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GENERALIZED LERAY–SCHAUDER NONLINEAR ALTERNATIVES FOR GENERAL CLASSES OF MAPS

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Abstract. New Leray–Schauder nonlinear alternatives are presented. These coincidence type results are established for set–valued maps.
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1. Introduction

In our main section we present a variety of coincidence results of Leray–Schauder type for general classes of maps. The ideas presented are elementary and are based on homotopy methods and on an Urysohn type lemma. Also in this paper we present generalized Furi–Pera type fixed point theorems. Our final section presents a very elementary approach to establishing Leray–Schauder nonlinear alternatives. The results presented in section 2 extend all previously known results in the literature (see [1, 4–9] and the references therein).

2. Main results

In this section we obtain a variety of Leray–Schauder type alternatives for certain classes of maps. First we consider the classes $\mathbf{A}$ and $\mathbf{B}$. Let E be a topological space and U an open subset of E.

Definition 2.1. We say $F \in B(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{B}(\overline{U}, E)$; here 2^E denotes the family of nonempty subsets of E and $\overline{U}$ denotes the closure of U in E. We say $F \in B(E, E)$ if $F : E \to 2^E$ and $F \in \mathbf{B}(E, E)$.

Definition 2.2. We say $F \in MA(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$. We say $F \in MA(E, E)$ if $F : E \to 2^E$ and $F \in \mathbf{A}(E, E)$.

In our next five results we <u>fix</u> a $\Phi \in B(E, E)$. Our first result is a nonlinear alternative motivated in part by [6, 9].

Theorem 2.3. *Let E be a topological space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in MA(E, E)$ and $F \in MA(\overline{U}, E)$. In addition assume*

$$\left\{ \begin{array}{l} \text{there exists a map } H : \overline{U} \times [0,1] \to 2^E \text{ with } \Phi(x) \cap H_t(x) = \emptyset \\ \text{for } x \in \partial U \text{ and } t \in [0,1),\ H_1 = F,\ H_0 = G; \text{ here } H_t(x) = H(x,t) \\ \text{and } \partial U \text{ denotes the boundary of } U \text{ in } E \end{array} \right. \tag{2.1}$$

$$\Omega = \{x \in \overline{U} :\ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is nonempty} \tag{2.2}$$

and there exists a completely regular (respectively normal) subset X of E with

$$\left\{ \begin{array}{l} \Omega \cap X \text{ compact (respectively closed) if } X \\ \text{is completely regular (respectively normal)} \end{array} \right. \tag{2.3}$$

$$\left\{ \begin{array}{l} \Phi(x) \cap G(x) = \emptyset \text{ for } x \in X \setminus (\overline{U \cap X}^{X}) \text{ (here} \\ \overline{U \cap X}^{X} = \overline{U \cap X} \cap X \text{ denotes the closure of } U \cap X \text{ in } X) \end{array} \right. \tag{2.4}$$

$$\left\{ \begin{array}{l} \text{for any map } J \in MA(X,X) \text{ there exists } x \in X \\ \text{with } J(x) \cap \Phi(x) \neq \emptyset \end{array} \right. \tag{2.5}$$

and

$$\left\{ \begin{array}{l} \text{for any continuous map } \mu : X \to [0,1] \text{ with} \\ \mu(\partial_X (U \cap X)) = 0 \text{ and } \mu(\Omega \cap X) = 1 \text{ the map} \\ J_\mu \in MA(X,X) \text{ where } J_\mu(x) = \left\{ \begin{array}{l} H(x,\mu(x)),\ x \in \overline{U \cap X} \cap X \\ G(x),\ x \in X \setminus (\overline{U \cap X} \cap X) \end{array} \right. \\ \text{and } \partial_X (U \cap X) \text{ denotes the boundary of } U \cap X \text{ in } X. \end{array} \right. \tag{2.6}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$ (otherwise we are finished). Let Ω be as in (2.2) and note $\Omega \neq \emptyset$. Next notice $\Omega \cap X$ and $\partial_X (U \cap X)$ are disjoint. To see this first note

$$\begin{aligned} \partial_X (U \cap X) &= (\overline{U \cap X} \cap X) \setminus (U \cap X)) \subseteq (\overline{U} \cap X)) \setminus (U \cap X) \\ &= (\overline{U} \cap X)) \setminus U \cup (\overline{U} \cap X) \setminus X = (\overline{U} \cap X) \setminus U \\ &\subseteq \overline{U} \setminus U = \partial U, \end{aligned}$$

so $\partial_X (U \cap X) \subseteq \partial U \cap X$ and also note

$$(\Omega \cap X) \cap \partial_X (U \cap X) \subseteq (\Omega \cap X) \cap (\partial U \cap X).$$

This together with $\Omega \cap \partial U = \emptyset$ (see (2.1)) guarantees that $\Omega \cap X$ and $\partial_X (U \cap X)$ are disjoint. Now from (2.3) there exists a continuous map $\mu : X \to [0,1]$ with $\mu(\Omega \cap X) = 1$ and $\mu(\partial_X (U \cap X)) = 0$. Define a map $J : X \to 2^X$ (see (2.6)) by

$$J(x) = \left\{ \begin{array}{l} H(x,\mu(x)),\ x \in \overline{U \cap X} \cap X \\ G(x),\ x \in X \setminus (\overline{U \cap X} \cap X). \end{array} \right.$$

Now (2.5) and (2.6) guarantee that there exists $x \in X$ with $\Phi(x) \cap J(x) \neq \emptyset$. Note (2.4) implies $x \in \overline{U \cap X} \cap X$ so $\Phi(x) \cap H_{\mu(x)}(x) \neq \emptyset$. As a result $x \in \Omega$ so $\mu(x) = 1$. Thus $\Phi(x) \cap H_1(x) \neq \emptyset$ i.e. $\Phi(x) \cap F(x) \neq \emptyset$. □

If (2.5) holds with $X = E$ then we have a special case of Theorem 2.3.

Corollary 2.4. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in MA(E,E)$ and $F \in MA(\overline{U},E)$. In addition suppose* (2.1) *and* (2.2) *hold, and also assume*

$$\begin{cases} \Omega \text{ compact (respectively closed) if } E \\ \text{is completely regular (respectively normal)} \end{cases} \tag{2.7}$$

$$\Phi(x) \cap G(x) = \emptyset \quad \text{for} \quad x \in E \setminus \overline{U} \tag{2.8}$$

$$\begin{cases} \text{for any map } J \in MA(E,E) \text{ there exists } x \in E \\ \text{with } J(x) \cap \Phi(x) \neq \emptyset \end{cases} \tag{2.9}$$

and

$$\begin{cases} \text{for any continuous map } \mu : E \to [0,1] \text{ with} \\ \mu(\partial U) = 0 \text{ and } \mu(\Omega) = 1 \text{ the map} \\ J_\mu \in MA(E,E) \text{ where } J_\mu(x) = \begin{cases} H(x,\mu(x)), & x \in \overline{U} \\ G(x), & x \in E \setminus \overline{U}. \end{cases} \end{cases} \tag{2.10}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Example 2.5. Let E be a Banach space, U an open subset of E and $0 \in U$.

In Theorem 2.3 we let $G = 0$ (the zero map) and $\Phi = i$ (the identity map).

We say $H \in MA(\overline{U},E)$ if $H : \overline{U} \to C(E)$ (here $C(E)$ denotes the family of nonempty convex subsets of E), $graph\, H$ is closed, H maps compact sets into relatively compact sets and the following condition is satisfied:

$$\begin{cases} M \subseteq \overline{U},\ M = co\,(\{0\} \cup H(M)) \text{ and } \overline{M} = \overline{C} \text{ with} \\ C \subseteq M \text{ countable implies } \overline{M} \text{ is compact.} \end{cases}$$

In Theorem 2.3 for F we assume $F \in MA(\overline{U},E)$ and in addition suppose the following two conditions hold (these are needed to guarantee (2.1) and (2.6)):

$$\begin{cases} M \subseteq \overline{U},\ M \subseteq co\,(\{0\} \cup F(M)) \text{ and } \overline{M} = \overline{C} \text{ with} \\ C \subseteq M \text{ countable implies } \overline{M} \text{ is compact} \end{cases} \tag{2.11}$$

and

$$x \notin \lambda F(x) \text{ for all } x \in \partial U \text{ and } \lambda \in (0,1). \tag{2.12}$$

Let $H(x,\lambda) = \lambda F(x)$. Clearly (2.1) (see (2.12)) and (2.2) hold (note $0 \in U$ and $0 \in \Omega$). Next let

$$X = \overline{co}\,(\{0\} \cup F(\overline{U})).$$

Note Ω is closed (see [9]) and (2.3) and (2.4) (note $0 \in U$) are immediate. Now [9 pp 601] (note (2.11) is needed for this) guarantees that (2.6) holds i.e. $J_\mu \in MA(X,X)$ where

$$J_\mu(x) = \begin{cases} \mu(x)\,F(x), & x \in \overline{U \cap X} \cap X \\ 0, & x \in X \setminus (\overline{U \cap X} \cap X). \end{cases}$$

Also (2.5) holds (this is the set valued analogue of Mönch's fixed point theorem due to O'Regan and Precup [9]). Now Theorem 2.3 guarantees that there exists $x \in \overline{U}$ with $x \in F(x)$.

Next we consider in addition a class **D**.

Definition 2.6. We say $F \in D(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$. We say $F \in D(E, E)$ if $F : E \to 2^E$ and $F \in \mathbf{D}(E, E)$.

Definition 2.7. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$, $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F. We say $F \in A(E, E)$ if $F : E \to 2^E$, $F \in \mathbf{A}(E, E)$ and there exists a selection $\Psi \in D(E, E)$ of F.

Theorem 2.8. *Let E be a topological space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in A(E, E)$ and $F \in A(\overline{U}, E)$. For any selection $\Psi \in D(\overline{U}, E)$ of F and any selection $\Lambda \in D(E, E)$ of G assume*

$$\begin{cases} \text{there exists a map } H : \overline{U} \times [0, 1] \to 2^E \text{ with } \Phi(x) \cap H_t(x) = \emptyset \\ \text{for } x \in \partial U \text{ and } t \in [0, 1),\ H_1 = \Psi,\ H_0 = \Lambda; \text{ here } H_t(x) = H(x, t) \end{cases} \tag{2.13}$$

$$\Omega = \{x \in \overline{U} :\ \Phi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\} \text{ is nonempty} \tag{2.14}$$

and there exists a completely regular (respectively normal) subset X of E with (2.3) *holding and*

$$\Phi(x) \cap \Lambda(x) = \emptyset \text{ for } x \in X \setminus (\overline{U \cap X} \cap X) \tag{2.15}$$

$$\begin{cases} \text{for any map } J \in D(X, X) \text{ there exists } x \in X \\ \text{with } J(x) \cap \Phi(x) \neq \emptyset \end{cases} \tag{2.16}$$

and

$$\begin{cases} \text{for any continuous map } \mu : X \to [0, 1] \text{ with} \\ \mu(\partial_X (U \cap X)) = 0 \text{ and } \mu(\Omega \cap X) = 1 \text{ the map} \\ J_\mu \in D(X, X) \text{ where } J_\mu(x) = \begin{cases} H(x, \mu(x)),\ x \in \overline{U \cap X} \cap X \\ \Lambda(x),\ x \in X \setminus (\overline{U \cap X} \cap X). \end{cases} \end{cases} \tag{2.17}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$. Let $\Psi \in D(\overline{U}, E)$ be any selection of F and $\Lambda \in D(E, E)$ be any selection of G. Note $\Phi(x) \cap \Psi(x) = \emptyset$ for $x \in \partial U$. Let Ω be as in (2.14) and note $\Omega \neq \emptyset$. Also $\Omega \cap X$ and $\partial_X (U \cap X)$ are disjoint. Then there exists a continuous map $\mu : X \to [0, 1]$ with $\mu(\Omega \cap X) = 1$ and $\mu(\partial_X (U \cap X)) = 0$. Define a map $J_\mu : X \to 2^X$ as in (2.17). Now there exists $x \in X$ with $\Phi(x) \cap J_\mu(x) \neq \emptyset$. Note (2.15) guarantee that $x \in \overline{U \cap X} \cap X$ so $\Phi(x) \cap H_{\mu(x)}(x) \neq \emptyset$. As a result $x \in \Omega$ so $\mu(x) = 1$. Thus $\Phi(x) \cap H_1(x) \neq \emptyset$ i.e. $\Phi(x) \cap \Psi(x) \neq \emptyset$. Thus $\Phi(x) \cap F(x) \neq \emptyset$. □

In fact if we slightly change Ω in (2.2) (or (2.14)) in Theorem 2.3 (or Theorem 2.8) (see [1]) then we can obtain similar type results.

Theorem 2.9. *Let E be a topological space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in MA(E, E)$ and $F \in MA(\overline{U}, E)$. Assume there exists a completely regular (respectively normal) subset X of E with*

$$\begin{cases} \text{there exists a map } H : (\overline{U \cap X} \cap X) \times [0, 1] \to 2^E \text{ with} \\ \Phi(x) \cap H_t(x) = \emptyset \text{ for } x \in \partial_X (U \cap X) \text{ and } t \in [0, 1), \\ H_1 = F,\ H_0 = G; \text{ here } H_t(x) = H(x, t) \end{cases} \tag{2.18}$$

$$\Omega = \{x \in \overline{U \cap X} \cap X :\ \Phi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\} \text{ is nonempty} \tag{2.19}$$

$$\left\{ \begin{array}{l} \Omega \ \ \textit{is compact (respectively closed) if} \ \ X \\ \textit{is completely regular (respectively normal)} \end{array} \right. \tag{2.20}$$

$$\left\{ \begin{array}{l} \textit{for any continuous map} \ \ \mu : X \to [0,1] \ \ \textit{with} \\ \mu(\partial_X (U \cap X)) = 0 \ \ \textit{and} \ \ \mu(\Omega) = 1 \ \ \textit{the map} \\ J_\mu \in MA(X,X) \ \ \textit{where} \ \ J_\mu(x) = \left\{ \begin{array}{l} H(x,\mu(x)), \ x \in \overline{U \cap X} \cap X \\ G(x), \ \ x \in X \setminus (\overline{U \cap X} \cap X) \end{array} \right. \end{array} \right. \tag{2.21}$$

and also suppose (2.4) *and* (2.5) *hold. Then there exists* $x \in \overline{U}$ *(in fact* $x \in \overline{U \cap X} \cap X$*) with* $\Phi(x) \cap F(x) \neq \emptyset$.

Remark 2.10. One could replace (2.18) with

$$\left\{ \begin{array}{l} \text{there exists a map} \ \ H : \overline{U} \times [0,1] \to 2^E \ \ \text{with} \ \ \Phi(x) \cap H_t(x) = \emptyset \\ \text{for} \ \ x \in \partial U \ \ \text{and} \ \ t \in [0,1), \ H_1 = F, \ H_0 = G \end{array} \right.$$

or

$$\left\{ \begin{array}{l} \text{there exists a map} \ \ H : (\overline{U \cap X} \cap X) \times [0,1] \to 2^E \ \ \text{with} \ \ \Phi(x) \cap H_t(x) = \emptyset \\ \text{for} \ \ x \in \partial U \cap X \ \ \text{and} \ \ t \in [0,1), \ H_1 = F, \ H_0 = G. \end{array} \right.$$

This is immediate since $\partial_X (U \cap X) \subseteq \partial U$ and $\partial_X (U \cap X) \subseteq \partial U \cap X$ (see Theorem 2.3).

Proof. Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial_X (U \cap X)$ (otherwise we are finished). Let Ω be as in (2.19) and note Ω and $\partial_X (U \cap X)$ are disjoint (see (2.18) and the above assumption). Then there exists a continuous map $\mu : X \to [0,1]$ with $\mu(\Omega) = 1$ and $\mu(\partial_X (U \cap X)) = 0$. Define a map $J_\mu : X \to 2^X$ as in (2.21). Now there exists $x \in X$ with $\Phi(x) \cap J_\mu(x) \neq \emptyset$. Note $x \in \overline{U \cap X} \cap X$ so $x \in \Omega$ and $\mu(x) = 1$. □

A similar argument establishes the following result.

Theorem 2.11. *Let* E *be a topological space,* U *an open subset of* E*,* $G : E \to 2^E$*,* $F : \overline{U} \to 2^E$*,* $G \in A(E,E)$ *and* $F \in A(\overline{U},E)$*. For any selection* $\Psi \in D(\overline{U},E)$ *of* F *and any selection* $\Lambda \in D(E,E)$ *of* G *assume there exists a completely regular (respectively normal) subset* X *of* E *with*

$$\left\{ \begin{array}{l} \textit{there exists a map} \ \ H : (\overline{U \cap X} \cap X) \times [0,1] \to 2^E \ \ \textit{with} \\ \Phi(x) \cap H_t(x) = \emptyset \ \ \textit{for} \ \ x \in \partial_X (U \cap X) \ \ \textit{and} \ \ t \in [0,1), \\ H_1 = \Psi, \ H_0 = \Lambda; \ \ \textit{here} \ \ H_t(x) = H(x,t) \end{array} \right. \tag{2.22}$$

$$\Omega = \{x \in \overline{U \cap X} \cap X : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \ \textit{for some} \ \ t \in [0,1]\} \ \ \textit{is nonempty} \tag{2.23}$$

and also suppose (2.15)*,* (2.16) *and* (2.20) *hold. In addition suppose*

$$\left\{ \begin{array}{l} \textit{for any continuous map} \ \ \mu : X \to [0,1] \ \ \textit{with} \\ \mu(\partial_X (U \cap X)) = 0 \ \ \textit{and} \ \ \mu(\Omega) = 1 \ \ \textit{the map} \\ J_\mu \in D(X,X) \ \ \textit{where} \ \ J_\mu(x) = \left\{ \begin{array}{l} H(x,\mu(x)), \ x \in \overline{U \cap X} \cap X \\ \Lambda(x), \ \ x \in X \setminus (\overline{U \cap X} \cap X). \end{array} \right. \end{array} \right. \tag{2.24}$$

Then there exists $x \in \overline{U}$ *with* $\Phi(x) \cap F(x) \neq \emptyset$.

Next we present a nonlinear alternative which is motivated in part by maps considered in [2, 8] (we consider the case when E is a topological vector space but the case when E is a topological space is noted in a remark).

Theorem 2.12. *Let E be a topological vector space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in \mathbf{A}(E, E)$ and $F \in \mathbf{A}(\overline{U}, E)$. Suppose there exists a set $K \subseteq E$ with $F(\overline{U}) \subseteq K$ and $G(E) \subseteq K$ and assume*

$$F \in A(\overline{U \cap L(K)} \cap L(K), L(K)) \quad \text{and} \quad G \in A(E, L(K)) \tag{2.25}$$

where $L(K)$ is the linear span of K (i.e. the smallest linear subspace of E that contains K). For any selection $\Psi \in D(\overline{U \cap L(K)} \cap L(K), L(K))$ of F and any selection $\Lambda \in D(E, L(K))$ of G assume

$$\begin{cases} \text{there exists a map } H : (\overline{U \cap L(K)} \cap L(K)) \times [0,1] \to 2^{L(K)} \text{ with} \\ \Phi(x) \cap H_t(x) = \emptyset \text{ for } x \in \partial_{L(K)}(U \cap L(K)) \text{ and } t \in [0,1), \\ H_1 = \Psi,\ H_0 = \Lambda; \text{ here } H_t(x) = H(x,t) \text{ and } \partial_{L(K)}(U \cap L(K)) \\ \text{denotes the boundary of } U \cap L(K) \text{ in } L(K) \end{cases} \tag{2.26}$$

$$\begin{cases} \Omega = \{x \in \overline{U \cap L(K)} \cap L(K) :\ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{is nonempty and compact} \end{cases} \tag{2.27}$$

$$\Phi(x) \cap \Lambda(x) = \emptyset \quad \text{for} \quad x \in L(K) \setminus (\overline{U \cap L(K)} \cap L(K)) \tag{2.28}$$

$$\begin{cases} \text{for any map } J \in D(L(K), L(K)) \text{ there exists } x \in L(K) \\ \text{with } J(x) \cap \Phi(x) \neq \emptyset \end{cases} \tag{2.29}$$

and

$$\begin{cases} \text{for any continuous map } \mu : L(K) \to [0,1] \text{ with} \\ \mu(\partial_{L(K)}(U \cap L(K))) = 0 \text{ and } \mu(\Omega) = 1 \text{ the map} \\ J_\mu \in D(L(K), L(K)) \text{ where} \\ J_\mu(x) = \begin{cases} H(x, \mu(x)),\ x \in \overline{U \cap L(K)} \cap L(K) \\ \Lambda(x),\ x \in L(K) \setminus (\overline{U \cap L(K)} \cap L(K)). \end{cases} \end{cases} \tag{2.30}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Remark 2.13. Note topological vector spaces are automatically completely regular. If $L(K)$ is normal then one can replace (2.27) with: Ω is nonempty and closed.

Proof. Note $U \cap L(K)$ is an open subset of $L(K)$ and $\overline{U \cap L(K)}^{\,L(K)} = \overline{U \cap L(K)} \cap L(K)$; here $\overline{U \cap L(K)}^{\,L(K)}$ denotes the closure of $U \cap L(K)$ in $L(K)$. Suppose $\Phi(x) \cap F(x) \neq \emptyset$ for $x \in \partial_{L(K)}(U \cap L(K))$. Let $\Psi \in D(\overline{U \cap L(K)} \cap L(K), L(K))$ be any selection of F and $\Lambda \in D(E, L(K))$ be any selection of G. Note Ω and $\partial_{L(K)}(U \cap L(K))$ are disjoint (see (2.26) and the above assumption). Then there exists a continuous map $\mu : L(K) \to [0,1]$ with $\mu(\Omega) = 1$ and $\mu(\partial_{L(K)}(U \cap L(K)) = 0$. Define a map $J_\mu : L(K) \to 2^{L(K)}$ as in (2.30). Now there exists $x \in L(K)$ with $\Phi(x) \cap J_\mu(x) \neq \emptyset$. Note $x \in \overline{U \cap L(K)} \cap L(K)$ (see (2.28)) so $x \in \Omega$ and $\mu(x) = 1$. □

Remark 2.14. One could replace (2.26) with

$$\begin{cases} \text{there exists a map } H : \overline{U} \times [0,1] \to 2^{L(K)} \text{ with } \Phi(x) \cap H_t(x) = \emptyset \\ \text{for } x \in \partial U \text{ and } t \in [0,1),\ H_1 = \Psi,\ H_0 = \Lambda \end{cases}$$

or

$$\begin{cases} \text{there exists a map } H : (\overline{U \cap L(K)} \cap L(K)) \times [0,1] \to 2^{L(K)} \text{ with} \\ \Phi(x) \cap H_t(x) = \emptyset \text{ for } x \in \partial U \cap L(K) \text{ and } t \in [0,1),\ H_1 = \Psi,\ H_0 = \Lambda. \end{cases}$$

This is immediate since $\partial_{L(K)}\,(U \cap L(K)) \subseteq \partial U$ and $\partial_{L(K)}\,(U \cap L(K)) \subseteq \partial U \cap L(K)$; as before note

$$\begin{aligned}\partial_{L(K)}\,(U \cap L(K)) &= (\overline{U \cap L(K)} \cap L(K)) \setminus (U \cap L(K)) \subseteq (\overline{U} \cap L(K)) \setminus (U \cap L(K)) \\ &= (\overline{U} \cap L(K)) \setminus U \,\cup\, (\overline{U} \cap L(K)) \setminus L(K) = (\overline{U} \cap L(K)) \setminus U \\ &\subseteq \overline{U} \setminus U = \partial U.\end{aligned}$$

Remark 2.15. There is an obvious analogue of Theorem 2.12 when E is a topological space: Let U be an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in \mathbf{A}(E, E)$ and $F \in \mathbf{A}(\overline{U}, E)$. Suppose there exists a set $X \subseteq E$ with $F(\overline{U}) \subseteq X$ and $G(E) \subseteq X$ and assume $F \in A(\overline{U \cap X} \cap X, X)$ and $G \in A(E, X)$. For any selection $\Psi \in D(\overline{U \cap X} \cap X, X)$ of F and any selection $\Lambda \in D(E, X)$ of G assume (2.15), (2.16), (2.22), (2.23), (2.20) and (2.24) hold. Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Before we discuss Theorem 2.12 we first recall the DKT maps from the literature [2]. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in DKT(Z, W)$ if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibres $S^{-1}(w) = \{z :\, w \in S(z)\}$ are open (in Z) for each $w \in W$.

Example 2.16. Let E be a Hausdorff locally convex linear topological space, U an open subset of E with $0 \in U$, $G = 0$ and $\Phi = i$.

We say $H \in \mathbf{A}(\overline{U}, E)$ if $H : \overline{U} \to 2^E$ and $H \in DKT(\overline{U}, E)$ is a compact map. Let $D(\overline{U}, E)$ denote the class of single valued continuous compact maps.

In Theorem 2.12 for F assume $F : \overline{U} \to 2^E$ and $F \in DKT(\overline{U}, E)$ is a compact map. Let K be a compact set with $F(\overline{U}) \subseteq K$ and note $L(K)$ is paracompact (see for example [3])[Also $L(K)$ is normal (recall paracompact spaces are normal)]. Now suppose

$$x \notin \lambda\, F(x) \quad \text{for all} \quad x \in \partial_{L(K)}(U \cap L(K)) \quad \text{and} \quad \lambda \in (0, 1). \tag{2.31}$$

Remark 2.17. We could replace $x \in \partial_{L(K)}(U \cap L(K))$ with $x \in \partial U$ in (2.31) since $\partial_{L(K)}\,(U \cap L(K)) \subseteq \partial U$.

Now if we show $F \in DKT(\overline{U \cap L(K)} \cap L(K), L(K))$ then [2] guarantees that (recall closed subsets of paracompact spaces are paracompact) there exists a selection $\Psi \in D(\overline{U \cap L(K)} \cap L(K), L(K))$ of F so $F \in A(\overline{U \cap L(K)} \cap L(K), L(K))$. Since $F \in DKT(\overline{U}, E)$ then there exists a map $\theta : \overline{U} \to E$ with $co\,(\theta(x)) \subseteq F(x)$ for $x \in \overline{U}$, $\theta(x) \neq \emptyset$ for each $x \in \overline{U}$ and $\theta^{-1}(y) = \{z \in \overline{U} :\, y \in \theta(z)\}$ is open (in $\overline{U}$) for each $y \in E$. Let $\theta^\star$ denote the restriction of θ to $\overline{U \cap L(K)} \cap L(K)$. Note $co\,(\theta^\star(x)) \subseteq F(x)$ for $x \in \overline{U \cap L(K)} \cap L(K)$ and $\theta^\star(x) \neq \emptyset$ for each $x \in \overline{U \cap L(K)} \cap L(K)$. If $y \in L(K)$ then (note $\overline{U \cap L(K)} \cap L(K) \cap \overline{U} = \overline{U \cap L(K)} \cap L(K)$ since $\overline{U \cap L(K)} \subseteq \overline{U}$),

$$\begin{aligned}(\theta^\star)^{-1}(y) &= \{z \in \overline{U \cap L(K)} \cap L(K) :\, y \in \theta^\star(z)\} \\ &= \{z \in \overline{U \cap L(K)} \cap L(K) :\, y \in \theta(z)\} \\ &= \overline{U \cap L(K)} \cap L(K) \cap \{z \in \overline{U} :\, y \in \theta(z)\} \\ &= \overline{U \cap L(K)} \cap L(K) \cap \theta^{-1}(y)\end{aligned}$$

which is open in $\overline{U \cap L(K)} \cap L(K)$. Thus $F \in DKT(\overline{U \cap L(K)} \cap L(K), L(K))$.

Let $\Lambda = 0$ and let $f : \overline{U \cap L(K)} \cap L(K) \to L(K)$ be any continuous compact selection of F. Now let $H(x,t) = t\, f(x)$. Clearly (2.26) (use (2.31)), (2.27) (note $0 \in U$), (2.28) (note $\Lambda = 0$ and $0 \in U$), (2.29) (Schauder-Tychonoff fixed point theorem) and (2.30) hold. Theorem 2.12 guarantees that there exists $x \in \overline{U}$ with $x \in F(x)$ (for another approach we refer the reader to [8]).

Next we present a Furi–Pera type result (see [4, 7] and the references therein).

Theorem 2.18. *Let E be a topological vector space, Q a closed subset of E, C a closed convex subset of E with $Q \subseteq C$, $F \in MA(Q,C)$ and $\Phi \in MA(Q,C)$. In addition assume:*

$$\textit{there exists a retraction } \; r : E \to Q \tag{2.32}$$

$$\Omega = \{x \in E : \; x \in F\, r(x)\} \;\; \textit{is nonempty and compact} \tag{2.33}$$

and

$$\begin{cases} \textit{the topology induced on } C \textit{ is metrizable;} \\ \textit{let } d^\star \textit{ denote the metric.} \end{cases} \tag{2.34}$$

For $i \in \{1,2,...\}$ let $U_i = \{x \in C : \; d^\star(x,Q) < \frac{1}{i}\}$. Suppose for each $i \in \{1,2,...\}$,

$$F\, r \in MA(\overline{U_i}, C) \;\; \textit{and} \;\; \Phi\, r \in MA(\overline{U_i}, E) \tag{2.35}$$

$$\begin{cases} \textit{either (A1). there exists } \; x \in \overline{U_i} \;\; \textit{with} \;\; x \in F\, r(x), \\ \textit{or (A2). there exists } \; x \in \partial U_i \;\; \textit{and} \;\; \lambda \in (0,1) \;\; \textit{with} \\ x \in \lambda\, F\, r(x) + (1-\lambda)\, \Phi\, r(x), \;\; \textit{hold} \end{cases} \tag{2.36}$$

$$\{x \in E : \; x \in \lambda\, F\, r(x) + (1-\lambda)\, \Phi\, r(x) \;\; \textit{for some} \;\; \lambda \in [0,1]\} \;\; \textit{is compact} \tag{2.37}$$

$$\begin{cases} \textit{if } \; \{(x_j, \lambda_j\}_{j=1}^{\infty} \;\; \textit{is a sequence in} \;\; \partial U_i \times [0,1] \;\; \textit{converging} \\ \textit{to } \; (x,\lambda) \;\; \textit{with} \;\; x \in \partial Q \;\; \textit{and} \;\; x_j \in \lambda_j\, F\, r\,(x_j) + (1-\lambda_j)\, \Phi\, r\,(x_j), \\ \textit{then } \; x \in \lambda\, F\, r\,(x) + (1-\lambda)\, \Phi\, r\,(x) = \lambda\, F\,(x) + (1-\lambda)\, \Phi\,(x) \end{cases} \tag{2.38}$$

and

$$\begin{cases} \textit{if } \; \{(x_j, \lambda_j\}_{j=1}^{\infty} \;\; \textit{is a sequence in} \;\; Q \times [0,1] \;\; \textit{converging} \\ \textit{to } \; (x,\lambda) \;\; \textit{with} \;\; x \in \lambda\, F\,(x) + (1-\lambda)\, \Phi\,(x) \;\; \textit{and} \;\; 0 \le \lambda < 1, \\ \textit{then } \; \{\lambda_j\, F\,(x_j) + (1-\lambda_j)\, \Phi\,(x_j)\} \subseteq Q \;\; \textit{for} \;\; j \;\; \textit{sufficiently large.} \end{cases} \tag{2.39}$$

Then F has a fixed point in Q.

Proof. Let Ω be as in (2.33) and note $\Omega \subseteq C$ (recall $F : Q \to 2^C$). We claim $\Omega \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose that $\Omega \cap Q = \emptyset$. Then since Ω is compact and Q is closed there exists $\delta > 0$ with

$$dist(\Omega, Q) = \inf\{d^\star(x,y) : \, x \in \Omega, \, y \in Q\} > \delta.$$

Choose $m \in \{1,2,...\}$ with $1 < \delta\, m$ and let (as in the statement of the theorem)

$$U_i = \left\{ x \in C : \; d^\star(x,Q) < \frac{1}{i} \right\} \text{ for } i \in \{m, m+1, ...\}.$$

Fix $i \in \{m, m+1, ...\}$. Since $dist(\Omega, Q) > \delta$ we see that $\Omega \cap \overline{U_i} = \emptyset$ (note $\overline{U_i} = \overline{U_i}^{d^\star} = \{x \in C : d^\star(x, Q) \leq \frac{1}{i}\}$ and $\partial U_i = \{x \in C : d^\star(x, Q) = \frac{1}{i}\}$). Now (2.36) guarantees that there exists $\lambda_i \in (0,1)$ and $y_i \in \partial U_i$ with

$$y_i \in \lambda_i \, F\, r(y_i) + (1 - \lambda_i)\, \Phi\, r(y_i).$$

Since $y_i \in \partial U_i$ we have

$$\{\lambda_i \, F\, r(y_i) + (1 - \lambda_i)\, \Phi\, r(y_i)\} \not\subseteq Q \quad \text{for} \quad i \in \{m, m+1, ...\}. \tag{2.40}$$

Let

$$K = \{x \in E : \; x \in \lambda \, F\, r(x) + (1 - \lambda)\, \Phi\, r(x) \;\; \text{for some} \;\; \lambda \in [0,1]\}.$$

Now $K \neq \emptyset$ is compact (see (2.33) and (2.37)) and $K \subseteq C$ since $F : Q \to 2^C$, $\Phi : Q \to 2^C$ and C is convex. This together with

$$d^\star(y_j, Q) = \frac{1}{j} \quad \text{and} \quad |\lambda_j| \leq 1 \quad \text{for} \quad j \in \{m, m+1, ...\}$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Now (2.38) implies $y^\star \in \lambda^\star \, F\, r(y^\star) + (1 - \lambda^\star)\, \Phi\, r(y^\star)$ i.e.

$$y^\star \in \lambda^\star \, F\, (y^\star) + (1 - \lambda^\star)\, \Phi\, (y^\star)$$

since $r(y^\star) = y^\star$. If $\lambda^\star = 1$ then $y^\star \in F\,(y^\star)$ which contradicts $\Omega \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. Now (2.39) with $x_j = r(y_j)$ and $x = y^\star = r(y^\star)$ implies

$$\{\lambda_j \, F\, r(y_j) + (1 - \lambda_j)\, \Phi\, r(y_j)\} \subseteq Q \quad \text{for} \quad j \;\; \text{sufficiently large}.$$

This contradicts (2.40).
Thus $\Omega \cap Q \neq \emptyset$ so there exists $x \in Q$ with $x \in F\, r(x) = F(x)$. □

Remark 2.19. Suppose in Theorem 2.18 we change (2.32) to:

$$\text{there exists a retraction} \quad r : E \to Q \quad \text{with} \quad r(z) \in \partial Q \quad \text{for} \quad z \in E \setminus Q. \tag{2.41}$$

Then the result in Theorem 2.18 again holds provided (2.39) is changed to (note $x_j = r(y_j) \in \partial\, Q$ in the above proof)

$$\begin{cases} \text{if} \;\; \{(x_j, \lambda_j\}_{j=1}^\infty \;\; \text{is a sequence in} \;\; \partial\, Q \times [0,1] \;\; \text{converging} \\ \text{to} \;\; (x, \lambda) \;\; \text{with} \;\; x \in \lambda \, F\,(x) + (1 - \lambda)\, \Phi\,(x) \;\; \text{and} \;\; 0 \leq \lambda < 1, \\ \text{then} \;\; \{\lambda_j \, F\,(x_j) + (1 - \lambda_j)\, \Phi\,(x_j)\} \subseteq Q \;\; \text{for} \;\; j \;\; \text{sufficiently large.} \end{cases}$$

A similar argument establishes the following result.

Theorem 2.20. *Let E be a topological vector space, Q a closed subset of E, C a closed convex subset of E with $Q \subseteq C$, $F \in A(Q, C)$ and $\Phi \in A(Q, C)$. In addition assume* (2.32), (2.34) *and*

$$\begin{cases} \textit{for any selection} \;\; \Psi \in D(Q, C) \;\; \textit{of} \;\; F \;\; \textit{assume} \;\; \Psi\, r \in D(E, C), \\ \Psi\, r \;\; \textit{has a fixed point in} \;\; C \;\; \textit{and} \;\; \Omega = \{x \in E : \; x \in \Psi\, r(x)\} \;\; \textit{is compact} \end{cases} \tag{2.42}$$

hold. For $i \in \{1, 2, ...\}$ let $U_i = \{x \in C : \; d^\star(x, Q) < \frac{1}{i}\}$. Suppose for each $i \in \{1, 2, ...\}$ and for any selection $\Psi \in D(Q, C)$ of F and any selection $\phi \in D(Q, C)$ of Φ we have the following:

$$\Psi\, r \in D(\overline{U_i}, C) \quad \textit{and} \quad \phi\, r \in D(\overline{U_i}, C) \tag{2.43}$$

$$\begin{cases} \textit{either (A1). there exists } x \in \overline{U_i} \textit{ with } x \in \Psi\, r(x), \\ \textit{or (A2). there exists } x \in \partial U_i \textit{ and } \lambda \in (0,1) \textit{ with} \\ x \in \lambda\, \Psi\, r(x) + (1-\lambda)\, \phi\, r(x), \textit{ hold} \end{cases} \tag{2.44}$$

$$\{x \in E : \ x \in \lambda\, \Psi\, r(x) + (1-\lambda)\, \phi\, r(x) \ \textit{ for some } \ \lambda \in [0,1]\} \ \textit{ is compact} \tag{2.45}$$

$$\begin{cases} \textit{if } \{(x_j, \lambda_j\}_{j=1}^{\infty} \textit{ is a sequence in } \partial U_i \times [0,1] \textit{ converging} \\ \textit{to } (x,\lambda) \textit{ with } x \in \partial Q \textit{ and } x_j \in \lambda_j\, \Psi\, r\,(x_j) + (1-\lambda_j)\, \phi\, r\,(x_j), \\ \textit{then } x \in \lambda\, \Psi\, r\,(x) + (1-\lambda)\, \phi\, r\,(x) = \lambda\, \Psi\,(x) + (1-\lambda)\, \phi\,(x) \end{cases} \tag{2.46}$$

and

$$\begin{cases} \textit{if } \{(x_j, \lambda_j\}_{j=1}^{\infty} \textit{ is a sequence in } Q \times [0,1] \textit{ converging} \\ \textit{to } (x,\lambda) \textit{ with } x \in \lambda\, \Psi\,(x) + (1-\lambda)\, \phi\,(x) \textit{ and } 0 \le \lambda < 1, \\ \textit{then } \{\lambda_j\, \Psi\,(x_j) + (1-\lambda_j)\, \phi\,(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently large.} \end{cases} \tag{2.47}$$

Then F has a fixed point in Q.

Remark 2.21. Suppose in Theorem 2.20 we change (2.32) to (2.41). Then the result in Theorem 2.20 again holds provided (2.47) is changed to

$$\begin{cases} \text{if } \{(x_j, \lambda_j\}_{j=1}^{\infty} \text{ is a sequence in } \partial\, Q \times [0,1] \text{ converging} \\ \text{to } (x,\lambda) \text{ with } x \in \lambda\, \Psi\,(x) + (1-\lambda)\, \phi\,(x) \text{ and } 0 \le \lambda < 1, \\ \text{then } \{\lambda_j\, \Psi\,(x_j) + (1-\lambda_j)\, \phi\,(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases}$$

We conclude this section by presenting another nonlinear alternative if we slightly change Ω in (2.2) (or (2.14)) and (2.19) (or (2.23)) in Theorem 2.3 (or Theorem 2.8) and Theorem 2.9 (or Theorem 2.11) (note $\overline{U \cap X}^{\,X} = \overline{U \cap X} \cap X \subseteq \overline{X} \cap \overline{U} \cap X = X \cap \overline{U} \subseteq \overline{U}$).

Theorem 2.22. *Let E be a topological space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in MA(E,E)$ and $F \in MA(\overline{U},E)$. Assume there exists a completely regular (respectively normal) subset X of E with*

$$\begin{cases} \textit{there exists a map } H : (X \cap \overline{U}) \times [0,1] \to 2^E \textit{ with} \\ \Phi(x) \cap H_t(x) = \emptyset \textit{ for } x \in \partial_X\,(X \cap \overline{U}) \textit{ and } t \in [0,1), \\ H_1 = F, \ H_0 = G; \textit{ here } H_t(x) = H(x,t) \end{cases} \tag{2.48}$$

$$\Omega = \{x \in X \cap \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \textit{ for some } \ t \in [0,1]\} \ \textit{ is nonempty} \tag{2.49}$$

$$\begin{cases} \Omega \textit{ is compact (respectively closed) if } X \\ \textit{is completely regular (respectively normal)} \end{cases} \tag{2.50}$$

$$\Phi(x) \cap G(x) = \emptyset \ \textit{ for } \ x \in X \setminus (X \cap \overline{U}) \tag{2.51}$$

$$\begin{cases} \textit{for any continuous map } \mu : X \to [0,1] \textit{ with} \\ \mu(\partial_X\,(X \cap \overline{U})) = 0 \textit{ and } \mu(\Omega) = 1 \textit{ the map} \\ J_\mu \in MA(X,X) \textit{ where } J_\mu(x) = \begin{cases} H(x,\mu(x)), \ x \in X \cap \overline{U} \\ G(x), \ x \in X \setminus (X \cap \overline{U}) \end{cases} \end{cases} \tag{2.52}$$

and also suppose (2.5) *holds. Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.*

Proof. Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial_X\,(X \cap \overline{U})$ (otherwise we are finished). Let Ω be as in (2.49) and note Ω and $\partial_X\,(X \cap \overline{U})$ are disjoint (see (2.48) and the above assumption). Then there exists a continuous map $\mu : X \to [0,1]$ with $\mu(\Omega) = 1$ and

$\mu(\partial_X(X \cap \overline{U})) = 0$. Define a map $J_\mu : X \to 2^X$ as in (2.52). Now (2.5) guarantees there exists $x \in X$ with $\Phi(x) \cap J_\mu(x) \neq \emptyset$. Note $x \in X \cap \overline{U}$ so $x \in \Omega$ and $\mu(x) = 1$. □

Remark 2.23. In (2.48) (and also (2.52)) we could replace $\partial_X(X \cap \overline{U})$ with $\partial_X(int_X(X \cap \overline{U}))$ (note also that $int_X(X \cap \overline{U}) = X \cap int_X(\overline{U})$).

A similar argument establishes the following result.

Theorem 2.24. *Let E be a topological space, U an open subset of E, $G : E \to 2^E$, $F : \overline{U} \to 2^E$, $G \in A(E, E)$ and $F \in A(\overline{U}, E)$. For any selection $\Psi \in D(\overline{U}, E)$ of F and any selection $\Lambda \in D(E, E)$ of G assume there exists a completely regular (respectively normal) subset X of E with*

$$\begin{cases} \text{there exists a map } H : (X \cap \overline{U}) \times [0,1] \to 2^E \text{ with} \\ \Phi(x) \cap H_t(x) = \emptyset \text{ for } x \in \partial_X(X \cap \overline{U}) \text{ and } t \in [0,1), \\ H_1 = \Psi,\ H_0 = \Lambda; \text{ here } H_t(x) = H(x,t) \end{cases} \tag{2.53}$$

$$\Omega = \{x \in X \cap \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is nonempty} \tag{2.54}$$

$$\begin{cases} \Omega \text{ is compact (respectively closed) if } X \\ \text{is completely regular (respectively normal)} \end{cases} \tag{2.55}$$

$$\Phi(x) \cap \Lambda(x) = \emptyset \text{ for } x \in X \setminus (X \cap \overline{U}) \tag{2.56}$$

and also suppose (2.16) *holds. In addition suppose*

$$\begin{cases} \text{for any continuous map } \mu : X \to [0,1] \text{ with} \\ \mu(\partial_X(X \cap \overline{U})) = 0 \text{ and } \mu(\Omega) = 1 \text{ the map} \\ J_\mu \in D(X, X) \text{ where } J_\mu(x) = \begin{cases} H(x, \mu(x)),\ x \in X \cap \overline{U} \\ \Lambda(x),\ x \in X \setminus (X \cap \overline{U}). \end{cases} \end{cases} \tag{2.57}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

3. Elementary approach

In this section we present an elementary approach to establishing Leray–Schauder nonlinear alternatives. In our first result we fix $\Phi \in B(E, E)$.

Theorem 3.1. *Let E be a topological space, U an open subset of E, $\Phi \in B(E, E)$ and $F \in A(\overline{U}, E)$. Suppose the following conditions are satisfied:*

$$\text{there exists a retraction } r : E \to \overline{U} \text{ with } r(w) \in \partial U \text{ if } w \in E \setminus U \tag{3.1}$$

$$\text{for any selection } \Psi \in D(\overline{U}, E) \text{ of } F \text{ assume } \Psi\, r \in D(E, E) \tag{3.2}$$

$$\text{for any map } J \in D(E, E) \text{ there exists } x \in E \text{ with } \Phi(x) \cap J(x) \neq \emptyset \tag{3.3}$$

and

$$\begin{cases} \text{for any selection } \Psi \in D(\overline{U}, E) \text{ of } F \text{ there} \\ \text{is no } x \in E \setminus \overline{U} \text{ and } y \in \partial U \text{ with } y = r(x) \\ \text{and } \Psi(y) \cap \Phi(x) \neq \emptyset. \end{cases} \tag{3.4}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. Let $\Psi \in D(\overline{U}, E)$ be any selection of F and let $G = \Psi\, r$. Now (3.2) and (3.3) guarantee that there exists $x \in E$ with $\Phi(x) \cap \Psi\, r(x) \neq \emptyset$. If $x \in E \setminus \overline{U}$ then if

$y = r(x)$, note $y \in \partial U$ and $\Phi(x) \cap \Psi(y) \neq \emptyset$, and this contradicts (3.4). Thus $x \in \overline{U}$ so $\emptyset \neq \Phi(x) \cap \Psi\, r(x) = \Phi(x) \cap \Psi\,(x) \subseteq \Phi(x) \cap F(x)$. □

Remark 3.2. We could replace (3.2) and (3.3) with: for any selection $\Psi \in D(\overline{U}, E)$ of F there exists $x \in E$ with $\Phi(x) \cap \Psi\, r(x) \neq \emptyset$.

Remark 3.3. Let E be a locally convex Hausdorff topological vector space, U an open convex subset of E, $0 \in U$ and $\Phi = i$ (the identity map). Let

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in E$$

where μ is the Minkowski functional on $\overline{U}$ (i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha\, \overline{U}\}$). Note (3.1) holds. Now assume

$$x \notin \lambda\, F x \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0, 1). \tag{3.5}$$

We now show (3.4) is true. To see this let $\Psi \in D(\overline{U}, E)$ be any selection of F and suppose there exists $x \in E \setminus \overline{U}$ and $y \in \partial U$ with $y = r(x)$ and $\Phi(x) \cap \Psi(y) \neq \emptyset$ (i.e. $x \in \Psi(y)$ i.e. $x \in \Psi\, r(x)$). Now

$$y = r(x) = \frac{x}{\mu(x)} \quad \text{with} \quad \mu(x) > 1 \quad \text{since} \quad x \in E \setminus \overline{U},$$

so

$$\frac{x}{\mu(x)} \in \frac{1}{\mu(x)}\, \Psi(y) \quad \text{i.e.} \quad y \in \lambda \Psi(y) \quad \text{with} \quad 0 < \lambda = \frac{1}{\mu(x)} < 1,$$

and this contradicts (3.5). Then (3.4) holds.

Theorem 3.4. *Let E be a Hausdorff topological space, U an open subset of E, $\Phi \in B(\overline{U}, E)$, $F \in A(\overline{U}, E)$ and suppose* (3.1) *holds. In addition assume the following:*

$$\textit{for any selection} \quad \Psi \in D(\overline{U}, E) \quad \textit{of} \quad F \quad \textit{assume} \quad r\, \Psi \in D(\overline{U}, \overline{U}) \tag{3.6}$$

$$\textit{for any map} \quad J \in D(\overline{U}, \overline{U}) \quad \textit{there exists} \quad x \in \overline{U} \quad \textit{with} \quad \Phi(x) \cap J(x) \neq \emptyset \tag{3.7}$$

and

$$\begin{cases} \textit{for any selection} \ \ \Psi \in D(\overline{U}, E) \ \ \textit{of} \ \ F \ \ \textit{there is no} \ \ z \in E \setminus \overline{U} \\ \textit{and} \ \ x \in \overline{U} \ \ \textit{with} \ \ z \in \Psi(x) \ \ \textit{and} \ \ r(z) \in \Phi(x). \end{cases} \tag{3.8}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. Let $\Psi \in D(\overline{U}, E)$ be any selection of F and let $G = r\, \Psi$. There exists a $x \in \overline{U}$ with $r\, \Psi(x) \cap \Phi(x) \neq \emptyset$. Let $w \in r\, \Psi(y) \cap \Phi(x)$. Then $w = r(z)$ for some $z \in \Psi(x)$ and note $w \in \Phi(x)$. If $z \in E \setminus \overline{U}$ then we have a contradiction with (3.8). Thus $z \in \overline{U}$ so $w = r(z) = z$ and as a result $w \in \Psi(x)$ and $z \in \Phi(x)$ i.e. $z \in \Psi(x) \cap \Phi(x)$. □

Remark 3.5. Let E be a locally convex Hausdorff topological vector space, U an open convex subset of E, $0 \in U$ and $\Phi = i$. Let

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in E$$

and suppose (3.5) holds. Then (3.8) holds. To see this let $\Psi \in D(\overline{U}, E)$ be any selection of F and suppose there exists $z \in E \setminus \overline{U}$ and $x \in \overline{U}$ with $z \in \Psi(x)$ and

$r(z) \in \Phi(x)$ (i.e. $r(z) = x$). Now

$$x = r(z) = \frac{z}{\mu(z)} \quad \text{with} \quad \mu(z) > 1 \quad \text{since} \quad z \in E \setminus \overline{U},$$

so $x \in \lambda\,\Psi(x)$ with $0 < \lambda = \frac{1}{\mu(z)} < 1$. Note $x = r(z) \in \partial U$ since $z \in E \setminus \overline{U}$. This contradicts (3.5), so (3.8) holds.

Remark 3.6. There are obvious analogues of Theorem's 3.1 and 3.4 for MA maps. For example the analogue of Theorem 3.1 is: Suppose $\Phi \in B(E,E)$, $F \in MA(\overline{U},E)$ with (3.1) and the following holding:

$$\text{there exists} \quad x \in E \quad \text{with} \quad \Phi(x) \cap F\,r(x) \neq \emptyset$$

and

$$\text{there is no} \quad x \in E \setminus \overline{U} \quad \text{and} \quad y \in \partial U \quad \text{with} \quad y = r(x) \quad \text{and} \quad F(y) \cap \Phi(x) \neq \emptyset.$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Recall $F \in A(\overline{U},E)$ if $F : \overline{U} \to 2^E$, $F \in \mathbf{A}(\overline{U},E)$ and there exists a selection $\Psi \in D(\overline{U},E)$ of F. However in some situations we have a map $F : \overline{U} \to 2^E$, $F \in \mathbf{A}(\overline{U},E)$ but we do not know if there exists a selection $\Psi \in D(\overline{U},E)$ of F. For example let $F : \overline{U} \to 2^E$ with $F \in DKT(\overline{U},E)$ a compact map (here $\mathbf{A}(\overline{U},E)$ denotes the class of compact DKT maps from $\overline{U}$ to 2^E). Suppose $D(\overline{U},E)$ denotes the class of single valued continuous compact maps. If $\overline{U}$ is paracompact then we know [2] that there exists a selection $\Psi \in D(\overline{U},E)$ of F. However if $\overline{U}$ is not necessarily paracompact it is also possible to obtain a result of Theorem 3.1 or 3.4 type as we will now indicate.

Theorem 3.7. *Let E be a topological space, U an open subset of E, $F : \overline{U} \to 2^E$ and there exists a set $K \subseteq E$ with $F(\overline{U}) \subseteq K$. Also assume $\Phi \in B(K,K)$ and $F \in A(\overline{U \cap K} \cap K, K)$. Suppose the following conditions are satisfied:*

$$\begin{cases} \textit{there exists a retraction} \;\; r : K \to \overline{U \cap K} \cap K \;\; \textit{with} \\ r(w) \in \partial_K (U \cap K) \;\; \textit{if} \;\; w \in K \setminus (U \cap K) \end{cases} \tag{3.9}$$

$$\textit{for any selection} \;\; \Psi \in D(\overline{U \cap K} \cap K, K) \;\; \textit{of} \;\; F \;\; \textit{assume} \;\; \Psi\, r \in D(K,K) \tag{3.10}$$

$$\textit{for any map} \;\; J \in D(K,K) \;\; \textit{there exists} \;\; x \in K \;\; \textit{with} \;\; \Phi(x) \cap J(x) \neq \emptyset \tag{3.11}$$

and

$$\begin{cases} \textit{for any selection} \;\; \Psi \in D(\overline{U \cap K} \cap K, K) \;\; \textit{of} \;\; F \;\; \textit{there is no} \\ x \in K \setminus (\overline{U \cap K} \cap K) \;\; \textit{and} \;\; y \in \partial_K (U \cap K) \;\; \textit{with} \;\; y = r(x) \\ \textit{and} \;\; \Psi(y) \cap \Phi(x) \neq \emptyset. \end{cases} \tag{3.12}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. The proof is as in Theorem 3.1 with U replaced by $U \cap K$ and E by K (note $U \cap K$ is an open subset of K and $\overline{U \cap K}^{\,K} = \overline{U \cap K} \cap K$. □

Remark 3.8. Returning to our example before Theorem 3.7 let $F : \overline{U} \to 2^E$ with $F \in DKT(\overline{U},E)$ a compact map; here E is a locally convex Hausdorff topological vector space, U an open convex subset of E, $0 \in U$ and $\Phi = i$. Now let C be a compact set with $F(\overline{U}) \subseteq C$ and note $K = L(C)$ is paracompact (see [3]); here $L(C)$ is the linear span of C. Note (see Example 2.16) that $F \in DKT(\overline{U \cap K} \cap K, K)$ so [2] guarantees that there exists a selection $\Psi \in D(\overline{U \cap K} \cap K, K)$ of F. Note (3.10) and (3.11) (Schauder–Tychonoff fixed point theorem) hold.

Theorem 3.9. *Let E be a topological space, U an open subset of E, $F : \overline{U} \to 2^E$ and there exists a set $K \subseteq E$ with $F(\overline{U}) \subseteq K$. Also assume $\Phi \in B(\overline{U \cap K} \cap K, K)$, $F \in A(\overline{U \cap K} \cap K, K)$ and (3.9) holds. Suppose the following conditions are satisfied:*

$$\begin{cases} \textit{for any selection } \ \Psi \in D(\overline{U \cap K} \cap K, K) \ \textit{ of } \ F \\ \textit{assume } \ r\,\Psi \in D(\overline{U \cap K} \cap K, \overline{U \cap K} \cap K) \end{cases} \tag{3.13}$$

$$\begin{cases} \textit{for any map } \ J \in D(\overline{U \cap K} \cap K, \overline{U \cap K} \cap K) \ \textit{ there exists} \\ x \in \overline{U \cap K} \cap K \ \textit{ with } \ \Phi(x) \cap J(x) \neq \emptyset \end{cases} \tag{3.14}$$

and

$$\begin{cases} \textit{for any selection } \ \Psi \in D(\overline{U \cap K} \cap K, K) \ \textit{ of } \ F \ \textit{ there is} \\ \textit{no } \ z \in K \setminus (\overline{U \cap K} \cap K) \ \textit{ and } \ x \in \overline{U \cap K} \cap K \ \textit{ with} \\ z \in \Psi(x) \ \textit{ and } \ r(z) \in \Phi(x). \end{cases} \tag{3.15}$$

Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Remark 3.10. The ideas in this section could be applied to other natural situations. Let E be a Hausdorff topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : dom\, L \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$. We say $F \in D(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L + T)^{-1}\,(F + T) \in D(\overline{U}, E)$ (respectively $(L + T)^{-1}\,(F + T) \in B(\overline{U}, E)$). We say $F \in A(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ and $(L + T)^{-1}\,(F + T) \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, Y; L, T)$ of F. For example the analogue of Theorem 3.1 is: Suppose $\Phi \in B(E, Y; L, T)$, $F \in A(\overline{U}, Y; L, T)$, (3.9) holds and the following conditions are satisfied:

$$\text{for any selection } \ \Psi \in D(\overline{U}, Y; L, T) \ \text{ of } \ F \text{ assume } \ (L + T)^{-1}(\Psi + T)\, r \in D(E, E)$$

$$\text{for any map } \ J \in D(E, E) \ \text{ there exists } \ x \in E \ \text{ with } \ J(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$$

and

$$\begin{cases} \text{for any selection } \ \Psi \in D(\overline{U}, Y; L, T) \ \text{ of } \ F \text{ there} \\ \text{is no } \ x \in E \setminus \overline{U} \ \text{ and } \ y \in \partial U \ \text{ with } \ y = r(x) \\ \text{and } \ (L + T)^{-1}(\Psi + T)(y) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset. \end{cases}$$

Then there exists $x \in \overline{U}$ with $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(F+T)(x) \neq \emptyset$. To see this let $\Psi \in D(\overline{U}, Y; L, T)$ be any selection of F and let $G = (L+T)^{-1}(\Psi+T)\, r$. Now there exists a $x \in E$ with $(L+T)^{-1}(\Psi+T)\, r(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. If $x \in E \setminus \overline{U}$ then if $y = r(x)$, note $y \in \partial U$ and $(L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(\Psi + T)(y) \neq \emptyset$, a contradiction. Thus $x \in \overline{U}$ so $\emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(\Psi + T)\, r(x) = (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(\Psi + T)\,(x)$.

References

[1] A. Ben Amar, D. O'Regan, *Topological Fixed Point Theory for Singlevalued and Multivalued Mappings with Applications*, Springer, Cham, 2016.

[2] X.P. Ding, W.K. Kim, K.K. Tan, *A selection theorem and its applications*, Bull. Austral. Math. Soc., **46**(1992), 205-212.

[3] G. Fournier, A. Granas, *The Lefschetz fixed point theorem for some classes on non-metrizable spaces*, J. Math. Pures Appl., **52**(1973), 271-283.

[4] M. Furi, P. Pera, *A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals*, Annales Polonici Mathematici, **47**(1987), 331-346.

[5] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[6] D. O'Regan, *Nonlinear alternatives for multivalued maps with applications to operator inclusions in abstract spaces*, Proc. Amer. Math. Soc., **127**(1999), 3557-3564.

[7] D. O'Regan, *Furi-Pera type theorems for the U_c^κ-admissible maps of Park*, Math. Proc. R. Ir. Acad., **102A**(2002), 163-173.

[8] D. O'Regan, *A nonlinear alternative for maps with continuous selections*, Communications in Applied Analysis, **16**(2012), 175-178.

[9] D. O'Regan, R. Precup, *Fixed point theorems for set-valued maps and existence principles for integral inclusions*, J. Math. Anal. Appl., **245**(2000), 594-612.

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