(y_t^*, Jy) \geq 0, \ \forall\ y \in C.$$

Letting $t \downarrow 0$, from (A3),

$$f(Jx^*, Jy) \geq 0, \ \forall\ y \in C.$$

Therefore, we have that $Jx^* \in JEP(f)$. This implies that $x^* \in EP(f)$.
Step 6. Finally, we show that $x_n \to R_{F_J(\Gamma)\cap EP(f)}x$.
From Lemma 2.4(ii), we obtain that

$$\phi(x, R_{F_J(\Gamma)\cap EP(f)}x) \leq \phi(x, R_{F_J(\Gamma)\cap EP(f)}x) + \phi(R_{F_J(\Gamma)\cap EP(f)}x, x^*) \leq \phi(x, x^*). \tag{3.7}$$

Again, using Lemma 2.4(ii), definition of x_{n+1}, and $x^* \in F_J(\Gamma) \cap EP(f) \subset C_n$, we compute as follows:

$$\begin{aligned} \phi(x, x_{n+1}) &\leq \phi(x, x_{n+1}) + \phi(x_{n+1}, R_{F_J(\Gamma)\cap EP(f)}x) \\ &= \phi(x, R_{C_{n+1}}x) + \phi(R_{C_{n+1}}x, R_{F_J(\Gamma)\cap EP(f)}x) \leq \phi(x, R_{F_J(\Gamma)\cap EP(f)}x). \end{aligned}$$

Since $x_n \to x^*$, taking limits on both sides of the last inequality, we obtain:

$$\phi(x, x^*) \leq \phi(x, R_{F_J(\Gamma)\cap EP(f)}x). \tag{3.8}$$

From inequalities (3.7) and (3.8), we obtain that $\phi(x, x^*) = \phi(x, R_{F_J(\Gamma)\cap EP(f)}x)$. By the uniqueness of $R_{F_J(\Gamma)\cap EP(f)}$, we obtain that $x^* = R_{F_J(\Gamma)\cap EP(f)}x$. This completes proof of the theorem.

4. Applications

Corollary 4.1 Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $T_n : C \to E^*, n = 1, 2, 3, ...$ be an infinite family of generalized J_*-nonexpansive maps and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$. Assume that $JF_J(\Gamma)$ is convex and $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT_n)x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}}x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(\Gamma)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(\Gamma)$.

Proof. Setting $f(Jx, Jy) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$, the result follows from Theorem 3.7.

Corollary 4.2 Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $T : C \to E^*$, be a generalized J_*-nonexpansive and J_*-closed map from C to E^* such that $F_J(T) \cap EP(f) \neq \emptyset$. Assume that $JF_J(T)$ is convex. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT)x_n), \\ u_n \in C, \ \textit{such that} \ f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall \ y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}}x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(T)\cap EP(f)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(T) \cap EP(f)$.

Proof. Set $T_n = T$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. The conclusion follows from Theorem 3.7.

We obtain the following corollary in classical Banach spaces.

Corollary 4.3 Let $E = L_p, \ l_p, or \ W_p^m(\Omega), 1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. Let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $T_n : C \to E^*, n = 1, 2, 3, ...$ be an infinite family of generalized J_*-nonexpansive maps and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$ and $F_J(\Gamma) \cap EP(f) \neq \emptyset$. Assume that $JF_J(\Gamma)$ is convex and $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT_n)x_n), \\ u_n \in C, \ \textit{such that} \ f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0, \ \forall \ y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}}x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(\Gamma)\cap EP(f)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(\Gamma) \cap EP(f)$.

Proof. E is uniformly smooth and uniformly convex. The result follows from Theorem 3.7.

Corollary 4.4 Let $E = L_p,\ l_p, or\ W_p^m(\Omega), 1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. Let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $T : C \to E^*$, be a generalized J_*-nonexpansive and J_*-closed maps from C to E^* such that $F_J(T) \cap EP(f) \neq \emptyset$. Assume that $JF_J(T)$ is convex. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT)x_n), \\ u_n \in C,\ such\ that\ f(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - y_n, Jy - Ju_n\rangle \geq 0,\ \forall\ y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}}x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(T)\cap EP(f)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(T) \cap EP(f)$.

Proof. Set $T_n = T$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. The conclusion follows from Theorem 3.7.

Remark 4.5 (see e.g., Alber and Ryazantseva, [4]; p. 36) The analytical representations of duality maps are known in a number of Banach spaces. For instance, in the spaces l_p, $L_p(G)$ and $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$Jx = ||x||_{l_p}^{2-p} y \in l_q,\ y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$$

$$J^{-1}x = ||x||_{l_q}^{2-q} y \in l_p,\ y = \{|x_1|^{q-2}x_1, |x_2|^{q-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$$

$$Jx = ||x||_{L_p}^{2-p}|x(s)|^{p-2}x(s) \in L_q(G),\ s \in G,$$

$$J^{-1}x = ||x||_{L_q}^{2-q}|x(s)|^{q-2}x(s) \in L_p(G),\ s \in G,\ and\ ,$$

$$Jx = ||x||_{W_m^p}^{2-p} \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha(|D^\alpha x(s)|^{p-2}D^\alpha x(s)) \in W_{-m}^q(G), m > 0, s \in G.$$

Recall that, under our setting, $J_* = J^{-1}$.

Corollary 4.6 Let $E = H$, a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $T_n : C \to H, n = 1, 2, 3, ...$ be an infinite family of generalized nonexpansive maps and Γ be a family of closed and generalized nonexpansive maps from C to H such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$ and $F(\Gamma) \cap EP(f) \neq \emptyset$. Assume that $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ u_n \in C, \ \textit{such that } f(u_n, y) + \frac{1}{r_n}\langle u_n - y_n, y - u_n\rangle \geq 0, \ \forall \ y \in C, \\ C_{n+1} = \{z \in C_n : ||z - u_n|| \leq ||z - x_n||\}, \\ x_{n+1} = P_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $P_{F(\Gamma)\cap EP(f)} x$, where P is the metric projection of H onto $F(\Gamma) \cap EP(f)$.

Proof. In a Hilbert space, J is the identity operator and $\phi(x,y) = ||x-y||^2 \ \forall x, y \in H$. The result follows from Theorem 3.7.

Corollary 4.7 Let $E = H$, a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $T : C \to H$, be a generalized nonexpansive and closed maps from C to H such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be generated by:

$$\begin{cases} x_1 = x \in C; C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ u_n \in C, \ \textit{such that } f(u_n, y) + \frac{1}{r_n}\langle u_n - y_n, y - u_n\rangle \geq 0, \ \forall \ y \in C, \\ C_{n+1} = \{z \in C_n : ||z - u_n|| \leq ||z - x_n||\}, \\ x_{n+1} = P_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \ \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $P_{F(\Gamma)\cap EP(f)} x$, where P is the metric projection of H onto $F(\Gamma) \cap EP(f)$.

Proof. Set $T_n = T$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. The conclusion follows from Corollary 4.6.

Remark 4.8 Theorem 3.7 is a complementary analogue of Theorem 1.1 in the sense that, while in Theorem 1.1 the family $\{T_n\}$ maps from a subset $C \subset E$ to the space E, in Theorem 3.7 the family $\{T_n\}$ maps from a subset $C \subset E$ to the dual E^*. Furthermore, in Hilbert spaces, both theorems virtually agree and yield the same conclusion.

Remark 4.9 Corollary 4.1 is an improvement and extension of the results of Nakajo and Takahashi [37], Qin and Su [42] in the following sense:

Corollary 4.1 extends the results in Nakajo and Takahashi [37] and, Qin and Su [42] from *a nonexpansive self-map* to *a generalized nonexpansive non-self map.*

Remark 4.10 It is well known that in a Hilbert space H, if $T : H \to H$ is any generalized nonexpansive mapping. Then, T is generalized J_*-nonexpansive. We now give some examples of generalized J_*-nonexpansive maps in more general Banach spaces.

Example 4.11 [The Lebesgue space] Let $E = L_p(G)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and G a measurable set in $\mathbb{R}^n$. Let $T : E \to E^*$ be a map defined by $Tx =$

$||x||_{L_p}^{2-p}|x(s)|^{p-2}x(s) \in L_q(G),\ s \in G$. Then, it is easy to see that, for each $x \in E$, $x \in F_J(T)$ and $\phi(p,(J_*oT)x) = \phi(p,x)$. Hence, T is generalized J_*-nonexpansive.

Example 4.12 [l_p space] Let $E = l_p$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $T : E \to E^*$ be defined by $Tx = ||x||_{l_p}^{2-p}y \in l_q,\ y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\ x = \{x_1, x_2, ...\}$. Then, clearly, for each $x \in E$, $x \in F_J(T)$ and $\phi(p,(J_*oT)x) = \phi(p,x)$. Hence, T is generalized J_*-nonexpansive.

Example 4.13 [l_p space] Let $E = l_p$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $T : E \to E^*$ be defined by $Tx = \frac{1}{2}||x||_{l_p}^{2-p}y \in l_q,\ y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\ x = \{x_1, x_2, ...\}$. Then, it follows that $F_J(T) = \{0\}$ and for $p \in F_J(T)$, $\phi(p,(J_*oT)x) < \phi(p,x)$. Hence, T is generalized J_*-nonexpansive.

Example 4.14 Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Then, the map $T : E \to E^*$ defined by $Tx = J\Pi_C x$, where Π_C is as given in Remark 2.1 is generalized J_*-nonexpansive.

Proof. This follows easily from Lemma 2.11.

Acknowledgments. The authors are thankful to the anonymous referees for their valuable and esteemed suggestions which have helped to improve the final version of the manuscript.

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Received: October 18, 2018; Accepted: December 16, 2019.