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ON THE SU-YAO THEOREM

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Abstract. Su and Yao [Fixed Point Theory Appl. 2015:120 (2015)] have proved a fixed point theorem for mappings in metric spaces satisfying a general contraction condition. In their paper numerous examples of important consequences of this theorem are given. Our main aim is to present an extension of the Su-Yao theorem to the case of dislocated metric spaces. The proof is short, the result is stronger also for metric spaces, and the theorem itself is a natural and elegant extension of the celebrated Banach fixed point theorem.
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1. Introduction

The celebrated fixed point theorem of Banach was generalized in many ways. The authors were interested in more advanced conditions presented in the inequality and/or they were interested in relaxing the assumptions concerning the space itself. The amount of papers devoted to the item is enormous. First of all let us note [15] with theorems for partial metric spaces, and [1] where the results for ordered metric spaces are presented. In both papers the "inequality" conditions are much advanced. Our paper is not so much ambitious as we develop more traditional ideas initiated by Boyd and Wong [2] and extended by Su and Yao [16].Instead of metric space we consider dislocated metric space from [3]. We apologize that our list of references is short and that the main ideas of those papers are not presented here. Nevertheless, we hope that the citations are satisfactory also for the most demanding reader.

2. Preliminaries

Su and Yao have proved a theorem [16, Theorem 2.1] which can be formulated as follows:

Theorem 2.1. *Let (X, d) be a complete metric space, and let $T\colon X \to X$ be a mapping such that*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)), \quad x, y \in X,$$

where $\varphi, \psi\colon [0,\infty) \to [0,\infty)$ *are mappings satisfying conditions:*

$$\psi(a) \le \varphi(b) \text{ implies } a \le b, \quad a, b \ge 0, \tag{1a}$$

$$a_n, b_n \to \alpha \text{ and } \psi(a_n) \le \varphi(b_n),\ n \in \mathbb{N}, \text{ imply } \alpha = 0. \tag{1b}$$

Then T *has a unique fixed point* x, *and* $T^n x_0 \to x$, $x_0 \in X$.

Our aim is to relax the assumptions of the above theorem and to prove shortly the respective version for the case of dislocated metric spaces (see Theorem 3.3). Some more advanced (or sophisticated) results are also obtained. In the final part of our paper Theorem 3.3 is extended to the case of dq-spaces (see Theorem 4.2).

At first, let us recall some basic notions.

A mapping $p\colon X \times X \to [0,\infty)$ is a *dq-metric* (or *dislocated quasi-metric*) [17], if the following conditions are satisfied:

$$p(x,y) = p(y,x) = 0 \text{ yields } x = y, \quad x, y \in X, \tag{2a}$$

$$p(x,z) \le p(x,y) + p(y,z), \quad x, y, z \in X. \tag{2b}$$

Then, (X,p) is a *dq-metric space* (its topology is generated by balls $B(x,r) = \{y \in X\colon p(x,y) < r\}$).

A dq-metric space (X,p) is 0-*complete* (see [12, Corollary 1.5]) if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, there exists an x for which $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x, x_n) = 0$ (with (2b) yields $p(x,x) = 0$).

A mapping $p\colon X \times X \to [0,\infty)$ is called a *d-metric* (or *dislocated metric*) [3], if the following conditions are satisfied:

$$p(x,y) = 0 \text{ yields } x = y, \quad x, y \in X, \tag{3a}$$

$$p(x,y) = p(y,x), \quad x, y \in X, \tag{3b}$$

$$p(x,z) \le p(x,y) + p(y,z), \quad x, y, z \in X. \tag{3c}$$

Then, (X,p) is a *d-metric space* (its topology is generated by balls).

Clearly, each d-metric is a dq-metric, and each d-metric space is a dq-metric space.

A d-metric space (X,p) is 0-*complete* [9, Definition 2.3], if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, there exists an $x \in X$ for which $\lim_{n\to\infty} p(x_n, x) = 0$ (with (3c) yields $p(x,x) = 0$).

A mapping $p\colon X \times X \to [0,\infty)$ is a *partial metric* [6, Definition 3.1], if the following conditions are satisfied:

$$x = y \text{ iff } p(x,x) = p(x,y) = p(y,y), \quad x, y \in X, \tag{4a}$$

$$p(x,x) \le p(x,y), \quad x, y \in X, \tag{4b}$$

$$p(x,y) = p(y,x), \quad x, y \in X, \tag{4c}$$

$$p(x,z) \le p(x,y) + p(y,z) - p(y,y), \quad x, y, z \in X. \tag{4d}$$

Then, (X,p) is a *partial metric space* (its topology is generated by "balls" $B(x,r) = \{y \in X\colon p(x,y) < p(x,x) + r\}$). From (4b) it follows that $p(x,y) = 0$ implies $p(x,x) = p(y,y) = 0$ and $x = y$ (see (4a)). Consequently, each partial metric is a d-metric (see (3a)).

Clearly, each metric is a partial metric, and each metric space is a partial metric space and a d-space. The convergence of sequences in partial metric spaces is discussed in details e.g. in [7].

A partial-metric space (X, p) is 0-*complete* (cp. [13, Definition 2.1]), if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, there exists an $x \in X$ for which $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$.

The next definition is a formal extension of [11, Definition 2.10] to include the cases of dq-metric spaces and partial metric space.

Definition 2.2. A self mapping f on a set X equipped with a dq-metric p is 0-continuous at x, if from $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x, x_n) = 0$ it follows that $\lim_{n\to\infty} p(fx_n, fx) = \lim_{n\to\infty} p(fx, fx_n) = 0$; we say that f is 0-continuous if it is 0-continuous at each $x \in X$.

3. New results for d-metric and partial metric

We write $\psi \succ \varphi$ if the following conditions are satisfied

$$\varphi, \psi\colon (0,\infty) \to [0,\infty) \text{ are mappings,} \tag{5a}$$

$$\alpha \geq \beta \text{ yields } \psi(\alpha) > \varphi(\beta), \quad \alpha, \beta > 0, \tag{5b}$$

$$\begin{aligned} &\text{for each } \alpha > 0 \text{ there exists an } \epsilon > 0 \text{ such that} \\ &\psi(s) > \varphi(t), \quad s, t \in (\alpha, \alpha + \epsilon). \end{aligned} \tag{5c}$$

In view of (5b), it is sufficient to consider $\alpha < s < t < \alpha + \epsilon$ in (5c).

Su and Yao assumed that φ, ψ are self mappings on $[0,\infty)$. From (1b) for $a_n = b_n = \alpha > 0$, $n \in \mathbb{N}$, we get $\psi(\alpha) > \varphi(\alpha)$, $\alpha > 0$. Clearly, (1a) is equivalent to $\alpha > \beta \geq 0$ implies $\psi(\alpha) > \varphi(\beta)$. Therefore, (1a) and (1b) yield (5b). If (5c) is not satisfied, then for some $\alpha > 0$ and all $n \in \mathbb{N}$ there exist $a_n, b_n \in (\alpha, \alpha + 1/n)$ such that $\psi(a_n) \leq \varphi(b_n)$. Consequently, $a_n, b_n \to \alpha$ holds, and (1b) cannot be satisfied. Therefore, (1b) yields (5c). Now, it is clear, that our system of conditions (5) is less restrictive than the one of Su and Yao (see also Remark 3.5).

Proposition 3.1. *If $\psi \succ \varphi$, $(a_n)_{n\in\mathbb{N}}$ is a nonnegative sequence, and*

$$a_{n+1} > 0 \text{ yields } \psi(a_{n+1}) \leq \varphi(a_n), \quad n \in \mathbb{N} \tag{6}$$

holds, then $(a_n)_{n\in\mathbb{N}}$ is nonincreasing and $a_n \to 0$.

Proof. If $a_n = 0$, then $a_{n+1} > 0$ contradicts (6). Consequently, we obtain $a_k = 0$, $k \geq n$. Assume $a_n > 0$, $n \in \mathbb{N}$. Then, from (6), (5b) it follows that $(a_n)_{n\in\mathbb{N}}$ decreases to some α. Suppose $\alpha > 0$. Then, $a_n < \alpha + \epsilon$ for large n, and (6) contradicts (5c). □

Lemma 3.2. *Let (X, p) be a d-metric (partial metric) space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of X such that for some $\psi \succ \varphi$, each $\alpha > 0$, and appropriately large $m, n \in \mathbb{N}$ the following conditions are satisfied*

$$p(x_{n+2}, x_{n+1}) > 0 \text{ yields } \psi(p(x_{n+2}, x_{n+1})) \leq \varphi(p(x_{n+1}, x_n)), \tag{7a}$$

$$p(x_{n+1}, x_{m+1}) > \alpha \text{ yields } \psi(p(x_{n+1}, x_{m+1})) \leq \varphi(p(x_n, x_m)). \tag{7b}$$

Then $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, and if (X, p) is 0-complete, then there exists an x such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$.

Proof. From Proposition (3.1) when applied to $a_n = p(x_{n+1}, x_n)$ for $n > n_0$, it follows that $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$. Suppose $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ is false. Then there exist $\alpha > 0$ and an infinite set $\mathbb{K} \subset \mathbb{N}$, such that for all $k \in \mathbb{K}$ there exist $n \in \mathbb{N}$ for which $\alpha < p(x_{k+n+1}, x_{k+1})$ holds. Let $n = n(k)$ be the smallest such number. We have (see (7b))

$$\psi(p(x_{k+n+1}, x_{k+1})) \le \varphi(p(x_{k+n}, x_k)),$$

and (see (5b))

$$\alpha < p(x_{k+n+1}, x_{k+1}) < p(x_{k+n}, x_k) \le$$
$$p(x_{k+n}, x_{k+1}) + p(x_{k+1}, x_k) \le \alpha + p(x_{k+1}, x_k) \to \alpha.$$

Consequently, we get $\lim_{k\in\mathbb{K}} p(x_{k+n}, x_k) = \alpha$, and

$$\alpha < p(x_{k+n+1}, x_{k+1}) < p(x_{k+n}, x_k) < \alpha + \epsilon$$

for large k. Now, (7b) contradicts (5c).
Therefore, $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, and if (X, p) is 0-complete, then there exists an x such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$. □

Now, we are ready to prove our main result. Also the contraction condition is more general than the one used by Su and Yao.

Theorem 3.3. *Assume that f is a self mapping on a d-metric (partial metric) space (X, p), $\psi \succ \varphi$, and the following condition is satisfied*

$$p(fy, fx) > 0 \text{ yields } \psi(p(fy, fx)) \le \varphi(p(y, x)), \quad x, y \in X. \tag{8}$$

Then f has at most one fixed point, and $fx = x$ yields

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0, \ x_0 \in X.$$

If (X, p) is 0*-complete, then f has a fixed point.*

Proof. Suppose x, y are two fixed points of f. Then we have (see (8))

$$\psi(p(y, x)) = \psi(p(fy, fx)) \le \varphi(p(y, x))$$

which contradicts (5b). Therefore, x is unique.

Lemma 3.2 applies to $x_n = f^n x_0$, $n \in \mathbb{N}$. Consequently, if (X, p) is 0-complete, then there exists a point x such that $\lim_{n\to\infty} p(x_n, x) = 0$. Now, it is sufficient to prove that $p(fx, x) = 0$. We have

$$p(fx, x) \le p(fx, x_{n+1}) + p(x_{n+1}, x) = p(fx, fx_n) + p(x_{n+1}, x).$$

If $p(fx, fx_n) > 0$ holds, then we get $\psi(p(fx, fx_n)) \le \varphi(p(x, x_n))$ (see (8)), and $p(fx, fx_n) < p(x, x_n)$ (see (5b)). Now, $p(fx, x) \le p(x, x_n) + p(x_{n+1}, x) \to 0$ means that $p(fx, x) = 0$, and x is a fixed point. □

Let us consider $\varphi, \psi \colon (0, \infty) \to [0, \infty)$ such that $\psi = id$, $\varphi(\alpha) < \alpha$, $\alpha > 0$, and for each $\alpha > 0$ $\varphi(\cdot) \le \alpha$ on some interval $(\alpha, \alpha + \epsilon)$. Clearly, (5b) is satisfied. Let $\alpha > 0$ be arbitrary. Then, $\psi(\alpha, \alpha + \epsilon) = (\alpha, \alpha + \epsilon)$, and for some small $\epsilon > 0$ we get $\varphi(\alpha, \alpha + \epsilon) \subset [0, \alpha]$, i.e. (5c) holds.

Corollary 3.4. *Theorem 3.3 is an extension of* [10, Theorem 3.1 (6)] *which in turn is a generalization of the Boyd-Wong theorem* [2].

Remark 3.5. Let us consider $\psi(\alpha) = \alpha$, $\alpha \geq 0$, and $\varphi(\alpha) = \alpha^2$ for $0 \leq \alpha < 1$, $\varphi(1) = 1/2$, and $\varphi(\alpha) = 1$ for $\alpha > 1$. It can be seen that (1b) is not satisfied (consider $a_n = {b_n}^2 < 1$, $b_n \to 1$), while (5b), (5c) hold. Therefore, [10, Theorem 3.1 (6)] cannot be derived from the Su-Yao theorem.

Proposition 3.6. *Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be sequences of points in a d-metric (partial metric) space (X, p) such that*

$$\lim_{n\to\infty} p(x_n, y_n) = \lim_{n\to\infty} p(x_{n+1}, x_n) = 0.$$

In addition, assume that for an infinite set $\mathbb{K} \subset \mathbb{N}$, and $n = n(k) \in \mathbb{N}$, $k \in \mathbb{K}$, we have $\lim_{k\in\mathbb{K}} p(x_{k+n}, y_k) = \alpha$. Then for any fixed $t \in \mathbb{N}$, and all $i, j = 0, \pm1, \ldots, \pm t$ the following conditions are satisfied

$$\lim_{m\to\infty} p(x_{m+i}, y_{m+j}) = 0, \tag{9a}$$

$$\lim_{k\in\mathbb{K}} p(x_{k+n+i}, y_{k+j}) = \alpha. \tag{9b}$$

Proof. Clearly, $\lim_{n\to\infty} p(x_{m+i}, x_{m+j}) = 0$ holds (see (3c)). Now,

$$p(x_{m+j}, y_{m+j}) - p(x_{m+j}, x_{m+i}) \leq p(x_{m+i}, y_{m+j}) \leq p(x_{m+i}, x_{m+j}) + p(x_{m+j}, y_{m+j})$$

yields (9a). In turn, (9a) and

$$p(x_{k+n}, y_k) - p(x_{k+n}, x_{k+n+i}) - p(y_{k+j}, x_k) - p(x_k, y_k) \leq p(x_{k+n+i}, y_{k+j})$$

$$\leq p(x_{k+n+i}, x_{k+n}) + p(x_{k+n}, y_k) + p(y_k, x_k) + p(x_k, y_{k+j})$$

yield (9b). □

Let us recall the notion of a cyclic mapping. The definition was formalized by Rus [14], while the idea itself is due to [5]. We adopt the notations from [10, Definition 2.5], because they are convenient and the case of one set ($t = 1$) is included.

For a $t \in \mathbb{N}$ we put $t{+}{+} = 1$, and $j{+}{+} = j + 1$, for $j \in \{1, \ldots, t-1\}$. Then $f\colon X \to X$ is *cyclic* if $X = X_1 \cup \cdots \cup X_t$, and $f(X_j) \subset X_{j++}$, $j = 1, \ldots, t$.

For a self mapping f on a d-metric (partial metric) space (X, p) we put

$$m_f(y, x) = \max\{p(y, x), p(fy, y), p(fx, x)\}, \quad x, y \in X.$$

Theorem 3.7. *Assume that f is a cyclic self mapping on a d-metric (partial metric) space (X, p), $\psi \succ \varphi$, and the following condition is satisfied*

$$\begin{gathered} p(fy, fx) > 0 \text{ yields } \psi(p(fy, fx)) \leq \varphi(m_f(y, x)), \\ x \in X_j,\ y \in X_{j++},\ j = 1, \ldots, t. \end{gathered} \tag{10}$$

Then f has at most one fixed point, and $fx = x$ yields

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0,\ x_0 \in X.$$

If (X, p) (or some X_j) is 0-complete, then for each $x_0 \in X$ there exists an $x \in X$ such that $\lim_{n\to\infty} p(f^n x_0, x) = p(x, x) = 0$; if f is 0-continuous at such an x, then $fx = x$.

Proof. Let x be a fixed point of f, and suppose $p(fx, x) > 0$. Then (see (10))

$$\psi(p(x,x)) = \psi(p(fx,fx)) \le \varphi(m_f(x,x)) = \varphi(p(x,x))$$

contradicts (5b). Therefore, we have $p(fx, x) = 0$. Now, if x, y are fixed points of f, then $m_f(y, x) = p(y, x)$, and as in the proof of Theorem 3.3 we get $x = y$.

We put $x_n = f^n x_0$, $n \in \mathbb{N}$. Then, (10), (5b) imply

$$\begin{aligned} &p(x_{n+2}, x_{n+1}) > 0 \textit{ yields } \psi(p(x_{n+2}, x_{n+1})) \le \varphi(m_f(x_{n+1}, x_n)) \\ &= \varphi(\max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\}) = \varphi(p(x_{n+1}, x_n)). \end{aligned}$$

Now, in view of Proposition 3.1 we get $a_n = p(x_{n+1}, x_n) \to 0$.

Suppose that for an infinite subset $\mathbb{K}$ of $\mathbb{N}$ and each $k \in \mathbb{K}$ there exist $n \in \mathbb{N}$ for which $0 < \alpha < p(x_{(n+1)t+k+2}, x_{k+1})$ holds. Let $n = n(k)$ be the smallest such number. In view of (9a), for m replaced by $nt + k + 2$, we have

$$\begin{aligned} \alpha &< p(x_{(n+1)t+k+2}, x_{k+1}) \le p(x_{(n+1)t+k+2}, x_{nt+k+2}) + p(x_{nt+k+2}, x_{k+1}) \\ &\le p(x_{(n+1)t+k+2}, x_{nt+k+2}) + \alpha \to \alpha. \end{aligned}$$

Thus, $\lim_{k\in\mathbb{K}} p(x_{(n+1)t+k+2}, x_{k+1}) = \lim_{k\in\mathbb{K}} p(x_{nt+k+2}, x_{k+1}) = \alpha$ holds. Now, for $y_k = x_{k+1}$ (9b) yields $\lim_{k\in\mathbb{K}} p(x_{(n+1)t+k+1}, x_k) = \alpha$, and therefore, we have $m_f(x_{(n+1)t+k+1}, x_k) = p(x_{(n+1)t+k+1}, x_k)$ for large k. In turn, from (10) it follows that

$$\psi(p(x_{(n+1)t+k+2}, x_{k+1}) \le \varphi(p(x_{(n+1)t+k+1}, x_k)),$$

which contradicts (5b) or (5c) for large k. Thus, $\lim_{k,n\to\infty} p(x_{nt+k+2}, x_{k+1}) = 0$ holds, and therefore, $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ (see (9b)).

If (X, p) (or some X_j) is 0-complete, then for each $x_0 \in X$ there exists an x such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$. If f is 0-continuous at such an x, then

$$p(fx, x) \le p(fx, x_{n+1}) + p(x_{n+1}, x)$$

and $\lim_{n\to\infty} p(x_n, x) = 0$ yield $p(fx, x) = 0$.

Now, assume $p(fy, y) = 0$, $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, and suppose $p(y, x_n) > \alpha > 0$, $n \in \mathbb{N}$. Then, from (10) we get

$$\begin{aligned} &\psi(p(y, x_{n+1})) = \psi(p(fy, x_{n+1})) \le \varphi(m_f(y, x_n)) = \\ &\varphi(\max\{p(y, x_n), p(fy, y), p(x_{n+1}, x_n)\}) = \varphi(p(y, x_n)) \end{aligned}$$

for large n. Therefore, we have $a_n = p(y, x_n) \to 0$ (see Proposition 3.1), a contradiction. Consequently, there exists a subsequence of $(p(y, x_n))_{n\in\mathbb{N}}$, which is convergent to 0. This fact, (3c), and $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ yield $\lim_{n\to\infty} p(y, x_n) = 0$, and $p(y, f^n x_0) \to 0$ for all $x_0 \in X$. □

Theorem 3.3 has the following "cyclic" extension. The proof of Theorem 3.7 for m_f, (10) replaced by p, (11), respectively, becomes a proof of Theorem 3.8 ("$\varphi(\max\{\dots\}) =$" should be then omitted).

Theorem 3.8. *Assume that f is a cyclic self mapping on a d-metric (partial metric) space (X, p), $\psi \succ \varphi$, and the following condition is satisfied*

$$\begin{aligned} &p(fy, fx) > 0 \textit{ yields } \psi(p(fy, fx)) \le \varphi(p(y, x)), \\ &x \in X_j,\ y \in X_{j++},\ j = 1, \dots, t. \end{aligned} \tag{11}$$

Then f has at most one fixed point, and $fx = x$ yields

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0,\ x_0 \in X.$$

If (X, p) (or some X_j) is 0-complete, then f has a fixed point.

For a selfmapping f on a d-metric (partial metric) space (X, p) we put

$$\begin{aligned} c_f(y, x) &= \max\{m_f(y, x), [p(fy, x) + p(fx, y)]/2\} \\ &= \max\{p(y, x), p(fy, y), p(fx, x), [p(fy, x) + p(fx, y)]/2\}, \quad x, y \in X. \end{aligned}$$

Theorem 3.9. *Assume that f is a cyclic self mapping on a partial metric space (X, p), $\psi \succ \varphi$, and the following condition is satisfied*

$$\begin{aligned} &p(fy, fx) > 0 \text{ yields } \psi(p(fy, fx)) \le \varphi(c_f(y, x)), \\ &x \in X_j,\ y \in X_{j++},\ j = 1, \ldots, t. \end{aligned} \tag{12}$$

Then f has at most one fixed point, and $fx = x$ yields

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0,\ x_0 \in X.$$

If (X, p) (or some X_j) is 0-complete, then for each $x_0 \in X$ there exists an $x \in X$ such that $\lim_{n\to\infty} p(f^n x_0, x) = p(x, x) = 0$*; if f is 0-continuous at such an x, then $fx = x$.*

Proof. If x, y are fixed points of f, then $c_f(y, x) = p(y, x)$ (see (4b)) and, as in the initial part of the proof of Theorem 3.3, we get $p(y, x) = 0$. Now, for $x_n = f^n x_0$, $n \in \mathbb{N}$, we have (see (4d))

$$\begin{aligned} &[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})]/2 \\ &\le [p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1}) + p(x_{n+1}, x_{n+1})]/2 \\ &= [p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]/2 \le \max\{p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_n)\}. \end{aligned}$$

Therefore, the following holds

$$\begin{aligned} c_f(x_{n+1}, x_n) &= \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_n), [p(x_{n+2}, x_n) \\ &\quad + p(x_{n+1}, x_{n+1})]/2\} \\ &= \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\}, \end{aligned}$$

and from (12), (5b) it follows that

$$\begin{aligned} &p(x_{n+2}, x_{n+1}) > 0 \text{ yields } \psi(p(x_{n+2}, x_{n+1})) \\ &\le \varphi(\max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\}) = \varphi(p(x_{n+1}, x_n)). \end{aligned}$$

Now, in view of Proposition 3.1 we get $a_n = p(x_{n+1}, x_n) \to 0$.

In what follows we apply the triangle inequality (3c) instead of (4d).

Suppose that for an infinite subset $\mathbb{K}$ of $\mathbb{N}$ and each $k \in \mathbb{K}$ there exist $n \in \mathbb{N}$ for which $0 < \alpha < p(x_{(n+1)t+k+2}, x_{k+1})$ holds. Let $n = n(k)$ be the smallest such number. Then in view of (9a), for m replaced by $nt + k + 2$, we obtain

$$\begin{aligned} \alpha &< p(x_{(n+1)t+k+2}, x_{k+1}) \le p(x_{(n+1)t+k+2}, x_{nt+k+2}) + p(x_{nt+k+2}, x_{k+1}) \\ &\le p(x_{(n+1)t+k+2}, x_{nt+k+2}) + \alpha \to \alpha, \end{aligned}$$

and $\lim_{k\in\mathbb{K}} p(x_{(n+1)t+k+2}, x_{k+1}) = \lim_{k\in\mathbb{K}} p(x_{nt+k+2}, x_{k+1}) = \alpha$. As regards

$$c_f(x_{(n+1)t+k+1}, x_k) = \max\{p(x_{(n+1)t+k+1}, x_k), p(x_{(n+1)t+k+2}, x_{(n+1)t+k+1}), p(x_{k+1}, x_k), [p(x_{(n+1)t+k+2}, x_k) + p(x_{k+1}, x_{(n+1)t+k+1})]/2\},$$

from (9b) for $y_k = x_{k+1}$ we get $\lim_{k\in\mathbb{K}} c_f(x_{(n+1)t+k+1}, x_k) = \alpha$. Now, (12) yields

$$\psi(p(x_{(n+1)t+k+2}, x_{k+1})) \le \varphi(c_f(x_{(n+1)t+k+1}, x_k)),$$

which contradicts (5b) or (5c) for large k. Thus, $\lim_{k,n\to\infty} p(x_{nt+k+2}, x_{k+1}) = 0$ holds, and therefore, $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ (see (9b)).

If (X, p) (or some X_j) is 0-complete, then for each $x_0 \in X$ there exists an x such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$. If f is 0-continuous at such an x, then

$$p(fx, x) \le p(fx, x_{n+1}) + p(x_{n+1}, x)$$

and $\lim_{n\to\infty} p(x_n, x) = 0$ yield $p(fx, x) = 0$.

Now, assume $p(fy, y) = 0$, $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, and suppose $p(y, x_n) > \alpha > 0$, $n \in \mathbb{N}$. Then, (12) implies

$$\begin{aligned}\psi(p(y, x_{n+1})) &= \psi(p(fy, x_{n+1})) \le \varphi(c_f(y, x_n))\\ &= \varphi(\max\{p(y, x_n), p(fy, y), p(x_{n+1}, x_n), [p(fy, x_n) + p(x_{n+1}, y)]/2\})\\ &= \varphi(\max\{p(y, x_n), [p(y, x_n) + p(x_{n+1}, y)]/2\})\end{aligned}$$

for large n. In view of (5b) the following inequality holds for large n

$$p(y, x_{n+1}) < \max\{p(y, x_n), [p(y, x_n) + p(y, x_{n+1})]/2\} = p(y, x_n).$$

Therefore, we have $a_n = p(y, x_n) \to 0$ (see Proposition 3.1), a contradiction. Consequently, there exists a subsequence of $(p(y, x_n))_{n\in\mathbb{N}}$, which is convergent to 0. This fact, (3c), and $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ prove that $\lim_{n\to\infty} p(y, x_n) = 0$, and $p(y, f^n x_0) \to 0$ for all $x_0 \in X$. □

From Theorem 3.7 for $t = 1$ we obtain

Theorem 3.10. *Assume that f is a self mapping on a d-metric (partial metric space) (X, p), $\psi \succ \varphi$, and the following condition is satisfied*

$$p(fy, fx) > 0 \text{ yields } \psi(p(fy, fx)) \le \varphi(m_f(y, x)), \quad x, y \in X.$$

Then f has at most one fixed point, and $fx = x$ yields

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx, x) = 0, \; x_0 \in X.$$

If (X, p) is 0-complete, then for each $x_0 \in X$ there exists an x such that

$$\lim_{n\to\infty} p(f^n x_0, x) = p(x, x) = 0;$$

if f is 0-continuous at such an x, then $fx = x$.

Now, [10, Theorem 3.1 (7)] is a consequence of Theorem 3.10 for φ, ψ preceding Corollary 3.4.

In turn, Theorem 3.9 for $t = 1$ yields

Theorem 3.11. *Assume that f is a self mapping on a partial metric space (X,p), $\psi \succ \varphi$, and the following condition is satisfied*

$$p(fy,fx) > 0 \text{ yields } \psi(p(fy,fx)) \le \varphi(c_f(y,x)), \quad x,y \in X.$$

Then f has at most one fixed point, and $fx = x$ yields

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx,x) = 0, \; x_0 \in X.$$

If (X,p) is 0-complete, then for each $x_0 \in X$ there exists an x such that

$$\lim_{n\to\infty} p(f^n x_0, x) = p(x,x) = 0;$$

if f is 0-continuous at such an x, then $fx = x$.

Theorems 3.3, 3.10, and also Theorems 3.8, 3.7 (for the related m,n) are far consequences of the following result.

Theorem 3.12. *Assume that f is a self mapping on a 0-complete d-metric (partial metric) space (X,p), $\psi \succ \varphi$, and the system of conditions (7) holds for $x_n = f^n x_0$, $n \in \mathbb{N}$, each $\alpha > 0$, and appropriately large $m,n \in \mathbb{N}$. Then there exists a point x such that $\lim_{n\to\infty} p(x_n,x) = p(x,x) = 0$; if f is 0-continuous at x, then $p(fx,x) = 0$.*

Proof. From Lemma 3.2 it follows that $\lim_{n\to\infty} p(x_n,x) = p(x,x) = 0$ for some $x \in X$. Now,

$$p(fx,x) \le p(fx,x_{n+1}) + p(x_{n+1},x) = p(fx,fx_n) + p(x_{n+1},x)$$

yields $p(fx,x) = 0$, as f is 0-continuous at x. □

4. Addendum: new results for dq-metric, and final remarks

Lemma 3.2 has the following "dq-metric" version.

Lemma 4.1. *Let (X,p) be a dq-metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of X such that for some $\psi \succ \varphi$, each $\alpha > 0$, and appropriately large $m,n \in \mathbb{N}$ the following conditions are satisfied*

$$p(x_{n+1},x_{n+2}) > 0 \text{ yields } \psi(p(x_{n+1},x_{n+2})) \le \varphi(p(x_n,x_{n+1})), \tag{13a}$$

$$p(x_{n+2},x_{n+1}) > 0 \text{ yields } \psi(p(x_{n+2},x_{n+1})) \le \varphi(p(x_{n+1},x_n)), \tag{13b}$$

$$p(x_{n+1},x_{m+1}) > \alpha \text{ yields } \psi(p(x_{n+1},x_{m+1})) \le \varphi(p(x_n,x_m)) \tag{13c}$$

Then $\lim_{m,n\to\infty} p(x_n,x_m) = 0$, and if (X,p) is 0-complete, then there exists an x such that $\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x,x_n) = p(x,x) = 0$.

Proof. The reasoning from the proof of Lemma 3.2 for (13b), (13c) (i.e. (7)) yields $\lim_{n\ge m\to\infty} p(x_n,x_m) = 0$; its symmetric version for (13a), (13c), $a_n = p(x_n,x_{n+1})$ and $\alpha < p(x_{k+1},x_{k+n+1})$ proves that $\lim_{m\ge n\to\infty} p(x_n,x_m) = 0$. Consequently, $\lim_{m,n\to\infty} p(x_n,x_m) = 0$ holds. □

Theorem 3.3 has the following "dq-metric" extension.

Theorem 4.2. *Assume that f is a self mapping on a dq-metric space (X,p), $\psi \succ \varphi$, and (8) is satisfied. Then f has at most one fixed point, and $fx = x$ yields*

$$\lim_{n\to\infty} p(f^n x_0, x) = p(fx,x) = 0 = p(x,fx) = \lim_{n\to\infty} p(x, f^n x_0), \; x_0 \in X.$$

If (X,p) is 0-complete, then f has a fixed point.

Proof. Clearly, f has at most one fixed point (see the proof of Theorem 3.3). We apply Lemma 4.1 to $x_n = f^n x_0$, $n \in \mathbb{N}$, and for x such that

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x, x_n) = 0$$

we obtain $p(fx, x) = 0$ as in the proof of Theorem 3.3, and $p(x, fx) = 0$ in a symmetric way. □

Also Theorem 3.12 has a "dq-metric" extension.

Theorem 4.3. *Assume that f is a self mapping on a 0-complete dq-metric space (X, p), $\psi \succ \varphi$, and the the system of conditions (13) holds for $x_n = f^n x_0$, $n \in \mathbb{N}$, each $\alpha > 0$, and appropriately large $m, n \in \mathbb{N}$. Then there exists a point x such that*

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x, x_n) = p(x, x) = 0;$$

if f is 0-continuous at x, then $p(fx, x) = p(x, fx) = 0$.

Proof. We apply Lemma 4.1. The final part of our theorem can be obtained as in the proof of Theorem 3.12 (also use the symmetric reasoning). □

Remark 4.4. The proofs of our theorems work also for the f orbitally 0-complete spaces (the respective sequences consist of points of any orbit of f). Therefore, all theorems of the present paper stay valid if we replace "0-complete" by "f orbitally 0-complete".

Remark 4.5. If g is a self mapping on X, and $f = g^s$ has a unique fixed point, then g has a unique fixed point, and moreover, $\lim_{n\to\infty} p(f^n x_0, x) = 0$ yields $\lim_{n\to\infty} p(g^n x_0, x) = 0$ (see [8, Lemma 29]). Consequently, if $f = g^s$ satisfies the assumptions of the respective theorem presented in this paper (see also Remark 4.4), then its conclusion concerns g. Let us note that Theorems 3.12, 4.3 do not ensure f to have a unique fixed point.

Remark 4.6. The final Remark from [4]: "..., it is a reasonable conjecture that every fixed point theorem for a single map defined on a metric space is extendable to the corresponding fixed point theorem on a d-metric space". The idea of Jungck and Rhoades is to remove the assumption $p(x, x) = 0$ from the proofs. Proving new theorems at once for d-metric spaces seems to be more efficient.

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References

[1] H. Aydi, W. Shantawi, M. Postolache, Z. Mustafa, N. Tahat, *Theorems for Boyd-Wong-type contractions in ordered metric spaces*, Abstr. Appl. Anal., **2012**, ID 359054.
[2] D.W. Boyd, J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 458-464.
[3] P. Hitzler, A.K. Seda, *Dislocated topologies*, J. Electr. Engin., **51(12)**(2000), 3-7.
[4] G.F. Jungck, B.E. Rhoades, *General fixed point results in dislocated metric spaces*, Fixed Point Theory, **18(2)**(2017), 615-624.

[5] W.A. Kirk, P.S. Srinivasan, P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4(1)**(2003), 79-89.

[6] S.G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728**(1994), 183-197.

[7] S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste, **36**(2004), 17-26.

[8] L. Pasicki, *Fixed point theorems for contracting mappings in partial metric spaces*, Fixed Point Theory Appl., **2014:185**(2014).

[9] L. Pasicki, *Dislocated metric and fixed point theorems*, Fixed Point Theory Appl., **2015:82**(2015).

[10] L. Pasicki, *The Boyd-Wong idea extended*, Fixed Point Theory Appl., **2016:63**(2016).

[11] L. Pasicki, *Some extensions of the Meir-Keeler theorem*, Fixed Point Theory Appl., **2017:1**(2017).

[12] L. Pasicki, *Dislocated quasi-metric and generalized contractions*, Fixed Point Theory, **19**(2018), no. 1, 359-368.

[13] S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl., art. ID 493298 (2010).

[14] I.A. Rus, *Cyclic representation and fixed points*, Ann. Tiberiu Popoviciu Semin. Funct. Eq. Approx. Convexity, **3**(2005), 171-178.

[15] W. Shantawi, M. Postolache, *Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces*, Fixed Point Theory Appl., **2013:54** (2013).

[16] Y. Su, J.C. Yao, *Further generalized contraction mapping principle and best proximity theorem in metric spaces*, Fixed Point Theory Appl., **2015:120**(2015).

[17] F.M. Zeyada, G.H. Hassan, M.A. Ahmed, *A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces*, Arab. J. Sci. Eng. Sect. A, **31**(2006), no. 1, 111-114.

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