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FIXED POINT OF MULTIVALUED CONTRACTIONS BY ALTERING DISTANCES WITH APPLICATION TO NONCONVEX HAMMERSTEIN TYPE INTEGRAL INCLUSIONS

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Abstract. A new contraction condition for multivalued maps in metric spaces is introduced and then, based on this new condition, we prove two fixed point theorems for such contractions. The new condition uses the altering distance technique and a Pompeiu type metric on the family of nonempty and closed subsets of a metric space. Our results essentially compliments and generalizes some well known results. As application, we model a nonconvex Hammerstein type integral inclusion and prove an existence theorem for this problem.

Key Words and Phrases: Fixed point, metric space, Hausdorff metric, multivalued contraction, Hammerstein type integral inclusion.
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1. Introduction and preliminaries

Nadler in his seminal paper [11] initiated the study of fixed point of multivalued maps and proved the existence of fixed point of closed bounded valued multivalued maps on a complete metric spaces. This theory of multivalued mappings has applications in convex optimization, control theory, economics and differential inclusions [7]. Reich [15] independently obtained fixed point results for the case of compact valued maps under general conditions. Reich [16] further asked the question whether his results are also true for the closed bounded valued maps The affirmative answer under some additional hypothesis was given by Mizoguchi and Takahashi [10]. Afterward Beg and Azam [1] and Feng and Liu [5] also obtained several extensions of Nadler's result in different directions. Petruşel [14] has discussed several operational inclusions in connection with fixed point of multivalued maps. Recently several researchers have generalized these results in different directions [2, 3, 6, 8, 9, 12]. In this article, we introduce a new contraction condition for multivalued maps in metric spaces, using the altering distance technique and a Pompeiu type metric on the family of nonempty and closed subsets of a metric space, and we prove two fixed point theorems in this framework. Our results essentially compliment and generalize the results of Nadler [11], Feng and Liu [5] and Klim and Wardowski [8], Kamran and Kiran [6] and is different from Reich [15]. As an application of our result we consider a nonconvex Hammerstein type integral inclusion and we prove an existence theorem for this problem.

Petruşel [14] has discussed several operational inclusions in connection with fixed point of multivalued maps.

Let $\tau \in (0, +\infty]$, and $\theta : [0, \tau) \to \mathbb{R}$ where $\mathbb{R}$ be the set of all real numbers. Let
(i) $\theta(t) > 0$ for each $t \in (0, \tau)$;
(ii) θ is nondecreasing on $[0, \tau)$;
(iii) $\theta(t_1 + t_2) \le \theta(t_1) + \theta(t_2)$ $\forall\, t_1, t_2 \in (0, \tau)$;
(iv) $\theta(t_1) < \theta(t_2) \Rightarrow t_1 < t_2,\ t_1, t_2 \in (0, \tau)$ strictly inverse isotone.
For the sake of clarity, one may consider $\theta : [0, \tau) \to \mathbb{R}$ defined by
(a) $\theta(t) = t^q$, $q \in (0, 1]$;
(b) $\theta(t) = \frac{t}{1+t}$;
(c) $\theta(t) = \tan^{-1} t\ \forall\, t \in [0, \tau)$.

By the mean value theorem, we note that $\tan^{-1} x - \tan^{-1} y < x - y$ in $[y, x]$. Clearly, each θ (in (a)-(c)) satisfies all the conditions (i)-(iv). As in [17], we define

$$\Theta[0, \tau) = \{\theta |\, \theta \text{ satisfies } (i) - (iv)\}.$$

The function $\Theta[0, \tau)$ is called *positive homogeneous* in $[0, \tau)$ if
(v) $\theta(at) \le a\theta(t)$ for all $a > 0, t \in [0, \tau)$.

The class of functions $\Theta[0, \tau)$ is denoted by $\Theta_h[0, \tau)$ if it satisfies condition (v).

Let (X, d) be a metric space. A subset K of X is said to be *proximal* if for each $x \in X$, there exists a $e \in K$ such that $d(x, e) = \inf_{y \in K} d(x, y) = d(x, K)$. Let 2^X denote a collection of all subsets of X, $CL(X)$ a collection of all nonempty closed subsets of X, $CB(X)$ a collection of all nonempty closed and bounded subsets of X, $K(X)$ a collection of all nonempty compact subsets of X, and $P(X)$ a collection of all proximal subsets of X.

For $E, F \in CB(X)$, let

$$H(E, F) = \max\Big\{ \sup_{x \in F} d(x, E), \sup_{y \in E} d(y, F) \Big\},$$

where $d(x, E) = \inf_{y \in E} d(x, y)$. The map H is called *Hausdorff metric* induced by d".

For $E, F \in CL(X)$, let

$$H(E, F) = \begin{cases} \max\Big\{ \sup\limits_{x \in F} d(x, E), \sup\limits_{y \in E} d(y, F) \Big\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

The map H is called *Hausdorff generalized functional* induced by d.
Denoting $\sup_{x\in F} d(x,E)$ by $\rho(F,E)$ we define

$$H^+(E,F)=\begin{cases} \frac{1}{2}[\rho(E,F)+\rho(F,E)], & \text{if both } \rho(E,F) \text{ and } \rho(F,E) \text{ exist} \\ \infty, & \text{otherwise.}\end{cases}$$

The map H^+ is a metric on $CL(X)$.
We also recall the following definitions:

(a) A map $f: X\to\mathbb{R}$ is said to be *lower semi-continuous*, if for $\{x_n\}$ in X and $x\in X$ such that $x_n\to x$, we have $f(x)\le \liminf_{n\to\infty} f(x_n)$.

(b) If, for $x_0\in X$, there exists x_n in X such that $x_n\in Tx_{n-1}$, then $O(T,x_0)=\{x_0,x_1,x_2,\cdots\}$ is called an orbit of $T:X\to CL(X)$.

(c) A map $f:X\to\mathbb{R}$ is called *T-orbitally lower semi-continuous* [9], if $\{x_n\}$ is a sequence in $O(T,x_0)$ and $x_n\to\xi$ implies $f(\xi)\le\liminf_{n\to\infty} f(x_n)$.

Let $T:X\to 2^X\setminus\{\emptyset\}$. For $b\in(0,1]$ and $x\in X$, we define

$$I_b^x=\big\{y\in Tx: bd(x,y)\le d(x,Tx)\big\},$$
$$M(b,x;\theta)=\big\{y\in Tx: b\,\theta(d(x,y))\le \theta(d(x,Tx))\big\}$$

and

$$M(1,x;\theta)=\big\{y\in Tx: \theta(d(x,y))=\theta(d(x,Tx))\big\}.$$

A point $x\in X$ is called a *fixed point* of a multivalued map $T:X\to 2^X\setminus\{\emptyset\}$, if $x\in Tx$.

2. Fixed point

Through out this section (X,d) is a complete metric space unless otherwise stated.

Lemma 2.1. *Suppose that $E,F\in CL(X)$ with $\rho(E,F)\le\rho(F,E)$. Then for each $a\in E$ and $\beta>1$ there exists an element $b\in F$ such that*

$$d(a,b)\le\beta\ H^+(E,F).$$

Proof. If, for some $a\in E$, $d(a,F)=0$ then $a\in F$, since F is a closed subset of X. Taking $b=a$ we see that (2.1) holds. Now suppose that $d(a,F)>0$, then $H^+(E,F)>0$. For any $\epsilon>0$, using the definition of $d(a,F)$, there exists $b=b(a,\epsilon)\in F$ such that

$$d(a,b)\le d(a,F)+\epsilon\le\sup_{a\in A} d(a,F)+\epsilon=\rho(E,F)+\epsilon. \tag{2.1}$$

As $\rho(E,F)\le\rho(F,E)$, from (2.1), we also have

$$d(a,b)\le\rho(F,E)+\epsilon. \tag{2.2}$$

Choose $\epsilon=(\beta-1)H^+(E,F)$. Then from (2.1) and (2.2), it follows that

$$\begin{aligned} d(a,b)&\le\frac{1}{2}[\rho(E,F)+\rho(F,E)]+\epsilon=H^+(E,F)+\epsilon\\ &\le\beta\ H^+(E,F)\ \text{(by inserting the value of}\,\epsilon).\end{aligned}$$

Next we state and prove our main result.

Theorem 2.2. *Suppose that φ is a function from $(0,\infty)$ to $[0,1)$ such that*

$$\limsup_{r\to t^+}\varphi(r)<1 \text{ for each } t\in[0,\infty). \tag{2.3}$$

Suppose that $T : X \to CL(X)$ *. Assume that for* $x \in X$, $y \in Tx$ *and* $\rho(Tx, Ty) \leq \rho(Ty, Tx)$, *we have:*

$$\theta(H^+(Tx, Ty)) \leq \varphi(d(x, y))\ \theta(d(x, y)) \text{ for each } x \in X \text{ and } y \in Tx, \tag{2.4}$$

where $\theta \in \Theta_h[0, +\infty)$. *Then:*

(i) for each $x_0 \in X$, *there exists an orbit* x_n *of* T *and* $\xi \in X$ *such that* $\lim_n x_n = \xi$;

(ii) ξ *is a fixed point of* T *if and only if the function* $f(x) := d(x, Tx)$ *is* T*-orbitally lower semi-continuous at* ξ.

Proof. Suppose $\limsup_{r \to t^+} \varphi(r) \leq \alpha_0$ for each $t \in [0, \infty)$. Then by (2.3) there exists $\alpha \in (\alpha_0, 1)$ such that $\varphi(t) < \alpha$ for each $t \in [0, \infty)$. Suppose that T has no fixed point, so $d(x, Tx) > 0$ for each $x \in X$. Let $x_0 \in X$ be arbitrary and fixed. Since $Tx_0 \neq \emptyset$, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. Clearly $x_0 \neq x_1$; using (iv) and taking

$$\beta = \frac{\alpha}{\varphi(d(x_0, x_1))},$$

it follows from Lemma 2.1 that there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq \frac{\alpha}{\varphi(d(x_0, x_1))} \theta(H^+(Tx_0, Tx_1)) \tag{2.5}$$

Proceeding with the same argument as above and noting that $x_{n-1} \neq x_n$, for otherwise x_{n-1} is a fixed point of T, we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that

$$\theta(d(x_n, x_{n+1})) \leq \frac{\alpha}{\varphi(d(x_{n-1}, x_n))} \theta(H^+(Tx_{n-1}, Tx_n)) \tag{2.6}$$

where $x_n \in Tx_{n-1}, n = 1, 2, \cdots$.

Using (2.4) it follows from (2.6) that

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \alpha \theta(d(x_{n-1}, x_n)) && (2.7) \\ &< \theta(d(x_{n-1}, x_n)). && (2.8) \end{aligned}$$

Hence $\theta(d(x_n, x_{n+1}))$ is a decreasing sequence of positive real numbers bounded below by 0. Since θ is strictly inverse isotone, $d(x_n, x_{n+1})$ is also a decreasing sequence of positive real numbers bounded below by 0, and thus convergent. Let $\theta(d(x_n, x_{n+1}))$ converges to some nonnegative real number, ℓ say. We claim that $\ell = 0$; for otherwise, by taking the limits in (2.7), we get

$$\ell \leq \alpha \ell < \ell$$

a contradiction. Further, we claim that $d(x_n, x_{n+1})$ also converges to 0. Suppose $\lim_{n \to \infty} d(x_n, x_{n+1}) = \gamma > 0$. Then for $0 < \epsilon < \gamma$, there exists a natural number n_0 such that

$$0 < \delta = \gamma - \epsilon < d(x_n, x_{n+1})\ \forall n \geq n_0.$$

Since θ is positive and nondecreasing, we obtain from the above

$$0 < \theta(\delta) \leq \theta(d(x_n, x_{n+1}))\ \forall n \geq n_0,$$

which is a contradiction, since $\theta(d(x_n, x_{n+1})) \to 0$. From (2.7), we get

$$\theta(d(x_n, x_{n+1})) \leq \alpha^n \theta(d(x_0, x_1)). \tag{2.9}$$

For any $m > n > n_0$, by (2.9), we have

$$\theta(d(x_m, x_n)) \leq \sum_{j=n}^{m-1} \theta(d(x_j, x_{j+1})) \leq \sum_{j=n}^{m-1} \alpha^j \theta(d(x_0, x_1)) = \Big(\sum_{j=n}^{m-1} \alpha^j\Big)\theta(d(x_0, x_1))$$
$$< \frac{\alpha^n}{1-\alpha}\theta(d(x_0, Tx_1)).$$

Therefore

$$\lim_{n,m\to\infty} \theta(d(x_m, x_n)) = 0.$$

We claim that

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Suppose not. Then there exist $\delta > 0$ and subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ respectively such that

$$d(x_{m_i}, x_{n_i}) > \delta \ \ \forall i.$$

Since θ is nondecreasing,

$$0 < \theta(\delta) \leq \theta(d(x_{m_i}, x_{n_i})) \to 0,$$

which is a contradiction. This proves our claim. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists $\xi \in X$ such that $x_n \to \xi$ and, by (2.4), we obtain

$$\theta\Big(\frac{1}{2}\Big\{\rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_{n-1})\Big\}\Big) = \theta(H^+(Tx_{n-1}, Tx_n)).$$

$$\theta\Big(\frac{1}{2}\Big\{\rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_{n-1})\Big\}\Big) \leq \varphi(d(x_{n-1}, x_n))\,\theta(d(x_{n-1}, x_n)).$$

It follows that

$$\theta\Big(\frac{1}{2}\Big\{\rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_{n-1})\Big\}\Big) < \theta(d(x_{n-1}, x_n)),$$

which further implies that

$$\frac{1}{2}\Big\{\rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_{n-1})\Big\} < d(x_{n-1}, x_n). \tag{2.10}$$

Now(2.10) yields

$$\lim_{n\to\infty} \frac{1}{2}\Big\{\rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_{n-1})\Big\} = 0.$$

It implies that

$$\lim_{n\to\infty} \Big(d(x_n, Tx_n) + \rho(Tx_n, Tx_{n-1})\Big) = 0.$$

Thus we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Suppose that $f(x) = d(x, Tx)$ is T-orbitally continuous at ξ; then

$$d(\xi, T\xi) = f(\xi) \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

It implies that $d(\xi, T\xi) = 0$, and since $T\xi$ is closed it must be the case that $\xi \in T\xi$. Conversely, if ξ is a fixed point, then $f(\xi) = 0 \leq \liminf_{n\to\infty} f(x_n)$.

Example 2.3. Let $X = \{0,1\}$ and $d : X \times X \to \mathbb{R}$ be a standard metric. Let $T : X \to CL(X)$ be such that

$$T(x) = \begin{cases} \{0,1\}, & \text{for} \quad x = 0, \\ \{0\}, & \text{for} \quad x = 1. \end{cases}$$

Define $\theta : (0,\infty) \to [0,1)$ by $\theta(t) = t$ for all $t \in (0,\infty)$. It is routine to check that condition (2.4) is satisfied for any $x \in X$ and $y \in Tx$. Notice that 0 is a fixed points of T.

Observe that the map T does not satisfy the assumptions of [6]. Indeed, for $x = 0$ and $y = 1 \in T(0)$ we have

$$\theta(d(1,T(1))) = \theta(d(1,\{0\})) = 1 > \varphi(d(0,1))\theta(d(0,1)).$$

The following result is a direct consequence of Theorem 2.2.

Corollary 2.4. *Let $T : X \to CL(X)$. Assume that conditions (2.4) of Theorem 2.2 holds. Also assume that there exist a $\theta \in \Theta_h[0,+\infty)$ and a $\Psi \in \Theta_h[0,+\infty)$ such that T satisfies*

$$\int_0^{\theta(H^+(Tx,Ty))} \psi(t)\,dt \le \varphi(d(x,y)) \int_0^{\theta(d(x,y))} \psi(t)\,dt, \tag{2.11}$$

where $\Psi(\epsilon) = \int_0^\epsilon \psi(t)dt$, and $\psi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying the following condition:

$$\int_0^\epsilon \psi(t)\,dt > 0 \text{ for all } \epsilon > 0. \tag{2.12}$$

Then T has a fixed point.

Since $H^+(Tx,Ty) \le H(Tx,Ty)$ for $y \in Tx$, we have the following:

Corollary 2.5. *Suppose that φ is a function from $(0,\infty)$ to $[0,1)$ such that*

$$\limsup_{r\to t^+} \varphi(r) < 1 \text{ for each } t \in [0,\infty). \tag{2.13}$$

Suppose that $T : X \to CL(X)$. Assume that the following condition holds:

$$\theta(H(Tx,Ty)) \le \varphi(d(x,y))\,\theta(d(x,y)) \text{ for each } x \in X \text{ and } y \in Tx, \tag{2.14}$$

where $\theta \in \Theta_h[0,+\infty)$. Then:

(i) for each $x_0 \in X$, there exists an orbit x_n of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is a fixed point of T if and only if the function $f(x) := d(x,Tx)$ is T-orbitally lower semi-continuous at ξ.

Now we state and prove a fixed point result in Banach spaces. Let $WCL(X)$ denote the collection of all nonempty weakly closed subsets of a Banach space X.

Theorem 2.6. *Suppose that φ is a function from $(0,\infty)$ to $[0,1)$ such that*

$$\limsup_{r\to t^+} \varphi(r) < 1 \text{ for each } t \in [0,\infty). \tag{2.15}$$

Let $T : X \to WCL(X)$. Also assume that the following condition holds:

$$\text{for all } x \in X, \ M(1,x:\theta) \text{ is nonempty} \tag{2.16}$$

and

$$\theta(\|y - Ty\|) \le \varphi(\|x-y\|)\,\theta(\|x-y\|) \text{ for each } x \in X \text{ and } y \in Tx, \tag{2.17}$$

where $\theta \in \Theta[0,+\infty)$. Then:

(1) for each $x_0 \in X$, there exists an orbit x_n of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(2) ξ is a fixed point of $T \iff$ the function $f(x) := d(x, Tx)$ is T-orbitally lower semi-continuous at ξ.

Proof. Assume that T has no fixed point, so $d(x, Tx) = \inf_{y \in Tx} \|x - y\| > 0$ for each $x \in X$. Let φ be as in condition *(ii)* and $x_0 \in X$. By *(2.15)*, using the same argument as in the proof of theorem 2.1 in [12], we obtain a Cauchy sequence $\{x_n\}$ such that $x_n \in Tx_{n-1}, x_n \neq x_{n-1}$, having

$$\theta(\|x_{n-1} - x_n\|) = \theta(d(x_{n-1}, Tx_{n-1})) \tag{2.18}$$

and

$$d(x_n, Tx_n) \leq \varphi\left(\|x_{n-1} - x_n\|\right)\|x_{n-1} - x_n\| \text{ and } \varphi(\|x_{n-1} - x_n\|) < 1. \tag{2.19}$$

Consequently, there exists $\xi \in X$ such that $x_n \to \xi$. Suppose that $f(x) = d(x, Tx)$ is T-orbitally continuous at ξ; then

$$d(\xi, T\xi) = f(\xi) \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

It implies that $d(\xi, T\xi) = 0$, and since $T\xi$ is weakly closed it further implies that $\xi \in T\xi$. It contradicts the fact that T has no fixed point. *Conversely*, if ξ is a fixed point of T, then $f(\xi) = 0 \leq \liminf_{n\to\infty} f(x_n)$.

Next we further compare and discuss our results with the known results in the existing literature. It is clear from Example 2.3, that Theorem 2.2 essentially compliments and generalizes the results of [6, 10].

Example 2.7. Let $X = [0, \infty)$ equipped with usual metric d. Define $T : X \to CL(X)$ by $Tx = [x, \infty)$ for all $x \in X$. Define $\theta : (0, \infty) \to [0, 1)$ by $\theta(t) = t$ for all $t \in (0, \infty)$ and $\varphi : (0, \infty) \to [0, 1)$ by

$$\varphi(t) = \begin{cases} \dfrac{1}{2} + \dfrac{x}{6}, & \text{if } x \in (0, 1] \\ \dfrac{2}{3}, & \text{otherwise..} \end{cases}$$

Clearly, $\limsup_{r\to t^+} \varphi(r) = \frac{2}{3} < 1$ for each $t \in [0, \infty)$. Further, for any $x \in X$ and $y \in Tx$, we have

$$H^+(Tx, Ty) = \frac{1}{2}|x - y| \leq \varphi(|x - y|)\ |x - y|.$$

Thus we see that all conditions of Theorem 2.2 are satisfied and 0 is a fixed point of T. Note that T does not fulfill the hypothesis of Theorems in [6, 10].

Example 2.8. For $1 < p < \infty$, let $\ell_p = \{(x_n) : |x_n|^p < \infty\}$ be the real Banach space equipped with the standard norm $\|\cdot\|_p$ defined by

$$\|x\|_p = (|x_n|^p)^{1/p},\ x = (x_n).$$

Note that for $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in \ell^p$ the usual metric d is defined by

$$d(x, y) = (|x_n - y_n|^p)^{1/p}.$$

For each $n = 1, 2, ...$, let e_n be the vector in ℓ_p with zeros as all its coordinates except the nth coordinate which is equal to 1 and e_0 be the vector in ℓ_p with zeros at all coordinates. Take $a = (-1, -\frac{1}{2}, ..., -\frac{1}{n}, \cdots)$, and $F = \{e_0, e_1, e_2, ..., e_n, ...\}$. Define $T : \ell_p \to WCL(\ell_p)$ by

$$Tx = F \ \ \forall x \in \ell_p.$$

Then, for $\varphi(t) = t$, Theorem 2.6 [condition (2.17)] is obviously satisfied. Further,

$$d(a, F) = \inf_{b \in F} \|a - b\|_p = \|a\|_p = \|a - e_0\|_p.$$

Observe that $F \in WCL(X)$ and there exists e_0 in F such that $\|a - e_0\|_p \leq d(a, F)$. Hence $M(1, x; \theta)$ is nonempty and Theorem 2.6 [condition (2.16)] is satisfied. Note that $e_0, e_1, e_2, ...$ are fixed points of T.

Applying the above results, we obtain the following significant result which plays an important role in the next section.

Proposition 2.9. Let X and $CL(X)$ be as given in Theorem 2.2, and let $\varphi : (0, \infty) \to [0, 1)$ be a function satisfying the condition (2.3). Suppose that $T_i : X \to CL(X), i = 1, 2$, are two H^+-type multi-valued mappings such that each one satisfies the condition (2.4). Then, for a given $\beta > 1$, if $Fix(T_1)$ and $Fix(T_2)$ denote the respective fixed point sets of T_1 and T_2,

$$H^+(Fix(T_1), Fix(T_2)) \leq \frac{\beta}{\beta - 1} \sup_{x \in X} H^+(T_1 x, T_2 x).$$

Proof. Notice that $Fix(T_i) \in CL(X)$ for $i = 1, 2$. Assume that $Fix(T_1) \leq Fix(T_2)$. Let $q > 1$ be given. Suppose $\limsup_{r \to t^+} \varphi(r) \leq \alpha_0$ for each $t \in [0, \infty)$. Then by (2.3) there exists $\alpha \in (\alpha_0, 1)$ such that $\varphi(t) < \alpha$ for each $t \in [0, \infty)$. Select $x_0 \in \mathrm{Fix}(T_1)$, and then select $x_1 \in T_2 x_0$. Set $\beta = \frac{1}{\sqrt{\alpha}}$. From Lemma 2.1 it follows that we can choose $x_2 \in T_2 x_1$ such that

$$\begin{aligned} \theta(d(x_1, x_2)) &\leqslant \frac{1}{\sqrt{\alpha}} \ \theta(H^+(T_2 x_0, T_2 x_1)) \\ &\leqslant \frac{1}{\sqrt{\alpha}} \ \varphi(d(x_0, x_1))\theta(d(x_0, x_1)) \\ &\leqslant \sqrt{\alpha} \ \theta(d(x_0, x_1)). \end{aligned} \tag{2.20}$$

Now define $\{x_n\}$ inductively so that $x_{n+1} \in T_2(x_n)$ and

$$\theta(d(x_n, x_{n+1})) \leqslant \frac{1}{\sqrt{\alpha}} \theta(H^+(T_2 x_{n-1}, T_2 x_n))) \ \forall n \in \mathbb{N}. \tag{2.21}$$

Repeating the same argument n-times as in (2.20), we get

$$\theta(d(x_n, x_{n+1})) \leqslant \alpha^{\frac{n}{2}} \ \theta(d(x_0, x_1)). \tag{2.22}$$

It implies that $\{x_n\}$ is a Cauchy sequence with limit, say z.

Since T_2 is continuous, we have

$$\lim_{n \to \infty} H^+(T_2 x_n, T_2 z) = 0.$$

Also, since $x_{n+1} \in T_2(x_n)$ it must be the case that $z \in T_2 z$; that is, $z \in \mathrm{Fix}(T_2)$. Furthermore, using defining property (ii) of θ and the inequality (2.22), we have

$$\theta(d(x_0, z)) \leqslant \sum_{n=0}^{\infty} \theta(d(x_{n+1}, x_n)) \leqslant (1 + \sqrt{\alpha} + (\sqrt{\alpha})^2 + \cdots)\theta(d(x_1, x_0)).$$

$$\theta(d(x_0, z)) \leqslant \frac{1}{1-\sqrt{\alpha}} H^+(T_2x_0, T_1x_0).$$

It leads to the conclusion that for each $y_0 \in \mathrm{Fix}(T_2)$ there exist $y_1 \in T_1y_0$ and $w \in \mathrm{Fix}(T_1)$ such that

$$d(y_0, w) \leqslant \frac{\beta}{\beta - 1} H^+(T_1y_0, T_2y_0).$$

Hence the conclusion follows.

Theorem 2.10. *Suppose that $\varphi : (0, \infty) \to [0, 1)$ is a function satisfying the condition (2.2), X and $CL(X)$ are as in Theorem 2.2. Suppose $T_i \colon X \to CL(X)$, $i = 1, 2, \ldots$ are H^+-type multi-valued contraction mappings satisfying the condition (2.4) and φ a function as defined in (2.3). If for a given $\beta > 1$, $\lim_{n\to\infty} H^+(T_nx, T_0x) = 0$ uniformly for $x \in X$, then*

$$\lim_{n\to\infty} H^+(Fix(T_n), Fix(T_0)) = 0.$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} H(T_nx, T_0x) = 0$ uniformly for $x \in X$ it is possible to choose $N \in \mathbf{N}$ so that for $n \geq N$, $\sup_{x\in X} H(T_nx, T_0x) < \frac{\beta-1}{\beta}\varepsilon$. By Proposition 2.9, $H(Fix(T_n), Fix(T_0)) < \varepsilon$ for all $n \geq N$. Hence the conclusion follows.

3. Application to nonconvex Hammerstein type integral inclusions

Let E be a real separable Banach space with the norm $\|\cdot\|$, $T = diam(E)$, and let $\mathcal{P}(E)$ denote the family of all nonempty subsets of E and $\mathcal{B}(E)$ the family of all Borel subsets of E. Then $0 < T < \infty$. Let $I := [0, T]$ and $\mathcal{L}(I)$ denote the σ-algebra of all Lebesgue measurable subsets of I.

In what follows, we denote by $C(I, E)$ the Banach space of all continuous functions $x(\cdot) : I \to E$ endowed with the norm $\|x(\cdot)\|_C = sup_{t\in I}\|x(t)\|$. Now consider the following integral equation

$$x(t) = \lambda(t) + \int_0^T k(t, s)\, g(t, s, u(s))\, ds \;\text{ on }\; [0, T]. \tag{3.1}$$

Here λ, k and g are given functions, where $\lambda(\cdot) : I \to E$ is a function with Banach space value, $k : I \times I \to \mathbf{R}_+=[0, \infty)$ is a positive real single valued function, while $g : I \times I \times E \to E$ is a map. Let $p \in [1, \infty)$, $q \in [1, \infty)$, and $r \in [1, \infty)$ be the conjugate exponent of q, that is $1/q + 1/r = 1$. Let $\|\cdot\|_p$ denote the p-norm of the space $L^p(I, E)$ and is defined by $\|u\|_p = (\int_0^T \|u(s)\|^p ds)^{1/p}$ for all $u \in L^p(I, E)$. Consider the Nemitskij operator associated to g, p, q and $G : L^p(I, E) \to L^q(I, E)$ given by

$$G(u) = g(t, s, u(s))\, a.e. \,\text{on}\; I.$$

Consider the linear integral operator of kernel $k, S : L^q(I, E) \to L^p(I, E)$ given by

$$S(u) = \lambda(t) + \int_0^T k(t, s)u(s)ds \; a.e. \,\text{on}\; I.$$

Thus the Hammerstein type integral equation (3.1) is transformed into the form

$$x = SG(u), \quad u \in L^p(I,E) \ a.e. \text{ on } I \tag{3.1'}$$

$$u(t) \in F(t, V(x)(t)) \quad a.e. \ (I := [0,T]), \tag{3.2}$$

where $V : C(I,E) \to C(I,E)$ is a given mapping. In the sequel, we also use the following: For any $x \in E$, $\lambda \in C(I,E), \sigma \in L^p(I,E)$, we define the multivalued maps

$$M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t)), \ t \in I,$$

$$T_\lambda(\sigma) := \{\psi(\cdot) \in L^p(I,E) : \psi(t) \in M_{\lambda,\sigma}(t) \ \ a.e. \ (I)\}.$$

In order to study problem (3.1)-(3.2) we introduce the following more general assumptions similar to [13].

Hypothesis 3.1. Let $F(\cdot,\cdot) : I \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty closed values that verify:

(H_1) The function $k : I \times I \to \mathbf{R}_+$ satisfies that $k(t,\cdot) \in L^r(I)$, and $t \to \|k(t,\cdot)\|_r \in L^p(I)$.

(H_2) The multivalued map $F(\cdot,\cdot)$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable.

(H_3) There exists $\varphi(\cdot) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I, F(t,\cdot)$ satisfies the condition

$$\theta(H^+(F(t,x), F(t,y))) \le \varphi(t) \ \theta(\|x-y\|) \tag{C1}$$

for all x, y in E, where φ and θ are as defined in (2.3) and (2.4), respectively.
For any $x, y \in E, w \in F(t,x)$ and any $\beta > 1$, there exists $z \in F(t,y)$ such that

$$\|w - z\|_p \le \beta \ H^+(F(t,x), F(t,y)) \tag{C2}$$

and $T_\lambda(\cdot)$ satisfies the conditions: For any $\sigma \in L^p(I,E)$ and $\sigma_1 \in T_\lambda(\sigma)$ there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_p \le \beta \ H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) \tag{C3}$$

and for all $t \in I$, there exists $\alpha_0 \in (0,1)$ such that

$$\limsup_{r \to t^+} \varphi(r) \le \alpha_0 \text{ and } \theta\Big(\varphi(t) \ t\Big) \le \varphi(t) \ \theta(t). \tag{C4}$$

(H_4) The mappings $k : I \times I \to \mathbf{R}_+, g : I \times I \times E \to E$ are continuous, $V : C(I,E) \to C(I,E)$ and there exist the constants $M_1, M_2, M_3 > 0$ such that

$$\|g(t,s,u_1) - g(t,s,u_2)\|_q \le M_1 \|u_1 - u_2\|^p, \ \forall u_1, u_2 \in E,$$
$$\|V(x_1)(t) - V(x_2)(t)\| \le M_2 \|x_1(t) - x_2(t)\|, \ \forall t \in I, \forall x_1, x_2 \in C(I,E),$$

and

$$\|k(t,s)\|_r \le M_3 \ \forall \, t, s \in I.$$

Note that the system (3.1)-(3.2) encompasses a large variety of differential inclusions and control systems including those defined by partial differential equations.

Assume that U be an open bounded subset of $\mathbf{R}^n$ (or Y, a subset of E homeomorphic to $\mathbf{R}^n$) and $U_T = (0,T] \times U$ for some fixed $T > 0$. We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(t,x)\xi_i\xi_j \geq \theta|\xi|^2 \text{ for all } (t,x) \in U_T, \xi \in \mathbf{R}^n.$$

The letter L denotes for each time t a second order partial differential operator, having either the divergence form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(t,x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(t,x)u_{x_i} + c(t,x)u$$

or else the nondivergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(t,x)u_{x_i} + c(t,x)u,$$

for given coefficients $a^{ij}, b^i, c \ (i,j = 1,2,\ldots,n)$.

A family$\{G(t) : t \in \mathbf{R}_+\}$ of bounded linear operators from X into E is a C_0-semigroup (also called linear semigroup of class (C_0)) on X if

(i) $G(0) =$ the identity operator, and $G(t+s) = G(t)G(s)\ \forall\, t,s \geq 0$;

(ii) $G(\cdot)$ is strongly continuous in $t \in \mathbf{R}_+$;

(iii) $\|G(t)\| \leq Me^{\omega t}$ for some $M > 0$, real ω and $t \in \mathbf{R}_+$.

Example 3.2. Set $k(t,\tau)g(t,\tau,u) = G(t-\tau)u, \Phi(x) = x, \lambda(t) = G(t)x_0$ where $\{G(t)\}_{t\geq 0}$ is a C_0-semigroup with an infinitesimal generator A. Then a solution of system (3.1)-(3.2) represents a mild solution of

$$x'(t) \in Ax(t) + F(t,x(t)), \qquad x(0) = x_0. \tag{3.3}$$

In particular, this problem includes control systems governed by parabolic partial differential equations as a special case. When $A = 0$, the relation (3.3) reduces to classical differential inclusions

$$x'(t) \in F(t,x(t)), \qquad x(0) = x_0. \tag{3.4}$$

Denote

$$\Phi(u)(t) = \int_0^T k(t,\tau)g(t,\tau,u(\tau))\,d\tau,\ t \in I. \tag{3.5}$$

Then the integral inclusion system (3.1)-(3.2) reduces to the form

$$x(t) = \lambda(t) + \Phi(u)(t) \qquad a.e.\ (I), \tag{S}$$

which may be written in more "compact" form as

$$u(t) \in F(t, V(\lambda + \Phi(u))(t)) \qquad a.e.\ (I).$$

Now we recall the following:

Definition 3.3. [13] A pair of functions (x,u) is called a solution pair of integral inclusion system (S), if $x(\cdot) \in C(I,E)$, $u(\cdot) \in L^p(I,E)$ and satisfy relation (S).

For our further discussion, we denote by $S(\lambda)$ the solution set of (3.1)-(3.2).

Notice that the integral operator in (3.5) plays a key role in the proofs of our main results.

For given $\beta \in \mathbf{R}$ we denote by $L^p(I,E)$ the Banach space of all Bochner integrable functions $u(\cdot): I \to E$ endowed with the norm

$$\|u(\cdot)\|_p = \Big(\int_0^T e^{-\beta M_1 M_2 M_3 m(t)} \|u(t)\|^p \, dt \Big)^{\frac{1}{p}},$$

where

$$m(t) = \int_0^t \varphi(s)\, ds, \ t \in I.$$

For our further discussion, we denote $L = m(T)$.

Theorem 3.4. *Let Hypothesis 3.1 be satisfied,* $\lambda(\cdot,\cdot), \mu(\cdot,\cdot) \in C(I \times E, E)$ *and* $v(\cdot) \in L^p(I,E)$ *be such that*

$$d(v(t), F(t, \Phi(y)(t)) \leq p(t) \qquad a.e. \qquad (I),$$

where $p(\cdot) \in L^p(I, \mathbf{R}_+)$ *and* $y(t) = \mu(t, v(t)) + \Phi(v)(t), \ \forall t \in I.$
Then for every $\beta > 1$*, there exists* $x(\cdot) \in S(\lambda)$ *such that for every* $t \in I$

$$\|x(t) - y(t)\| \leq \|\lambda - \mu\|_C + M_1 M_3 e^{\beta M_1 M_2 M_3 L} \Big[\frac{\beta^{1/q}}{(\beta-1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C + \frac{\beta}{\beta - 1} \Big(\int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t) dt \Big)^{\frac{1}{p}} . \Big]^p .$$

Proof. For $\lambda \in C(I,E)$ and $u \in L^p(I,E)$ define

$$x_{u,\lambda}(t) = \lambda(t) + \int_0^T k(t,s)\, g(t,s,u(s))\, ds, \ t \in I.$$

Let us consider that $\lambda \in C(I,E), \sigma \in L^p(I,E)$ and define the multivalued maps

$$M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t)), \ t \in I, \tag{3.6}$$

$$T_\lambda(\sigma) := \{\psi(\cdot) \in L^p(I,E) : \psi(t) \in M_{\lambda,\sigma}(t) \ \ a.e. \ (I)\}. \tag{3.7}$$

Further, in view of condition (C2) of Hypothesis 3.1(H_3), $T_\lambda(\cdot)$ satisfies the condition: For any $\sigma \in L^p(I,E)$, $\sigma_1 \in T_\lambda(\sigma)$ and any given $\beta > 1$ there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_p \leq \beta \ H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)). \tag{3.8}$$

Now we claim that $T_\lambda(\sigma)$ is nonempty, bounded and closed for every $\sigma \in L^p(I,E)$.
It is well known that the multivalued map $M_{\lambda,\sigma}(\cdot)$ is measurable. For example the map $t \to M_{\lambda,\sigma}(t)$ can be approximated by step functions and so we can apply Castaing and Valadier [4, Theorem III.40]. As the values of F are closed, with the measurable selection theorem we infer that $M_{\lambda,\sigma}(\cdot)$ is nonempty.
Further, we note that the set $T_\lambda(\sigma)$ is bounded and closed. Indeed, if $\psi_n \in T_\lambda(\cdot)$ and $\|\psi_n - \psi\|_p \to 0$, then there exists a subsequence ψ_{n_k} such that $\psi_{n_k}(t) \to \psi(t)$ for a.e. $t \in I$ and we find that $\psi \in T_\lambda(\sigma)$.

Let $\sigma_1, \sigma_2 \in L^p(I,E)$ be given. Let $\psi_1 \in T_\lambda(\sigma_1)$ and let $\delta > 0$. Consider the following multi-valued map:

$$\mathcal{G}(t) := M_{\lambda,\sigma_2(t)} \cap \Big\{ z \in E : \|\psi_1(t) - z\|^p.$$

$$\mathcal{G}(t) \leq M_1 M_2 M_3 \varphi(t)[\varphi(\|\sigma_1 - \sigma_2\|_p)]^p \int_0^t \|\sigma_1(s) - \sigma_2(s)\|^p \, ds + \delta \Big\}.$$

By (3.8), it follows that

$$\begin{aligned} &\theta(d(\psi_1(t), M_{\lambda,\sigma_2}(t))) \leq \beta^{-1}\theta\left(H^+\left(F(t, V(x_{\sigma_1,\lambda})(t)), F(t, V(x_{\sigma_2,\lambda})(t))\right)\right) \\ &\leq \beta^{-1}\varphi(\|V(x_{\sigma_1,\lambda})(t) - V(x_{\sigma_2,\lambda})(t)\|)\ \theta(\|V(x_{\sigma_1,\lambda})(t)) - V(x_{\sigma_2,\lambda})(t))\|) \\ &\leq \alpha_0\beta^{-1}M_2\ \ \theta(\|x_{\sigma_1,\lambda}(t) - x_{\sigma_2,\lambda}(t)\|) \\ &\leq \alpha_0\beta^{-1}M_2\ \theta\Big(\int_0^T \|k(t,s)\|_r \|g(t,s,x_1(s))\, ds - g(t,s,x_2(s))\|_q \, ds\Big) \\ &\leq \alpha_0\beta^{-1}M_1M_2M_3\ \theta\Big(\int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p ds\Big). \end{aligned}$$

From this we deduce that $\mathcal{G}(\cdot)$ is nonempty bounded and has closed values. Further, according to [4, Proposition III.4], $\mathcal{G}(\cdot)$ is measurable.
Let $\psi_2(\cdot)$ be a measurable selector of $\mathcal{G}(\cdot)$. It follows that $\psi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned} &\|\psi_1 - \psi_2\|_p^p = \int_0^T e^{-\beta M_1M_2M_3m(t)}\|\psi_1(t) - \psi_2(t)\|^p dt \\ &\leq \int_0^T e^{-\beta M_1M_2M_3m(t)}\Big(M_1M_2M_3\varphi(t)[\varphi(\|\sigma_1 - \sigma_2\|_p)]^p \int_0^t \|\sigma_1(s) - \sigma_2(s)\|^p ds\Big)dt \\ &+ \delta\int_0^T e^{-\beta M_1M_2M_3m(t)}dt \\ &\leq \beta^{-1}[\varphi(\|\sigma_1 - \sigma_2\|_p)]^p\|\sigma_1 - \sigma_2\|_p^p + \delta\int_0^T e^{-\beta M_1M_2M_3m(t)}dt \\ &< [\varphi(\|\sigma_1 - \sigma_2\|_p)]^p\|\sigma_1 - \sigma_2\|_p^p + \delta\int_0^T e^{-\beta M_1M_2M_3m(t)}dt. \end{aligned}$$

Because δ is arbitrary, so letting $\delta \to 0$ we deduce from the above inequality that

$$\|\psi_1 - \psi_2\|_p^p < \varphi(\|\sigma_1 - \sigma_2\|_p)]^p\|\sigma_1 - \sigma_2\|_p^p$$

Therefore

$$\|\psi_1 - \psi_2\|_p < \varphi(\|\sigma_1 - \sigma_2\|_p)\|\sigma_1 - \sigma_2\|_p.$$

This yields

$$d(\psi_1, T_\lambda(\sigma_2)) < \varphi(\|\sigma_1 - \sigma_2\|_p)\|\sigma_1 - \sigma_2\|_p.$$

Thus, we have

$$\rho(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) = \sup_{\psi_1 \in T_\lambda(\sigma_1)} d(\psi_1, T_\lambda(\sigma_2)) \leq \varphi(\|\sigma_1 - \sigma_2\|_p)\|\sigma_1 - \sigma_2\|_p. \tag{3.9}$$

Now replacing $\sigma_1(\cdot)$ with $\sigma_2(\cdot)$ and arguing as above, we obtain

$$\rho(T_\lambda(\sigma_2), T_\lambda(\sigma_1)) \leq \varphi(\|\sigma_1 - \sigma_2\|_p)\|\sigma_1 - \sigma_2\|_p. \tag{3.10}$$

Using (3.9), (3.10), (C4) and defining properties of θ, we obtain

$$
\begin{aligned}
\theta(H^+(T_\lambda(\sigma_1), T_\lambda(\sigma_2))) &= \theta\Big(\frac{\rho(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) + \rho(T_\lambda(\sigma_2), T_\lambda(\sigma_1))}{2}\Big) \\
&\le \frac{1}{2}\Big[\theta(\rho(T_\lambda(\sigma_1), T_\lambda(\sigma_2)))) + \theta(\rho(\rho(T_\lambda(\sigma_2), T_\lambda(\sigma_1)))\Big] \\
&\le \theta\Big(\varphi(\|\sigma_1 - \sigma_2\|_p)\|\sigma_1 - \sigma_2\|_p\Big) \\
&\le \varphi(\|\sigma_1 - \sigma_2\|_p)\theta(\|\sigma_1 - \sigma_2\|_p).
\end{aligned}
$$

Hence we conclude that $T_\lambda(\cdot)$ is an H^+-type multivalued mapping on $L^p(I, E)$.
Next, we consider the following multivalued maps

$$
\begin{aligned}
\widetilde{F}(t, x) &:= F(t, x) + p(t), \\
\widetilde{M}_{\lambda,\sigma}(t) &:= \widetilde{F}(t, \phi(x_{\sigma,\lambda})(t)), \qquad t \in I, \\
\widetilde{T}_\lambda(\sigma) &:= \{\psi(\cdot) \in L^p(I, E); \psi(t) \in \widetilde{M}_{\lambda,\sigma}(t) \ a.e.\,(I)\}.
\end{aligned}
$$

It is obvious that $\widetilde{F}(\cdot,\cdot)$ satisfies Hypothesis 3.1.
Let $\phi \in T_\lambda(\sigma), \delta > 0$ and define

$$
\mathcal{G}_1(t) := \widetilde{M}_{\lambda,\sigma(t)} \cap \Big\{z \in X : \|\phi(t) - z\|^p \le M_2\ \varphi(t)\|\lambda - \mu\|_C^p + p(t) + \delta\Big\}.
$$

Using the same argument as used for the set valued map $\mathcal{G}(\cdot)$, we deduce that $\mathcal{G}_1(\cdot)$ is measurable with nonempty closed values.
Next, we prove the following estimate:

$$
H^+(T_\lambda(\sigma), \widetilde{T}_\mu(\sigma)) \le \frac{1}{\beta^{\frac{1}{p}} M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}}\|\lambda - \mu\|_C + \Big(\int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t)dt\Big)^{\frac{1}{p}}. \tag{3.11}
$$

Let $\psi(\cdot) \in T_\mu(\sigma)$. Then

$$
\begin{aligned}
\|\phi - \psi\|_p^p &= \int_0^T e^{-\beta M_1 M_2 M_3 m(t)}\|\phi(t) - \psi(t)\|^p dt \\
&\le \int_0^T e^{-\beta^{-1} M_1 M_2 M_3 m(t)}[M_2\varphi(t)\|\lambda - \mu\|_C^p + p(t) + \delta]dt \\
&\le M_2\|\lambda - \mu\|_C^p \int_0^T e^{-\beta M_1 M_2 M_3 m(t)}\varphi(t)dt \\
&\quad + \int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t)dt + \delta\int_0^T e^{-\beta M_1 M_2 M_3 m(t)}dt \\
&\le \frac{1}{\beta M_1 M_3}\|\lambda - \mu\|_C^p + \int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t)dt \\
&\quad + \delta\int_0^T e^{-\beta M_1 M_2 M_3 m(t)}dt.
\end{aligned}
$$

Because δ is arbitrary, so letting $\delta \to 0$ we deduce from the above inequality that

$$\|\phi - \psi\|_p^p \leq \frac{1}{\beta M_1 M_3} \|\lambda - \mu\|_C^p + \int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t) dt.$$

Thus, by taking $\frac{1}{p}$th power on both sides of the above inequality breaking the right hand side, one obtains (3.11).
Now applying Proposition 2.9 we obtain

$$\begin{aligned} H^+(Fix(T_\lambda), Fix(\widetilde{T}_\mu)) \leq & \frac{\beta^{1/q}}{(\beta-1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C \\ & + \frac{\beta}{\beta - 1} \Big(\int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t) dt \Big)^{\frac{1}{p}}. \end{aligned}$$

Because $v(\cdot) \in Fix(\widetilde{T}_\mu)$, it follows that there exists $u(\cdot) \in Fix(T_\mu)$ such that

$$\|v - u\|_p \leq \frac{\beta^{1/q}}{(\beta-1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C + \frac{\beta}{\beta - 1} \Big(\int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t) dt \Big)^{\frac{1}{p}}. \quad (3.12)$$

We define

$$x(t) = \lambda(t) + \int_0^T k(t,s)\, g(t,s,u(s))\, ds.$$

Then we have the following inequality:

$$\begin{aligned} \|x(t) - y(t)\| &\leq \|\lambda(t) - \mu(t)\| + M_1 M_3 \int_0^T \|u(s) - v(s)\|^p \, ds \\ &\leq \|\lambda - \mu\|_C + M_1 M_3 e^{\beta M_1 M_2 M_3 L} \|u - v\|_p^p. \end{aligned}$$

Combining the last inequality with (3.12) we obtain

$$\begin{aligned} \|x(t) - y(t)\| \leq \|\lambda - \mu\|_C + M_1 M_3 e^{\beta M_1 M_2 M_3 L} \Big[& \frac{\beta^{1/q}}{(\beta-1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C \\ + \frac{\beta}{\beta - 1} \Big(\int_0^T e^{-\beta M_1 M_2 M_3 m(t)} p(t) dt \Big)^{\frac{1}{p}} \Big]^p. & \end{aligned}$$

This completes the proof.

Pathak and Shahzad [13, Theorem 4.4] is a particular case of our Theorem 3.4.

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