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INTERSECTION OF NONEXPANSIVE MAPPINGS WITH RESPECT TO A FINITE NUMBER OF RENORMINGS

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Abstract. Given a Banach space $(X, \|\cdot\|)$ and a subset C of X, we consider the family of bounded Lipschitzian mappings $BLip(C, X)$. This family is endowed with a norm and a topology that does not depend on renormings. With this topology we prove that it is not enough to consider the family of mappings that are nonexpansive with respect to finitely many renormings, to get the family of mappings that are nonexpansive w.r.t. all renormings.
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1. Introduction

One of the main problems in Fixed Point Theory is the relation between the Fixed Point Property (FPP) and reflexivity, the question if FPP implies reflexivity was open until 2008, when P. K. Lin [14] proved that there exists a non reflexive Banach space with the FPP. For this, Lin used a renorming $\|\cdot\|_L$ of the space ℓ_1 such that $(\ell_1, \|\cdot\|_L)$ has the FPP.

After that, many works have appeared on the relation between renormings and the Fixed Point Property. T. Domínguez-Benavides [4] in 2009 proved that every reflexive Banach space can be renormed to enjoy the FPP, and there is a great diversity of papers that provide examples of non reflexive Banach spaces through the renorming technique as [3], [9], [10], [11] and [15]. Nevertheless, the structure of the set of nonexpansive mappings and norms with or without the FPP has not been widely studied, and there are only a few contributions in that direction, such as [5], [6], [7], and [8] in which a generic approach was used to study the family of norms, [12] and [13] in which rays of norms without the FPP was studied, and [1] and [2] in which invariant families of nonexpansive mappings was studied.

What really happens is that under renorming, the collection of nonexpansive mappings changes. For example, the right shift operator R defined over the positive face F of the unit ball of $(\ell_1, \|\cdot\|_1)$ to itself, does not have any fixed point. This implies that under the norm $\|\cdot\|_L$ the operator R is not nonexpansive, because with respect to $\|\cdot\|_L$ its domain is still convex closed and bounded, and it is fixed point free.

The structure of nonexpansive mappings that are invariant under renormings was studied in [2] from the geometric point of view. It was proved that for two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$, and a convex set C with at least two points, there is a nonexpansive mapping $T: C \to C$ which is $\|\cdot\|_1$ nonexpansive, but is not $\|\cdot\|_2$ nonexpansive, and vice versa. This is a type of separation result for nonexpansive mappings with respect to two equivalent norms. Another result proved in [1] provides a characterization of the intersection of nonexpansive mappings $T: C \to C$ over all equivalent norms, the so called $S'(C)$, which corresponds to a subfamily of certain affine mappings.

In this work we prove that in order to get $S'(C)$ it is not enough to consider a finite set of renormings of the space X: in the intersection of families of nonexpansive mappings with respect to a finite number of renormings, we always have mappings which do not belong to $S'(C)$. This result is proved by giving a topology on the set of Lipschitzian mappings, the so called Lip topology, and by using Baire's Category Theorem.

2. The Banach space $BLip(C, X)$

Let (X, d) and (C, ϕ) be two metric spaces. It is well known that a function $T : (C, \phi) \to (X, d)$ is said to be Lipschitz if there exists $0 \leq r < \infty$ such that

$$d(Tx, Ty) \leq r\phi(x, y)$$

for each $x, y \in Y$. So we define:

$$BLip(C, X) = \{T : (C, \phi) \to (X, d) \,|\, T \text{ is Lipschitz and } \sup_{x,y\in C} d(Tx, Ty) < \infty\}$$

as the family of bounded Lipschitzian mappings.

If (X, d) is a normed space $(X, \|\cdot\|)$ and C is a subset of X, then for each $T \in BLip(C, X)$ we denote by $K(T, \|\cdot\|)$ its Lipschitz constant with respect to $\|\cdot\|$, which is defined by

$$K(T, \|\cdot\|) = \sup\left\{ \frac{\|Tx - Ty\|}{\|x - y\|} \,\middle|\, x, y \in C, x \neq y \right\}$$

and we define $\|T\|_\infty = \sup_{x\in C} \|Tx\|$, hence

$$BLip(C, X) = \{T : C \to X \,|\, K(T, \|\cdot\|) < \infty, \|T\|_\infty < \infty\}.$$

Let $\mathcal{C}(C, X)$ be the collection of all continuous functions from C to X. If X is a Banach space, then it is a known result that $\mathcal{C}(C, X)$ is a Banach space with the norm $\|\cdot\|_\infty$, and although $BLip(C, X)$ is a vector subspace of $\mathcal{C}(C, X)$, it is not necessarily a Banach subspace.

There exist a wide variety of norms that endow $BLip(C, X)$ with a Banach space structure, the ones used most frequently are max-norm and 1-norm, defined by

$$\|T\| = \max\{\|T\|_\infty, K(T, \|\cdot\|)\}$$

and

$$\|T\| = \|T\| + K(T, \|\cdot\|)$$

for each $T \in BLip(C, X)$. In this paper we use the 1-norm, however all results hold for any equivalent norm due to their topological nature. The proof of the next lemma as well as a detailed exposition of Lipschitz algebras can be found in [16].

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$, $C \neq \emptyset$. Then $(BLip(C, X), \|\cdot\|)$ is a Banach space with $\|T\| := \|T\|_\infty + K(T, \|\cdot\|)$ for each $T \in BLip(C, X)$.*

Remark 2.2. Assuming that $T_n \to T$ with respect to $\|\cdot\|$, we obtain that $\|T_n - T\| \to 0$, so $\|T_n - T\|_\infty \to 0$, therefore $\|T_n\|_\infty \to \|T\|_\infty$. Applying the continuity of the norm $\|\cdot\|$ we deduce that $K(T, \|\cdot\|) = \lim_n K(T_n, \|\cdot\|)$.

Remark 2.3. Note that Lemma 2.1 has weaker forms as follows:

1) If C and X are arbitrary metric spaces such that X is complete and C is not necessary a subset of X, then $BLip(C, X)$ is a complete metric space for the analogously metric induced in that case.
2) If X is a Banach space and C is a metric space, which is not necessary a subset of X, then $BLip(C, X)$ is a Banach space.

The proof of previous observations are similar as the proof of Lemma 2.1.

Remark 2.4. In some sense, the Banach space $(BLip(C, X), \|\cdot\|)$ is a generalization of some Sobolev spaces, for example:

$$(\mathcal{W}^{1,\infty}(0,1), \|\cdot\|_{1,\infty}) = (BLip((0,1), \mathbb{R}), |\cdot|).$$

In this direction, let us consider C open and convex subset of X. Then the family of Fréchet differentiable functions $T : C \to X$ with bounded image and derivative is, by the generalized mean value Theorem, a subspace of $BLip(C, X)$. Another example is given by replacing Fréchet by Gâteaux in the previous example.

Let $(X, \|\cdot\|)$ be a normed space and $C \subset X$ with $C \neq \emptyset$. Then we define $\mathfrak{N}(X)$ as the collection of equivalent norms over X. For each norm $\|\cdot\|_1 \in \mathfrak{N}(X)$ we denote by X_1 the normed space $(X, \|\cdot\|_1)$ and by $\|\cdot\|_{1,\infty}$ the infinity norm in $BLip(C, X)$ associated to $\|\cdot\|_1$. In this context, the normed space $BLip(C, X_1)$ is considered endowed with the norm $\|T\|_1 = \|T\|_{1,\infty} + K(T, \|\cdot\|_1)$. With this notation the following lemma shows a relation between the topologies in $BLip(C, X_1)$ and $BLip(C, X_2)$.

Lemma 2.5. *Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be Banach spaces and C a nonempty subset of X. Then the following are equivalent:*

1) *$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.*
2) *The sets $BLip(C, X_1)$ and $BLip(C, X_2)$ are equal, and the Banach spaces $BLip(C, X_1)$ and $BLip(C, X_2)$ are isomorphic.*

Proof. We show that 1) implies 2). Let $\|\cdot\|_1, \|\cdot\|_2 \in \mathfrak{N}(X)$. Then for $T : C \to X$ we have:

$$\sup_{x \in C} \|Tx\|_1 < \infty \text{ if and only if } \sup_{x \in C} \|Tx\|_2 < \infty$$

and

$$T \text{ is } \|\cdot\|_1\text{-Lipschitz if and only if } T \text{ is } \|\cdot\|_2\text{-Lipschitz.}$$

Thus $BLip(C, X_1)$ and $BLip(C, X_2)$ are equal.

Let l and u be constants such that

$$l\|x\|_2 \leq \|x\|_1 \leq u\|x\|_2$$

for all $x \in X$.

We consider the identity operator $I : (BLip(C, X_1), \|\cdot\|_1) \to (BLip(C, X_2), \|\cdot\|_2)$. Note that taking $m = \max\{l^{-1}, \frac{u}{l}\}$ we have

$$\begin{aligned} \|I(T)\|_2 &= \|T\|_{2,\infty} + K(T, \|\cdot\|_2) \\ &\leq l^{-1}\|T\|_{1,\infty} + \frac{u}{l}K(T, \|\cdot\|_1) \\ &\leq m(\|T\|_{1,\infty} + K(T, \|\cdot\|_1)) \\ &= m\|T\|_1. \end{aligned}$$

Then I is a bounded operator. Therefore $BLip(C, X_1)$ and $BLip(C, X_2)$ are isomorphic.

Now we prove the reciprocal. We assume that $BLip(C, X_1)$ equals $BLip(C, X_2)$ and that they are isomorphic Banach spaces, that is, the identity function $I : (BLip(C, X_1), \|\cdot\|_1) \to (BLip(C, X_2), \|\cdot\|_2)$ is a bounded operator.

For each $x \in X$ we define $f_x : C \to X$, as $f_x(y) = x$ for all $y \in C$. It is clear that

$$K(f_x, \|\cdot\|_1) = K(f_x, \|\cdot\|_2) = 0 \text{ for every } x \in X.$$

Then

$$\|f_x\|_1 = \|f_x\|_{1,\infty} = \|x\|_1$$

and

$$\|f_x\|_2 = \|f_x\|_{2,\infty} = \|x\|_2$$

for each $x \in X$. So,

$$\frac{1}{\|I^{-1}\|}\|f_x\|_1 \leq \|f_x\|_2 \leq \|I\|\|f_x\|_1,$$

thus

$$\frac{1}{\|I^{-1}\|}\|x\|_1 \leq \|x\|_2 \leq \|I\|\|x\|_1 \quad \text{for all } x \in X,$$

which concludes the proof. □

Remark 2.6. In a natural way, the previous theorem has a weaker form in which the Banach space condition is changed by normed space. To do this, it is enough consider a normed space isomorphism as a linear bounded operator with bounded inverse.

Remark 2.7. Regardless the norm in the normed space X, the topological vector space $BLip(C, X)$ is unique under equivalent norms. Then the closedness of a set and the limit of a sequence are defined without ambiguity, on the understanding that all norms considered are equivalent.

Given a normed space $(X, \|\cdot\|)$ and nonempty $C \subset X$. We define τ_{Lip} as the topology in $BLip(C, X)$ induced by $\|\cdot\|$, and τ_∞ the topology in $BLip(C, X)$ as a subspace of $\mathcal{C}(C, X)$.

Remark 2.8. By Remark 2.2 it follows that $\tau_\infty \subset \tau_{Lip}$. Thus every τ_∞-closed set is τ_{Lip}-closed.

3. Families of nonexpansive mappings in $BLip(C, X)$

The aim of this section is to study some properties inherited by $BLip(C, X)$ to some families of nonexpansive mappings as subsets of $BLip(C, X)$.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed space, and C a nonempty and closed subset of X. Then*

1) *If C is bounded, then the family of isometries $T : C \to X$ is τ_{Lip}-closed.*
2) *If C is convex, then the family of affine Lipschitz mappings $T : C \to X$ with bounded image is convex and τ_{Lip}-closed.*

In particular

3) *If C is a convex subset of X, then the family of functions $T : C \to X$ such that there exists a bounded linear operator $S : \overline{span(C)} \to X$ with $S|_C = T$ is convex and τ_{Lip}-closed.*

If we consider the case $T : C \to C$, then the families in 1), 2) and 3) are closed under composition.

Proof. In all of the three cases, the only non trivial fact is the τ_{Lip}-closedness, and this follows from Remark 2.8 and the τ_∞-closedness. □

Let $(X, \|\cdot\|)$ be a normed space and C a nonempty subset of X. Then for each norm $\|\cdot\|$ over X we define

$$NE(C, \|\cdot\|) = \{T : C \to C \mid K(T, \|\cdot\|) \leq 1\},$$

and we call

$$BLip(C) = BLip(C, C).$$

It is clear that for each $\|\cdot\| \in \mathfrak{N}(X)$ we have that $NE(C, \|\cdot\|) \subset BLip(C)$.

Lemma 3.2. *If C is a nonempty, convex, and closed subset of a normed space X, then for each $\|\cdot\| \in \mathfrak{N}(X)$ both $NE(C, \|\cdot\|)$ and $BLip(C)$ are convex, closed under compositions, and τ_{Lip}-closed in $BLip(C, X)$.*

Additionally, if C is bounded and $\|\cdot\|_1 \in \mathfrak{N}(X)$ is a fixed norm, then $NE(C, \|\cdot\|)$ is a bounded subset of $(BLip(C, X), \|\cdot\|_1)$.

Proof. The convexity and closure under compositions of $NE(C, \|\cdot\|)$ and $BLip(C)$ follow directly from the convexity of C.

The proof of the τ_{Lip}-closedness of $NE(C, \|\cdot\|)$ and $BLip(C)$ in $BLip(C, X)$ follows from Remark 2.2 and the closedness of C.

If additionally C is bounded in X, then given $\|\cdot\|_1 \in \mathfrak{N}(X)$, there exists a constant $M \geq 0$ such that

$$\sup_{x \in C} \|x\|_1 = M,$$

and there are constants $0 < l \leq u$ with $l\|x\|_1 \leq \|x\| \leq u\|x\|_1$ for each $x \in X$. Hence for each $T \in NE(C, \|\cdot\|)$ we have

$$\begin{aligned}\|T\|_1 &= \|T\|_{1,\infty} + K(T, \|\cdot\|_1) \\ &\leq M + \frac{u}{l} K(T, \|\cdot\|) \\ &\leq M + \frac{u}{l}.\end{aligned}$$

Thus $NE(C, \|\cdot\|)$ is bounded in $(BLip(C, X), \|\cdot\|_1)$. □

We also would like to say something about the question: what happens with the family of nonexpansive mappings with another equivalent norm?

In this sense, the first approximation is to characterize the set:

$$\mathcal{S}'(C) = \bigcap_{\|\cdot\| \in \mathfrak{N}(X)} NE(C, \|\cdot\|),$$

when C is a convex, closed, and bounded set in X.

Let X be a normed space and C a nonempty convex subset of X. Then for each $x \in C$ we define $f_x : C \to C$ by $f_x(y) = x$, and we denote by $I : C \to C$ the identity map. Thus we define

$$\mathcal{S}(C) = conv(\{f_x \mid x \in C\} \cup \{I\}).$$

Remark 3.3. From Lemma 3.2, it follows that $\mathcal{S}'(C)$ is convex, closed under composition, and τ_{Lip}-closed.

It is straightforward that the set $\mathcal{S}(C)$ is convex, closed under compositions, and each of its elements is a nonexpansive mapping with respect to any norm. Therefore $\mathcal{S}(C) \subset S'(C)$, thus by Remark 3.3, we have that $\overline{\mathcal{S}(C)}^{\tau_{Lip}} \subset \mathcal{S}'(C)$.

In order to describe the set $\mathcal{S}(C)$, we will give the following technical lemma.

Lemma 3.4. *Let X be a normed space and C a nonempty convex subset of X. Then $\overline{\mathcal{S}(C)}^{\tau_{Lip}} = \mathcal{L}$, where $\mathcal{L}$ is the subset of $BLip(C, X)$ with elements T of the form:*

A) *$T = x + \lambda I$ for some $0 \leq \lambda \leq 1$ and x a limit point of elements $\sum_{k=1}^{m_n} \lambda_{n,k} x_{n,k}$ with $x_{n,k} \in C$, $\sum_{k=1}^{m_n} \lambda_{n,k} \leq 1$, $0 \leq \lambda_{n,k} \leq 1$ and $\lim_n \sum_{k=1}^{m_n} \lambda_{n,k} = 1 - \lambda$.*

Proof. First we prove that $\overline{\mathcal{S}(C)}^{\tau_{Lip}} \subset \mathcal{L}$. Let $\|\cdot\| \in \mathfrak{N}(X)$ and T a limit point of $\mathcal{S}(C)$. Then there exists a sequence (S_n) in $\mathcal{S}(C)$ such that (S_n) $\|\cdot\|$-converges to T. Thus for each $n \in \mathbb{N}$, we have that

$$S_n = \sum_{k=1}^{m_n} \lambda_{k,n} x_{k,n} + \lambda_{0,n} I,$$

for some $x_{k,n} \in C$, and $\lambda_{k,n} \geq 0$ with $\sum_{k=0}^{m_n} \lambda_{k,n} = 1$. Hence we define

$$x_n = \sum_{k=1}^{m_n} \lambda_{k,n} x_{k,n}$$

and

$$\lambda_n = \lambda_{0,n}.$$

Then $S_n = x_n + \lambda_n I$, and $\sum_{k=1}^{m_n} \lambda_{k,n} = 1 - \lambda_n$.

For each $n \in \mathbb{N}$, we have that $0 \leq \lambda_n \leq 1$. Then without loss of generality we may assume, passing by a subsequence if necessary, that (λ_n) converges to some $\lambda \in [0,1]$, therefore $\left(\sum_{k=1}^{m_n} \lambda_{k,n}\right)$ converges to $1 - \lambda$.

We fix $y \in C$. By Remark 2.2, the $\|\cdot\|$-convergence implies the $\|\cdot\|_\infty$-converge. Then

$$\begin{aligned} Ty &= \lim_n (x_n + \lambda_n I y) \\ &= \lim_n (x_n + \lambda_n y) \\ &= \lim_n (x_n + \lambda y), \end{aligned}$$

hence $\lim_n x_n$ exists. Thus we call $x = \lim_n x_n$.

We note that x does not depend on y, then we conclude that $T = x + \lambda I$, that is, $T \in \mathcal{L}$. Therefore $\overline{\mathcal{S}(C)}^{\tau_{Lip}} \subset \mathcal{L}$.

Now we prove that $\mathcal{L} \subset \overline{\mathcal{S}(C)}^{\tau_{Lip}}$, for this we take $T \in \mathcal{L}$, given that T satisfies A), then $T = x + \lambda I$ and $x = \lim_n \sum_{k=1}^{m_n} \lambda_{n,k} x_{n,k}$.

For each $n \in \mathbb{N}$ we define $\lambda_n = 1 - \sum_{k=1}^{m_n} \lambda_{n,k}$ and $S_n = \sum_{k=1}^{m_n} \lambda_{n,k} x_{n,k} + \lambda_n I$, we note that $S_n \in \mathcal{S}(C)$ for each $n \in \mathbb{N}$, thus $T \in \overline{\mathcal{S}(C)}^{\tau_{Lip}}$ and $\overline{\mathcal{S}(C)}^{\tau_{Lip}} = \mathcal{L}$. □

Theorem 3.5. *Let X be a normed space and C a non empty closed convex subset of X. Then $\mathcal{S}(C)$ is convex closed under compositions and τ_{Lip}-closed.*

Proof. By Remark 3.3, it is sufficient to prove that $\mathcal{S}(C)$ is τ_{Lip}-closed.
Let $T \in \overline{\mathcal{S}(C)}^{\tau_{Lip}}$. Then by Lemma 3.4, we have that $T = x + \lambda I$, then it is enough to prove that there exists $y \in C$ such that $x = (1-\lambda)y$.
We have that $x = \lim_n \sum_{k=1}^{m_n} \lambda_{n,k} x_{n,k}$, then for each $n \in \mathbb{N}$ we call $\delta_n = \sum_{k=1}^{m_n} \lambda_{n,k}$. We note that

$$\lim_n \sum_{k=1}^{m_n} \lambda_{n,k} = 1 - \lambda$$

and for each $n \in \mathbb{N}$

$$\sum_{k=1}^{m_n} \frac{\lambda_{n,k}}{\delta_n} = 1.$$

Therefore for each $n \in \mathbb{N}$, we have that $\sum_{k=1}^{m_n} \frac{\lambda_{n,k}}{\delta_n} x_{n,k} \in C$.

Hence by the closedness of C

$$\frac{x}{1-\lambda} = \lim_n \sum_{k=1}^{m_n} \frac{\lambda_{n,k}}{\delta_n} x_{n,k} \in C,$$

Thus

$$T = x + \lambda I = (1-\lambda)\frac{x}{1-\lambda} + \lambda I \in \mathcal{S}(C).$$

□

Given $T : C \to C$, as usual, $Fix(T)$ will denote the set of its fixed points. Then we define

$$BLip^f(C,X) = \{T \in BLip(C,X) : Fix(T) \neq \emptyset\}$$

Thus in a natural way we define $NE^f(C, \|\cdot\|) = NE(C, \|\cdot\|) \cap BLip^f(C,X)$, and $BLip^f(C) = BLip(C) \cap BLip^f(C,X)$.

Theorem 3.6. *Let $(X, \|\cdot\|)$ be a normed space and C a complete convex subset of X. Then $NE^f(C, \|\cdot\|)$ is τ_{Lip}-dense in $NE(C, \|\cdot\|)$.*

Proof. Let $\varepsilon > 0$, $T \in NE(C, \|\cdot\|)$, and $y \in C$. If $f_y : C \to C$ is the constant function with value y, then for each $0 < \lambda < 1$ we define

$$S_\lambda = \lambda T + (1-\lambda) f_y.$$

By the convexity of C we have that $S_\lambda : C \to C$. It is not hard to prove that $K(S_\lambda, \|\cdot\|) \leq \lambda$. Thus for each $0 < \lambda < 1$ by the Banach's Fixed Point Theorem we have that $Fix(S_\lambda) \neq \emptyset$. Hence for each $0 < \lambda < 1$ we have

$$\begin{aligned} \|T - S_\lambda\| &= \|T - \lambda T - (1-\lambda) f_y\| \\ &= (1-\lambda)\|T - f_y\| \end{aligned}$$

So, if $\lambda \to 1$, then $\|T - S_\lambda\| \to 0$, Hence $NE^f(C, \|\cdot\|)$ is τ_{Lip}-dense in $NE(C, \|\cdot\|)$. □

Given a normed space $(X, \|\cdot\|)$, $C \subset X$ with $C \neq \emptyset$, and $T : C \to C$. Then an approximate fixed point sequence (a.f.p.s.) for T is a sequence (x_n) in C such that

$$\lim_{n \in \mathbb{N}} \|Tx_n - x_n\| = 0.$$

We define

$$BLip^{af}(C,X) = \{T \in BLip(C,X) \,|\, T \text{ has an a.f.p.s.}\},$$
$$NE^{af}(C, \|\cdot\|) = NE(C, \|\cdot\|) \cap BLip^{af}(C,X),$$

and

$$BLip^{af}(C) = BLip(C) \cap BLip^{af}(C,X).$$

Theorem 3.7. *Let $(X, \|\cdot\|)$ be a normed space and C a nonemepty subset of X. Then $BLip^f(C,X)$ is τ_{Lip}-dense in $BLip^{af}(C,X)$, i.e.*

$$\overline{BLip^f(C,X)}^{\tau_{Lip}} = BLip^{af}(C,X).$$

Proof. To show that $\overline{BLip^f(C,X)}^{\tau_{Lip}} \subset BLip^{af}(C,X)$, let (T_n) be a sequence in $BLip^f(C,X)$ that converges to some $T \in BLip(C,X)$. Without loss of generality we may assume that $\|T - T_n\| < \frac{1}{n}$ for each $n \in \mathbb{N}$. Then there exists a sequence (x_n) in C such that for each $n \in \mathbb{N}$ we have $T_n x_n = x_n$. We claim that $\lim_{n\in\mathbb{N}} \|Tx_n - x_n\| = 0$, in fact,

$$\begin{aligned}\|Tx_n - x_n\| &= \|Tx_n - T_n x_n\| \\ &\le \|T - T_n\|_\infty \\ &\le \|T - T_n\| < \frac{1}{n},\end{aligned}$$

Thus $T \in BLip^{af}(C,X)$.

Now we prove the reverse inclusion. Let $T \in BLip^{af}(C,X)$ and $\varepsilon > 0$. Then there exists $x \in C$ such that $\|Tx - x\| < \varepsilon$. We call $w = x - Tx$ and $f_w : C \to X$ the function $f_w(y) = w$. If $S = T + f_w$, then

$$\begin{aligned}\|T - S\| &= \|f_w\| \\ &= \|w\| + K(f_w, \|\cdot\|) \\ &< \varepsilon,\end{aligned}$$

and $Sx = Tx + f_w(x) = Tx + x - Tx = x$. Hence $S \in BLip^f(C,X)$. □

Corollary 3.8. *Let $(X, \|\cdot\|)$ be a normed space and C a nonempty closed subset of X. Then $BLip^{af}(C)$ is τ_{Lip}-closed.*

In the following theorem we study the interior of the family of nonexpansive mappings with respect to the interior of its domain of definition.

Theorem 3.9. *Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$ and consider the following statements:*

1) *C has empty interior in $(X, \|\cdot\|)$.*
2) *$NE(C, \|\cdot\|)$ has empty τ_{Lip}-interior in $BLip(C,X)$*
3) *$BLip(C)$ has empty τ_{Lip}-interior in $BLip(C,X)$.*
4) *$NE^f(C, \|\cdot\|)$ has empty τ_{Lip}-interior in $BLip(C,X)$.*
5) *$NE^f(C, \|\cdot\|)$ has empty τ_{Lip}-interior in $BLip^f(C,X)$.*
6) *$BLip^f(C)$ has empty τ_{Lip}-interior in $BLip(C,X)$*
7) *$BLip^f(C)$ has empty τ_{Lip}-interior in $BLip^f(C,X)$.*
8) *For each $\varepsilon > 0$ and $T \in BLip^{af}(C)$, there exists $S \in BLip^f(C,X) \setminus BLip^f(C)$ such that $\|T - S\| < \varepsilon$.*
9) *For each $\varepsilon > 0$ and $T \in NE(C, \|\cdot\|)$, there exists $S \in BLip^f(C,X) \setminus NE^f(C, \|\cdot\|)$ such that $\|T - S\| < \varepsilon$.*

Then we have:

a) *1) to 3) are equivalent.*

b) *If C is complete and it has at least two points, then* 1) *to* 8) *are equivalent.*
c) *If C is a complete convex set with at least two points, then* 1) *to* 9) *are equivalent.*

Proof. Proof of Statement a).

First we prove that 2) implies 1) by contraposition. Let $x \in C$ be an interior point. Then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset C$. We define $f_x : C \to C$ by $f_x(y) = x$, and $\delta = \min\{\varepsilon, 1\}$. Thus for each $T \in B(f_x, \delta)$ we have

$$\|T - f_x\| = \|T - f_x\|_\infty + K(T - f_x, \|\cdot\|) < \delta.$$

We note that

i) $T : C \to B(x, \varepsilon) \subset C$, since $\|T - f_x\|_\infty < \varepsilon$, and
ii) $K(T, \|\cdot\|) \leq K(T - f_x, \|\cdot\|) < 1$, since for each $y, z \in C$

$$\begin{aligned}\|Ty - Tz\| &= \|Ty - x - Tz + x\| \\ &= \|(T - f_x)y - (T - f_x)z\| \\ &\leq K(T - f_x, \|\cdot\|)\|x - y\|,\end{aligned}$$

From i) and ii) it follows that $T \in NE(C, \|\cdot\|)$. Then $NE(C, \|\cdot\|)$ has nonempty τ_{Lip}-interior.

Now we prove that 1) implies 2). Let $\varepsilon > 0$, $T \in NE(C, \|\cdot\|)$, and $x \in C$. Then there exists $w \in X \setminus C$ such that $0 < \|w - Tx\| < \varepsilon$. We call $w' = w - Tx$ and define the function $f_{w'} : C \to X$ by $f_{w'}(y) = w'$. It is clear that $K(f_{w'}, \|\cdot\|) = 0$. Thus $S = T + f_{w'}$ implies

$$\begin{aligned}\|T - S\| &= \|f_{w'}\| \\ &= \|w'\| + K(f_{w'}, \|\cdot\|) \\ &= \|w - Tx\| < \varepsilon.\end{aligned}$$

We note that

$$\begin{aligned}Sx &= Tx + f_{w'}x \\ &= Tx + (w - Tx) \\ &= w \notin C\end{aligned}$$

Then $S \notin NE(C, \|\cdot\|)$. Hence $NE(C, \|\cdot\|)$ has empty interior in $BLip(C, X)$.

It is clear that 3) implies 2), and 1) implies 3) can be proved by a similar way as 1) implies 2).

Proof of statement b):

Now we suppose that C is a complete metric space with at least two elements. We are going to prove the rest of biconditionals.

By a contrapositive argument it follows that 5) implies 4), and 7) implies 6). In order to prove 4) implies 1) we proceed by a similar argument of 2) implies 1). Let $\varepsilon > 0$ and $f_x \in NE^f(C, \|\cdot\|)$ as in the proof of 2) implies 1). Then from ii) it follows that for each $T \in B(f_x, \varepsilon)$ we have that $K(T, \|\cdot\|) < 1$. Then by Banach's Fixed Point Theorem $Fix(T) \neq \emptyset$, we have that $T \in NE^f(C, \|\cdot\|)$ for each $T \in B(f_x, \varepsilon)$. Hence $NE^f(C, \|\cdot\|)$ has non empty interior in $BLip(C, \|\cdot\|)$.

Now we prove 1) implies 5). Let $\varepsilon > 0$ and $T \in NE^f(C, \|\cdot\|)$. Then there exists $x \in C$ such that $Tx = x$. By hypothesis there exists $y \in C$ whit $y \neq x$, then there exists $w \in X \setminus C$ such that $\|w - Tx\| < \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\|x - y\|\}$. We call $w' = w - Tx$, thus, by a retract technique, there exist $R : C \to [0, w']$ with $Rx = 0$, $Ry = w'$, and $K(R, \|\cdot\|) < \frac{\varepsilon}{2}$. We define $S = T + R$. Since

$$\begin{aligned} \|T - S\| &= \|R\| \\ &= \|w'\| + K(R, \|\cdot\|) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and $Sx = Tx + Rx = x + 0 = x$, then $T \in BLip^f(C, X)$. Thus $NE^f(C, \|\cdot\|)$ has empty interior in $BLip^f(C, X)$.

The proof of 6) implies 1) is similar as the proof of 4) implies 1), and the proof of 1) implies 7) is similar as the proof of 1) implies 5).

Now we will prove the equivalence between 1) and 8). First we prove 1) implies 8). We suppose that C has empty interior, and $T \in BLip^{af}(C) \subset BLip^{af}(C, X)$. By Theorem 3.7, there exists $S_1 \in BLip^f(C, X)$ such that

$$\|T - S_1\| < \frac{\varepsilon}{2}$$

Since, by 1) implies 7), we have that $BLip^f(C)$ has empty τ_{Lip}-interior in $BLip^f(C, X)$, then there exists $S_2 \in BLip^f(C, X) \setminus BLip^f(C)$ with

$$\|S_2 - S_1\| < \frac{\varepsilon}{2}$$

Hence $\|T - S_2\| < \varepsilon$.

Now we prove 8) implies 1), and we proceed by a contrapositive argument. Since C has nonempty interior, then, by 7) implies 1), we have that $BLip^f(C)$ has nonemtpy τ_{Lip}-interior in $BLip^f(C, X)$. Then there exist $\varepsilon > 0$ and $T \in BLip^f(C) \subset BLip^{af}(C)$ such that for each S with $\|T - S\| < \varepsilon$ we have that $S \in BLip^f(C)$.

Proof of statement c):

Now we suppose that C is a complete convex set and prove 1) implies 9). We note that by Theorem 3.6, $NE^f(C, \|\cdot\|)$ is dense in $NE(C, \|\cdot\|)$, and by 1) implies 5), we have that $NE^f(C, \|\cdot\|)$ has empty interior in $BLip^f(C, X)$. Let $T \in NE(C, \|\cdot\|)$ and $\varepsilon > 0$. Then there exists $S_1 \in NE^f(C, \|\cdot\|)$ such that

$$\|T - S_1\| < \frac{\varepsilon}{2},$$

and there exists $S_2 \in BLip^f(C, X) \setminus NE^f(C, \|\cdot\|)$ with

$$\|S_2 - S_1\| < \frac{\varepsilon}{2}$$

Hence $\|T - S_2\| < \varepsilon$.

In order to prove 9) implies 1), we proceed by a contrapositive argument. We suppose C has nonempty interior. Then by 5) implies 1), it follows that there exist $\varepsilon > 0$ and $T \in NE^f(C, \|\cdot\|) \subset NE(C, \|\cdot\|)$ such that for each $S \in BLip^f(C, X)$ with $\|T - S\| < \varepsilon$ we have that $S \in NE^f(C, \|\cdot\|)$. □

4. $BLip(C, X)$ in the framework of fixed point theory

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a normed space and C a nonempty subset of X. Then $NE(C, \|\cdot\|)$ has nonempty τ_{Lip}-interior in $BLip(C)$.*

Proof. Let $x \in C$ and $f_x \in NE(C, \|\cdot\|)$ the constant function with value x. Then for each $T \in B(f_x, 1) \cap BLip(C)$ we have that

$$K(T, \|\cdot\|) = K(f_x - T, \|\cdot\|) < 1,$$

that is, $NE(C, \|\cdot\|)$ has nonempty τ_{Lip}-interior in $BLip(C)$. □

Remark 4.2. From the proof of the previous theorem it follows that for each $\|\cdot\| \in \mathfrak{N}(X)$, the constant functions are τ_{Lip}-interior points of $NE(C, \|\cdot\|)$ in the topological space $BLip(C)$.

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a normed space and C be a closed subset of X with at least two elements. Then:*

1) *For each $r \geq 0$ the family $\{T \in BLip(C, X) \,|\, K(T, \|\cdot\|) = r\}$ is τ_{Lip}-closed with empty interior in $BLip(C, X)$.*

If additionally C is convex then

2) *For each $r \geq 0$ the family $\{T \in BLip(C) \,|\, K(T, \|\cdot\|) = r\}$ is τ_{Lip}-closed with empty interior in $BLip(C)$.*
3) *For each $0 \leq r \leq 1$ the family $\{T \in NE(C, \|\cdot\|) | K(T, \|\cdot\|) = r\}$ is τ_{Lip}-closed with empty interior in $NE(C, \|\cdot\|)$.*

If additionally C is bounded then

4) *The family $\{T \in NE(C, \|\cdot\|) \,|\, K(T, \|\cdot\|) = 1\}$ is a boundary set of $NE(C, \|\cdot\|)$ in $BLip(C, X)$.*

Proof. In all of three first cases the respective sets are closed by Remark 2.2 and Lemma 3.2, thus we only need to prove the emptiness of interiors.

First we prove 1). We suppose $r = 0$ and $T \in BLip(C, X)$ with $K(T, \|\cdot\|) = 0$. Then T is a constant function. Let $\varepsilon > 0$. By a retract technique, there exists $S \in BLip(C, X)$ such that $K(S, \|\cdot\|) > 0$, and $\|S\| = \|S\|_\infty + K(S, \|\cdot\|) < \epsilon$. We call $R = S + T$, then $K(R, \|\cdot\|) = K(S, \|\cdot\|)$ and $\|R - T\| = \|S\| < \varepsilon$.

We assume $r > 0$ and $T \in BLip(C, X)$ with $K(T, \|\cdot\|) = r$. Let $\varepsilon > 0$. By the continuity of the scalar product, we can find $0 < \delta < 1$ such that $\|T - \delta T\| < \varepsilon$. Hence $K(\delta T, \|\cdot\|) = \delta r < r$.

Now we prove 2). Let $T \in BLip(C)$ with $K(T, \|\cdot\|) = r$ and $\varepsilon > 0$. If $r = 0$, then by a retract argument, we can prove that there exists a non constant $S \in BLip(C)$. Then by the continuity of the scalar product, there exists $\lambda \in (0, 1)$ such that $R = \lambda T + (1 - \lambda)S$ satisfies $\|T - R\| < \varepsilon$. It follows that $K(R, \|\cdot\|) > 0$.

For $r > 0$. Let $y \in C$ and f_y the constant function y with domain C. Again by the continuity of the scalar product, there exists $\lambda \in (0, 1)$ such that $R = \lambda f_y + (1 - \lambda)T$ satisfies $\|T - R\| < \varepsilon$. It is not hard to prove that $K(R, \|\cdot\|) = \lambda r < r$.

The proof of 3) is analogous to the proof of 2).

Now we prove 4). Let $0 < \varepsilon < 1$ and $T \in NE(C, \|\cdot\|)$ with $K(T, \|\cdot\|) = 1$. Then there exist $x, y \in C$ such that

$$(1-\varepsilon)\|x-y\| < \|Tx - Ty\| \le \|x-y\|.$$

We define

$$w = \varepsilon\|x-y\|\frac{Tx-Ty}{\|Tx-Ty\|},$$

since it is collinear with $Tx - Ty$, we have

$$\begin{aligned}\|Tx - Ty + w\| &= \|Tx - Ty\| + \|w\| \\ &= \|Tx - Ty\| + \varepsilon\|x-y\| \\ &> \|x-y\|.\end{aligned}$$

By a retract technique, we may construct $S' : C \to [0, -w]$, such that for each $\lambda \in [0,1]$

$$S'(\lambda x + (1-\lambda)y) = -(1-\lambda)w$$

and $K(S', \|\cdot\|) = \varepsilon$. We define $S = T + S'$ and we note that

$$\begin{aligned}\|Sx - Sy\| &= \|Tx + S'x - Ty - S'y\| \\ &= \|Tx - Ty + w\| \\ &> \|x-y\|\end{aligned}$$

Thus $S \notin NE(C, \|\cdot\|)$, and

$$\begin{aligned}\|T - S\| &= \|S'\| \\ &= \|w\| + \varepsilon \\ &= \varepsilon\|x-y\| + \varepsilon \\ &\le \varepsilon(diam(C) + 1)\end{aligned}$$

On other hand. By the proof of 3), it follows that for each $\varepsilon > 0$ there exists $S \in NE(C, \|\cdot\|)$ such that $K(S, \|\cdot\|) < 1$ and $\|T - S\| < \varepsilon$. □

Corollary 4.4. *Let $(X, \|\cdot\|)$ be a Banach space and C a nonempty closed and convex subset of X. Then the family of $T \in NE(C, \|\cdot\|)$ such that T is an isometry or is a fixed point free mapping is a meager set in $NE(C, \|\cdot\|)$.*

Proof. If T is an isometry or is a fixed point free map, then $K(T, \|\cdot\|) = 1$. Thus by Theorem 4.3 we obtain the conclusion. □

Remark 4.5. In the previous corollary the Banach space condition cannot be avoided. In order to show this, let C be the unitary ball in c_{00} with respect to $\|\cdot\|_1$. If for each $(x_n) \in C$ we call

$$\#(x_n) = \min\{m \in \mathbb{N} \mid x_m \ne \frac{1}{2^m}\}$$

and for each $m \in \mathbb{N}$

$$P_m(x_n) = (x_1, \dots, x_m, 0, \dots),$$

with

$$P_0(x_n) = (0, 0, \dots),$$

then we define

$$T(x_n) = P_{\#(x_n)-1}(x_n) + \frac{1}{2}R(I - P_{\#(x_n)-1})(x_n) + \frac{1}{2^{\#(x_n)}}e_{\#(x_n)}$$

where R is the right shift operator and e_k is the k^{th}-basic element of c_{00}.

It can be shown that $T : C \to C$ and $K(T, \|\cdot\|_1) = \frac{1}{2}$. But T is a fixed point free mapping.

Remark 4.6. In general, given a Banach space X and a convex closed bounded $C \subset X$, the problem of finding a fixed point free nonexpansive operator defined from C to itself is a too hard problem. Thus the previous corollary in some way justifies this difficulty. If we study the literature, we note that in some essential way, the fixed point free nonexpansive mappings defined from a convex closed and bounded set to itself are like the right shift operator R in $(\ell_1, \|\cdot\|_1)$ where

$$R(x_n) = (0, x_1, x_2, \cdots)$$

and

$$\|(x_n)\| = \sum_{n=1}^{\infty}|x_n|.$$

Thus the previous corollary reinforces that feeling.

Theorem 4.7. *Let $(X, \|\cdot\|)$ be a normed space and C a nontrivial convex and closed subset of X. Then*

1) *The family $\{T \in BLip(C, X) \,|\, T \text{ is affine}\}$ is τ_{Lip}-closed with empty interior in $BLip(C, X)$.*
2) *The family $\{T \in BLip(C) \,|\, T \text{ is affine}\}$ is τ_{Lip}-closed with empty interior in $BLip(C)$.*
3) *The family $\{T \in NE(C, \|\cdot\|) \,|\, T \text{ is affine}\}$ is τ_{Lip}-closed with empty interior in $NE(C, \|\cdot\|)$.*

Proof. By Lemmas 3.1, and 3.2, in all three cases the families are closed. We only make the proof of 3), since all others are similar. Let $T \in NE(C, \|\cdot\|)$ be an affine mapping. Then by Corollary 19 of [2] there exists a non affine mapping $S \in NE(C, \|\cdot\|)$. Then for all $\lambda \in (0, 1)$ the operator $\lambda T + (1 - \lambda)S$ is non affine. Thus by the continuity of the scalar product, the proof is over. □

Remark 4.8. It is curious that when asking for examples of mappings defined on normed spaces, the natural ones are linear, affine, and in the remaining of the cases isometries. Nevertheless Corollary 4.4 and Theorem 4.7 tell us the opposite, that is, almost every one of the examples are non affine and non isometries, thus we can ask:

What are the operators that we can construct?,
What are the operators that we can describe?

If in some way we have an answer to some of the previous questions, then

In the set of constructible or describable operators,
How big is the family of affine mappings or isometries?

Finally we have the following

Theorem 4.9. *Let X be a Banach space and C a nonempty closed and convex subset of X such that for each $x, y \in C$ there exists $z \in C$ with $z \neq \lambda x + (1-\lambda)y$ for all $\lambda \in \mathbb{F}$. Then for $\|\cdot\|_1, \cdots, \|\cdot\|_n \in \mathfrak{N}(X)$ there exists*

$$T \in (NE(C, \|\cdot\|_1) \cap \cdots \cap NE(C, \|\cdot\|_n)) \setminus \mathcal{S}'(C).$$

Moreover, T can be chosen in a such way it is not affine.

Proof. By Theorem 4.1 and Remark 4.2, for all $k = 1, \cdots, n$, the sets $NE(C, \|\cdot\|_k)$ has, as interior points in $BLip(C)$, the constant functions. Thus

$$\mathcal{C} = \bigcap_{k=1}^{n} NE(C, \|\cdot\|_k)$$

has nonempty interior in $BLip(C)$. Since X is a Banach space, then by Lemmas 2.1 and 3.2, and Baire's Category Theorem follows that $\mathcal{C}$ is of second category in $BLip(C)$. Nevertheless by Theorem 4.7 the family $\mathcal{A}$ of affine mappings $T \in BLip(C)$ is of first category in $BLip(C)$. Then $\mathcal{C} \setminus \mathcal{A}$ is of second category in $BLip(C)$. Hence there exists $T \in \mathcal{C} \setminus \mathcal{A}$. By Theorem 5 and Corollary 8 of [2] follows that $\mathcal{S}'(C) \subset \mathcal{A}$ which concludes the proof. □

Last theorem states that, under these assumptions, it is not enough finite intersections in order to get the set $S'(C)$.

Remark 4.10. If C lies inside a one dimensional affine subspace of X, then by Remarks 22 and 24, and Corollary 23 of [2], it follows that the previous theorem is not true, since $NE(C, \|\cdot\|_1) = \cdots = NE(C, \|\cdot\|_n) = \mathcal{S}'(C)$.

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