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AN EXTENSION OF THE POINCARÉ-BIRKHOFF FIXED POINT THEOREM TO NONINVARIANT ANNULI

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Abstract. An extension of the Poincaré-Birkhoff fixed point theorem to noninvariant under area-preserving homeomorphism annuli is considered. Unlike the well-known W.-Y. Ding's theorem [7], the inner boundary component of an annulus is not assumed to be star-shaped, while the outer boundary component is star-shaped. The existence of at least two fixed points for area preserving homeomorphism satisfying some twist condition is proved.
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1. Introduction

In 1912, H. Poincaré published the unproved theorem [21], which now is known as "Poincaré's last geometric theorem" or the Poincaré-Birkhoff fixed point theorem. It asserts that the area-preserving homeomorphism of the planar circular annulus onto itself, keeping both boundary components invariant, admits at least two fixed points if the points of the inner and the outer boundary circles are advanced in opposite angular directions (twist condition).

The invariance of the boundary components under the homeomorphism mentioned above is too severe assumption which does not permit the use of this theorem in order to prove the existence of periodic solutions for nonautonomous ordinary differential equations (particularly, for planar nonautonomous hamiltonian systems).

In 1925, G. Birkhoff, who published in 1913 the erroneous proof of Poincaré's theorem [1] (corrected in [3]), presented the topological generalization of this theorem [2], but in this paper we consider the results concerning only an area-preserving homeomorphism. Accordingly, the result of H. Jacobowitz [11] should be noted. He considered an area preserving homeomorphism of a noninvariant annulus with arbitrary (not star-shaped) external boundary component and a circular or a degenerate internal boundary component. By applying Birkhoff's method [2], he proved the existence of at least two fixed points for homeomorphism.

W.-Y. Ding [7] considered the noninvariant annulus with the star-shaped inner boundary component and an arbitrary simple curve as an outer boundary component. His proof of the existence of at least two fixed points was based on the theorem of Jacobowitz.

C. Rebelo doubted the correctness of Jacobowitz's proof [22] and, as a result, the validity of Ding's theorem. The proof of the existence of at least two fixed points for homeomorphism of the noninvariant annulus, presented in [22], was based directly on the Poincaré-Birkhoff theorem without invoking Jacobowitz theorem. In addition, both boundary components of an annulus were assumed to be star-shaped.

Afterwards, the doubts of Rebelo were justified. R. Martins and A.J. Urena constructed [18] the twist area-preserving free fixed point homeomorphism of the annulus with both not star-shaped boundary components. A while later, P. Le Calvez and J. Wang [15] considered the example of annulus with not star-shaped outer boundary component and, by applying the Oxtoby-Ulam theorem [19], constructed the twist area-preserving annulus homeomorphism without fixed points. In fairness it must be said that numerous applications of Ding's theorem, to the author's knowledge, are related to star-shaped annuli.

Also, it is worth to mention the following results. J. Franks in [9] proved the existence of two fixed points for symplectic diffeomorphism of noninvariant circular annulus. The authors of [8] in order to study the periodic solutions of Hamiltonian systems considered the periodic annulus, i.e. the annulus which is comprised of periodic trajectories. This particular case permitted them to deal with neither invariant nor star-shaped annulus.

The goal of this paper is to prove the extension of the Poincaré-Birkhoff fixed point theorem to a noninvariant under the area-preserving homeomorphism annulus which inner boundary component is not assumed to be star-shaped, unlike Ding's and Rebelo's theorems. Obviously, in view of the counterexamples mentioned above, removing the star-shaped condition we have to introduce another assumption. We assume the existence of a set of curves by which we introduce the curvilinear coordinates which permits us to formulate the boundary twist condition and, as a result, to obtain at least two fixed points.

2. Preliminaries

Let us formulate the modern version of the Poincaré-Birkhoff fixed point theorem. Denote $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x > 0\}$. Let (ϱ, θ) be the polar coordinates, $\Pi : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{O\}$ is the covering projection, $\Pi(\varrho, \theta) = (\varrho\cos\theta, \varrho\sin\theta)$. Then a continuous map $\widetilde{f} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+ \times \mathbb{R}$ is a lifting of a homeomorphism $f : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$ if

$$\Pi \circ \widetilde{f} = f \circ \Pi.$$

We call $\widetilde{f}$ a polar lifting of f. A lifting $\widetilde{f}$ is not unique but all other polar liftings $\widetilde{f}^*$ are expressed as $\widetilde{f}^*(\varrho, \theta) = \widetilde{f}(\varrho, \theta) + (0, 2\pi k)$, $k \in \mathbb{Z}$.

Let $A_0 = \{z \in \mathbb{R}^2 : a \leq |z| \leq b,\ 0 < a < b\}$ be a circle annulus with

$$C_{10} = \{z \in \mathbb{R}^2 : |z| = a\} \text{ and } C_{20} = \{z \in \mathbb{R}^2 : |z| = b\}$$

as its inner and outer boundary components, respectively. The following result is the modern version of the Poincaré-Birkhoff fixed point theorem.

Theorem 2.1. Let $f : A_0 \to A_0$ be an area preserving homeomorphism which admits a lifting $\widetilde{f} : \Pi^{-1}(A_0) \to \mathbb{R}_0^+ \times \mathbb{R}$ of the form

$$\widetilde{f}(\varrho, \theta) = (h(\varrho, \theta), \theta + g(\varrho, \theta)), \tag{2.1}$$

where $g(\varrho, \theta),\ h(\varrho, \theta)$ are continuous 2π-periodic in θ functions.
Assume that $f(C_{10}) = C_{10},\ f(C_{20}) = C_{20}$ and $g(a, \theta) \cdot g(b, \theta) < 0,\ \forall \theta \in \mathbb{R}$.
Then f has two fixed points in A_0.

If $g(a, \theta) \cdot g(b, \theta) < 0$ then f is called *a twist map.*

We say that $g(\theta, \varrho)$ and $I := h(\varrho, \theta) - \varrho$ are the angle and the radial displacements, respectively.

There are many various results in the direction of weakening the assumptions of the Poincaré-Birkhoff theorem (besides mentioned above, see [4], [5], [6], [10], [12], [13], [14], [16], [17], [20], [23]).

In this paper, we consider area-preserving homeomorphisms and noninvariant under these homeomorphisms topological annuli. A topological annulus $A \subset \mathbb{R}^2$ is a homeomorphic image of a circular annulus A_0. Let C_1, C_2 be the inner and the outer, respectively, boundary components of A. Assume that C_1, C_2 are simple closed curves. Denote by D_i the open domain bounded by $C_i : \partial D_i = C_i$. $\bar{D}_i$ is the closure of D_i, $i = 1, 2$. In what follows, we suppose that $O = (0, 0) \in D_1$.

H. Jacobowitz was the first who obtained, in 1976, a modified version of the Poincaré-Birkhoff theorem for a topological annulus and an area-preserving homeomorphism [11].

Theorem 2.2 (Jacobowitz, [11]**).** Let $f : A \to f(A) \subset \mathbb{R}^2 \setminus O$ be an area-preserving homeomorphism which admits a lifting $\widetilde{f} : \Pi^{-1}(A) \to \mathbb{R}_0^+ \times \mathbb{R}$ of the form (2.1). Assume that:

1) the inner boundary component C_1 is a circle $|z| = R, R \geq 0$, invariant under f (in the degenerate case the origin is the inner boundary);

2) $\liminf_{\varrho \to R} g(\varrho, \theta) > 0$ on $\Pi^{-1}(C_1)$ and $g(\varrho, \theta) < 0$ on $\Pi^{-1}(C_2)$.

Then f has at least two fixed points in A.

It should be noted that the condition 1) of this theorem admits the degenerate inner boundary component C_1 (if $R = 0$), which is an invariant circle and C_2 is a noninvariant simple curve not assumed to be star-shaped. W.-Y. Ding [7] presented the proof of the existence of at least two fixed points for area preserving twist homeomorphism without invariance of C_1 and C_2. In addition, the inner boundary component C_1 was supposed to be star-shaped.

Theorem 2.3 (W.-Y. Ding, [7]**).** Let $f : A \to f(A) \subset \mathbb{R}^2 \setminus O$ be an area-preserving homeomorphism which admits a lifting $\widetilde{f} : \Pi^{-1}(A) \to \mathbb{R}_0^+ \times \mathbb{R}$ of the form (2.1). Assume that:

1) the inner boundary component C_1 is star-shaped around the origin;

2) $g(\varrho, \theta) > 0$ on $\Pi^{-1}(C_1)$ and $g(\varrho, \theta) < 0$ on $\Pi^{-1}(C_2)$;

3) there exists an area-preserving homeomorphism $f_0 : \bar{D}_2 \to \mathbb{R}^2$ which satisfies $f_0|A = f$ and $O \in f_0(D_1)$.

Then f has at least two fixed points in A.

Remark 2.4. By the Oxtoby-Ulam theorem [19] the assumption 3) may be replaced by the following one: the areas of D_1 and $f(D_1)$ are equal and O belongs to a domain bounded by $f(C_1)$. Then, according to [19], an area-preserving homeomorphism f_0 (condition 3)) exists.

Ding's proof is based on an extension of f to D_1, and the construction of this extension is based essentially on the starlikeness of C_1.

C. Rebelo, doubting the correctness of Jacobowitz's theorem, proved the analogous theorem [22] basing her proof directly on the Poincaré-Birkhoff theorem for a circular annulus. In addition, in [22] C_2 was supposed to be a star-shaped curve. Denote $\hat{A} = \bar{D}_2 \setminus \{O\}$.

Theorem 2.5 (C. Rebelo, [22]**, Theorem 1).** Assume the outer boundary component C_2 is strictly star-shaped around the origin O. Let $f : \hat{A} \to f(\hat{A})$ be an area-preserving homeomorphism which admits a lifting $\widetilde{f} : \Pi^{-1}(\hat{A}) \to \mathbb{R}_0^+ \times \mathbb{R}$ of the form (2.1), where $g,\ h$ are continuous in $\hat{A}$ functions which are 2π-periodic in θ. Assume that

1) $g(\varrho, \theta) < 0$ for any point $(\varrho, \theta) \in \Pi^{-1}(C_2)$;
2) $\liminf_{\varrho \to 0} g(\varrho, \theta) > 0$ uniformly in θ;
3) $O \in f(\hat{A})$.

Then f has at least two fixed points $(x_i, y_i) = \Pi(\varrho_i, \theta_i) \in \hat{A}$, $i = 1, 2$, such that $g(\varrho_i, \theta_i) = 0$.

Note, that in contrast to Jacobowitz's theorem, the outer boundary component C_2 is a strictly star-shaped curve around the origin. Next, basing on the above theorem, C.Rebelo proved [22] the following theorem, assuming, in contrast to the Ding's theorem, the starlikeness not only of C_1 but as well as the starlikeness of C_2. The rest assumptions of her theorem, namely 2) and 3), coincide with Ding's ones.

Theorem 2.6 (C. Rebelo, [22]**, Corollary 2)**. Let $f : A \to f(A) \subset \mathbb{R}^2 \setminus \{O\}$ be an area-preserving homeomorphism which admits a lifting $\widetilde{f} : \Pi^{-1}(A) \to \mathbb{R}_0^+ \times \mathbb{R}$ of the form (2.1). Assume that:

1) the inner and the outer boundary components C_1, C_2 are strictly star-shaped around the origin;
2) $g(\varrho, \theta) > 0$ on $\Pi^{-1}(C_1)$ and $g(\varrho, \theta) < 0$ on $\Pi^{-1}(C_2)$;
3) there exists an area-preserving homeomorphism $f_0 : \bar{D}_2 \to \mathbb{R}^2$ which satisfies $f_0|A = f$ and $O \in f_0(D_1)$.

Then f has at least two fixed points $P_i = \Pi(\varrho_i, \theta_i) \in A$ such that $g(\varrho_i, \theta_i) = 0$, $i = 1, 2$.

Remark 2.7. It is worth to note that C. Rebelo asserts that $g(\varrho_i, \theta_i) = 0$ for fixed points $P_i = \Pi(\varrho_i, \theta_i)$. Really, otherwise it may occur that $P_i = \Pi(\varrho_i, \theta_i)$, $i = 1, 2$, turn out to be the periodical points of f_0 in D_1. The latter was not taken into account in the Ding's proof of the absence of fixed points for f_0 in D_1.

In this article, we prove the existence of at least two fixed points for an area-preserving homeomorphism of a topological annulus which inner boundary component C_1 is an arbitrary (not star-shaped) smooth simple curve and the outer boundary component C_2 is strictly star-shaped around the origin. Thus, we prove an extension of Rebelo's Theorem 2.6.

3. The main result

Denote $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, $I = [0, 2\pi)$. Let $\{\Gamma_\varphi\}$, $\varphi \in I$, be a set of smooth simple curves Γ_φ: $\mathbb{R}^+ \to \mathbb{R}^2$, such that Γ_φ is a continuously differentiable map, $\lim_{\varphi\to 2\pi-0} \Gamma_\varphi = \Gamma_0$ uniformly in $s \in \mathbb{R}^+$, $\Gamma_\varphi(0) = O$ and

$$\cup_{\varphi\in[0,2\pi)}\Gamma_\varphi = \mathbb{R}^2, \qquad \Gamma_\varphi \cap \Gamma_\psi = O,\ \varphi, \psi \in [0, 2\pi),\ \varphi \neq \psi.$$

Thus, O is the unique starting point of the curves $\Gamma_\varphi, \varphi \in I$. Let $s \in \mathbb{R}^+$ be a natural parameter, i.e. s is the length of the arc $O\Gamma_\varphi(s) \subset \Gamma_\varphi(\mathbb{R}^+)$, where $\Gamma_\varphi(s)$ is the endpoint of the arc $O\Gamma_\varphi(s)$, for some $\varphi \in [0, 2\pi)$. Assume that

$$\varphi = Arg\left(\frac{d\Gamma_\varphi(s)}{ds}(0)\right).$$

We assume that each curve Γ_φ, $\varphi \in I$, intersects the inner boundary component C_1 at a unique point.

Next, along with a polar lifting $\widetilde{f}$ we introduce one more lifting corresponding to Γ_φ. Consider a continuously differentiable map $\Gamma : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{O\}$ such that $\Gamma(s, \varphi)$ is 2π-periodic in φ, $\Gamma(s, \varphi) = \Gamma_\varphi(s)$ for $\varphi \in [0, 2\pi)$. We may consider $\mathbb{R}_0^+ \times \mathbb{R}$ and $\mathbb{R}^2 \setminus \{O\}$ as covering and base spaces, respectively, and Γ as a covering map. Really, for any point $z^* \in \mathbb{R}^2 \setminus \{O\}$ there exists such unique point $(s^*, \varphi^*) \in \mathbb{R}_0^+ \times \mathbb{R}$, where $\varphi^* \in [0, 2\pi)$, that $z^* = \Gamma^*_\varphi(s^*) = \Gamma(s^*, \varphi^*)$. At first, suppose $\varphi^* \neq 0$. Consider $\varphi_1, \varphi_2 \in (0, 2\pi)$, $\varphi_1 < \varphi^* < \varphi_2$, $|\varphi^* - \varphi_i| < \varepsilon_1$ for some sufficiently small $\varepsilon_1 > 0$, and s_1, s_2, $s_i \in \mathbb{R}_0^+$, $s_1 < s^* < s_2$, $|s^* - s_i| < \varepsilon_2$ for some sufficiently small $\varepsilon_2 > 0$ $i = 1, 2$. Then Γ is the homeomorphic map of an open set

$$\widetilde{U}_0 = \{(s, \varphi) : s_1 < s < s_2,\ \varphi_1 < \varphi < \varphi_2\}$$

onto an open set $U = \{z : z \in \mathbb{R}^2 \setminus \{O\},\ z = \Gamma(s, \varphi), (s, \varphi) \in \widetilde{U}_0\}$. Besides, for each $k \in \mathbb{Z}$, Γ is the homeomorphic map of an open set

$$\widetilde{U}_k = \{(s, \varphi) : s_1 < s < s_2,\ \varphi_1 + 2\pi k < \varphi < \varphi_2 + 2\pi k\}$$

onto U and $\Gamma^{-1}(U)$ is the disjoint union of open subsets

$$\widetilde{U}_k = \{(s, \varphi) : s_1 < s < s_2,\ \varphi_1 + 2\pi k < \varphi < \varphi_2 + 2\pi k\}$$

of $\mathbb{R}_0^+ \times \mathbb{R}$:

$$\Gamma^{-1}(U) = \bigcup_{k=-\infty}^{\infty} \widetilde{U}_k.$$

Now, suppose $\varphi^* = 0$. Consider φ_i, s_i, $i = 1, 2$, such that $-2\pi < \varphi_1 < 0$, $0 < \varphi_2 < 2\pi$, and assume that $|\varphi_i| < \varepsilon_1$ for some sufficiently small $\varepsilon_1 > 0$, and s_1, s_2 satisfying the above assumptions. Preserving the preceding notations, consider

$$\widetilde{U}_0 = \{(s, \varphi) : s_1 < s < s_2, \varphi_1 < \varphi < \varphi_2\}.$$

Let us represent $\widetilde{U}_0$ as $\widetilde{U}_0 = \widetilde{U}_0^- \cup \widetilde{U}_0^+$, where

$$\widetilde{U}_0^- = \{(s, \varphi) : s_1 < s < s_2,\ \varphi_1 < \varphi < 0\},\ \widetilde{U}_0^+ = \{(s, \varphi) : s_1 < s < s_2,\ 0 \leq \varphi < \varphi_2\}.$$

Let us construct the following homeomorphism

$$\Gamma(s,\varphi)=\Gamma_\varphi(s),\quad (s,\varphi)\in U_0^+,$$
$$\Gamma(s,\varphi)=\Gamma(s,\varphi+2\pi)=\Gamma_{\varphi+2\pi}(s),\quad (s,\varphi)\in U_0^-.$$

Consider the set $U=\{z : z\in\mathbb{R}^2\setminus\{O\}, z=\Gamma_\varphi(s)\}$. Thus, taking in account the properties of curves Γ_φ, we obtain that Γ is the homeomorphic map of an open set $\widetilde{U}_0$ onto an open set U. The concluding reasoning, analogous to the preceding one, implies that $\Gamma:\mathbb{R}_0^+\times\mathbb{R}\to\mathbb{R}^2\setminus\{O\}$ is the covering map. Then a map $\widetilde{f}_\Gamma:\mathbb{R}\times\mathbb{R}_0^+\to\mathbb{R}\times\mathbb{R}_0^+$ is a lifting of f if

$$\Gamma\circ\widetilde{f}_\Gamma=f\circ\Gamma.$$

We call $\widetilde{f}_\Gamma$ a Γ-lifting of f.

Let us introduce the following notations:

$$\widetilde{C}_i=\Pi^{-1}(C_i),\ \widetilde{D}_i=\Pi^{-1}(D_i),\ \widetilde{C}_{i\Gamma}=\Gamma^{-1}(C_i),\ \widetilde{D}_{i\Gamma}=\Gamma^{-1}(D_i),\ i=1,2,$$
$$\widetilde{A}=\Pi^{-1}(A),\ \widetilde{A}_\Gamma=\Gamma^{-1}(A).$$

Suppose that $\widetilde{f}$ has the form (1) and $\widetilde{f}_\Gamma$ admits the following form

$$\widetilde{f}_\Gamma(s,\varphi)=(h_\Gamma(s,\varphi),\varphi+g_\Gamma(s,\varphi)),$$

where g_Γ, h_Γ are continuous functions, 2π-periodic in φ. Next, consider the question of compatibility of liftings $\widetilde{f}$ and $\widetilde{f}_\Gamma$. For $M\in A$ denote $M_1=f(M)$, $M=\Pi(\varrho,\theta)=\Gamma(s,\varphi)$, $M_1=\Pi(\varrho_1,\theta_1)=\Gamma(s_1,\varphi_1)$, where second variables (angles) are defined up to $2\pi n, n\in\mathbb{Z}$. For $\theta\in[0,2\pi)$ and $\varphi\in[0,2\pi)$ there exists a one-to-one correspondence (diffeomorphism) between points (ϱ,θ) and (s,φ), which follows, in this case, from one-to-one correspondences between $(x,y)\in\mathbb{R}^2$ and (ϱ,θ), on the one hand, and between (x,y) and (s,φ), on the other hand. Thus, for $\theta\in[0,2\pi)$ and $\varphi\in[0,2\pi)$, there exists a one-to-one map $q:\mathbb{R}_0^+\times[0,2\pi)\to\mathbb{R}_0^+\times[0,2\pi)$, $q(s,\varphi)=(\varrho(s,\varphi),\theta(s,\varphi))$.

Let us extend q from $\mathbb{R}_0^+\times[0,2\pi)$ to $\mathbb{R}_0^+\times\mathbb{R}$.

First, assume that $\varrho(t,\varphi+2\pi n)=\varrho(t,\varphi)$, $\theta(t,\varphi+2\pi n)=\theta(t,\varphi)+2\pi n$, $n\in\mathbb{Z}$. Next, if $\varphi\notin[0,2\pi)$ then there exists a unique $k\in\mathbb{Z}$ and a unique $\varphi^*\in[0,2\pi)$ such that $\varphi=\varphi^*+2\pi k$, and we have

$$q(s,\varphi)=q(s,\varphi^*+2\pi k)=(\varrho(s,\varphi^*+2\pi k),\theta(s,\varphi^*+2\pi k))=(\varrho(s,\varphi^*),\theta(s,\varphi^*)+2\pi k).$$

Note, that if $\widetilde{f}(s,\varphi^*)=(s_1,\varphi_1^*)$, then

$$\begin{aligned}\widetilde{f}(s,\varphi^*+2\pi k)&=(s_1,\varphi^*+2\pi k+g_\Gamma(s,\varphi^*+2\pi k)=(s_1,\varphi^*+2\pi k+g_\Gamma(s,\varphi^*))\\&=(s_1,\varphi_1^*+2\pi k).\end{aligned}$$

Denote $\varphi_k=\varphi^*+2\pi k$, $\varphi_{1k}=\varphi_1^*+2\pi k$. Then

$$\theta(s_1,\varphi_{1k})-\theta(s,\varphi_k)=\theta(s_1,\varphi_1^*+2\pi k)-\theta(s,\varphi^*+2\pi k)=\theta(s_1,\varphi_1^*)-\theta(s,\varphi^*).$$

Hence, $\theta(s_1,\varphi_{1k})-\theta(s,\varphi_k)=\theta(s_1,\varphi_{1m})-\theta(s,\varphi_m)$ for any $k,m\in\mathbb{Z}$.

Thus, the difference $\theta(s_1,\varphi_1)-\theta(s,\varphi)$ is correctly defined if $\varphi_1=\varphi+g_\Gamma(t,\varphi)$, $\varphi=\varphi_k$, $\varphi_1=\varphi_{1k}$, $\varphi_1^*=\varphi^*+g_\Gamma(t,\varphi^*)$. Using the above notations, consider the equality

$$\theta_1=\theta+g(\varrho,\theta)=\theta(s,\varphi)+g(\varrho(s,\varphi),\theta(s,\varphi)).$$

On the other hand

$$\theta_1 = \theta(s_1, \varphi_1) = \theta(h_\Gamma(s,\varphi), \varphi + g_\Gamma(s,\varphi)).$$

Therefore

$$g(\varrho(s,\varphi), \theta(s,\varphi)) = \theta(h_\Gamma(s,\varphi), \varphi + g_\Gamma(s,\varphi)) - \theta(s,\varphi),$$

where $g(\varrho(s,\varphi), \theta(s,\varphi))$ is well defined and does not depend on $k \in \mathbb{Z}$ such that $\varphi = \varphi_k$, $\varphi_1 = \varphi_{1k}$. The main theorem is the following one.

Theorem 3.1. Let $f : A \to f(A)$ be an area-preserving homeomorphism.

Assume that

1) the outer boundary component C_2 is a strictly star-shaped curve around the origin, the inner boundary component C_1 is a smooth simple curve;

2) f has the following liftings

$$\widetilde{f}(\varrho,\theta) = (h(\varrho,\theta), \theta + g(\varrho,\theta),), \quad \widetilde{f}_\Gamma(s,\varphi) = (h_\Gamma(s,\varphi), \varphi + g_\Gamma(s,\varphi)),$$

where $h(\varrho,\theta), g(\varrho,\theta), h_\Gamma(s,\varphi), g_\Gamma(s,\varphi)$ are continuous 2π-periodic in θ and φ, respectively, functions such that

$g(\varrho(s,\varphi), \theta(s,\varphi)) < 0$ for any $(s,\varphi) \in \widetilde{C}_{2\Gamma}$;

$g_\Gamma(s,\varphi) > 0$ for any $(s,\varphi) \in \widetilde{C}_{1\Gamma}$;

3) there exists an area-preserving homeomorphism $f_0 : \bar{D}_2 \to \mathbb{R}^2$ such that

$$f_0|_A = f, \ O \in f_0(D_1).$$

Then f has at least two fixed points in A, $(x_i, y_i) = \Gamma(s_i, \varphi_i) \in A$, $i = 1, 2$, such that $g(s_i, \varphi_i) = 0$.

Proof. The idea of the proof is as follows: we will construct, modifying Ding's method [7], an area-preserving map $f_2 : \bar{D}_2 \to \mathbb{R}^2$ such that $f_2|_A = f$, $f_2(O) = O$, f_2 has no fixed points in D_1, and if $\widetilde{f}_2 = (h_2(\varrho,\theta), \theta + g_2(\varrho,\theta))$ is a polar lifting of f_2 then $\liminf_{\varrho\to 0} g_2(\varrho,\theta) > 0$. Then, we apply Rebelo's Theorem 2.5 to the map f_2. Since $f_2|_A = f$ and f_2 has no fixed points in D_1 then f has at least two distinct fixed points in A. We will construct f_2 as a composition of two maps: $f_2 = f_1 \circ G$ which will be constructed in the following lemmas.

Lemma 3.2. Under the above assumptions concerning $\Gamma(s,\varphi), C_1, C_2$, there exists an area-preserving homeomorphism $f_1 : \bar{D}_2 \to \mathbb{R}^2$ such that $f_1|_A = f$, $f_1(O) = O$.

Proof. In what follows, we will construct the homeomorphism f_1. Denote $P = f_0^{-1}(O) \in D_1$. There exists a curve Γ_φ such that $P \in \Gamma_\varphi$ for some φ. Consider a closed arc $\widehat{OP} \subset \Gamma_\varphi$ and a sequence of points $P_i \in \widehat{OP}$ such that $P_i = \Gamma(s_i, \varphi)$, where $s_{i-1} < s_i$, $i = 1, ..., n$, $P_n = P$, $P_0 = O$. Recall that s_i is the length of the arc $\widehat{OP_i}$. Let us fix points $O_i \in \widehat{P_{i-1}P_i}$, $i = 1, ..., n$, such that $|O_iP_{i-1}| = |O_iP_i| = d_i$, where d_i is the length of segments O_iP_{i-1}, O_iP_i. If O_i is not a unique point then we can take any point satisfying the above condition. Denote by d the distance between $\widehat{OP}$ and C_1. For sufficiently large n the points P_i, $i = 1, ..., n$, can be chosen in such a way that $2d_i < d$. For each $i \in \{1, ..., n\}$, let us choose any polar coordinate system (ϱ_i, θ_i) in $\mathbb{R}^2$ with O_i as its pole. Let us introduce a map $S_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 1, ..., n$, such that if $\widetilde{S}_i$ is a polar lifting of S_i then

$$\widetilde{S}_i(\varrho_i, \theta_i) = (\varrho_i, \theta_i + \eta_i(\varrho_i)),$$

where $\eta_i \in C^\infty$, $\eta_i(\varrho_i) = 0$ for $\varrho_i \geq \frac{1}{2}(d_i + d)$, $\eta_i(\varrho_i) = \alpha_i$ for $0 < \varrho_i \leq d_i$, where $\alpha_i > 0$ is the angle of rotation of the segment O_iP_{i-1} till its coincidence with O_iP_i. Hence, $S_i(P_{i-1}) = P_i$, $S_i|_A = Id$ (the identity map). It is obvious that S_i is an area-preserving diffeomorphism. Consider the composition $S = S_n \circ S_{n-1} \circ ... \circ S_1$. Then $S|_A = Id$, $S(O) = P$, S is an area-preserving diffeomorphism. Let us define a map $f_1 = f_0 \circ S$. Then f_1 is an area-preserving homeomorphism,

$$f_1|_A = f, \ f_1(O) = f_0 \circ S(O) = f_0(P) = O.$$

Thus, Lemma 3.2 is proved.

Before proceeding to the next Lemma 3.3, denote by $\widetilde{f}_{1\Gamma}$ a Γ-lifting of restriction $f_1|_{\bar{D}_2 \setminus \{O\}}$. Let us represent $\widetilde{f}_{1\Gamma}$ as

$$\widetilde{f}_{1\Gamma}(s, \varphi) = (h_{1\Gamma}(s, \varphi), \varphi + g_{1\Gamma}(s, \varphi),$$

where $g_{1\Gamma}, h_{1\Gamma}$ are continuous 2π-periodic in φ functions. Hence $f_1|_A = f$ then $h_{1\Gamma}(s, \varphi)$ and $g_{1\Gamma}$ may be chosen in such a way that

$$h_{1\Gamma}(s, \varphi) = h_\Gamma(s, \varphi), \quad g_{1\Gamma}(s, \varphi) = g_\Gamma(s, \varphi) \tag{3.1}$$

for $(s, \varphi) \in \widetilde{A}_\Gamma$, i.e. $\widetilde{f}_{1\Gamma}|_{\widetilde{A}_\Gamma} = \widetilde{f}_\Gamma$. According to the assumption 2) of the Theorem, $g_\Gamma(s, \varphi) > 0$ for any $(s, \varphi) \in \widetilde{C}_{1\Gamma}$. Then $g_{1\Gamma}(s, \varphi) > 0$ for any $(s, \varphi) \in \widetilde{C}_{1\Gamma}$. The periodicity of $g_{1\Gamma}(s, \varphi)$ in φ implies that there exists a neighborhood U of C_1 and a constant $a > 0$ such that

$$g_{1\Gamma}(s, \varphi) \geq a > 0$$

for any $(s, \varphi) \in \Gamma^{-1}(U)$.

Theorem 2.6 is valid not only for the standard area measure $dxdy$. It will remain valid for any appropriate measure (see [11], p.46-47, [3], p.31). Therefore, in what follows we assume that f_1 is the homeomorphism preserving the measure $\frac{dxdy}{J_\Gamma}$. With this assumption we construct a homeomorphism satisfying the twist condition 2) of Theorem 2.6.

Denote $P_\varphi = \Gamma_\varphi \cap C_1$. Note that, according to the assumption, P_φ is a unique point for each Γ_φ. Suppose, that $P_\varphi = \Gamma(s_\varphi, \varphi)$ for some s_φ and $\varphi \in [0, 2\pi)$.

Consider a function $\lambda(s, \varphi) = \frac{s}{s_\varphi}$, where $s \in \mathbb{R}^+$, $\varphi \in [0, 2\pi)$. Using λ we get a set of curves $C_\mu \subset \bar{D}_1$, $\mu \in [0, 1]$ such that $\Gamma^{-1}(C_\mu) = \{(s, \varphi) : \lambda(s, \varphi) = \mu\}$. Note, that $\lambda^{-1}([0, 1)) = D_1$ and each Γ_φ has a unique point with each C_μ. Really, if C_μ, for some $\mu \in (0, 1]$, has two common points $P_{1\varphi} \neq P_{2\varphi}$ with some Γ_φ such that $P_{1\varphi} = \Gamma(s_1, \varphi)$, $P_{2\varphi} = \Gamma(s_2, \varphi)$, $s_1 \neq s_2$, then $s_1 = \mu s_\varphi = s_2$ and we get a contradiction.

Let us choose $\varepsilon > 0$ such small that $\lambda^{-1}((1 - \varepsilon, 1)) \subset U$. For $\mu \in [0, 1]$ denote

$$\widetilde{C}_{\mu\Gamma} = \Gamma^{-1}(C_\mu),$$

$$\beta(\mu) = \max\{s : (s, \varphi) \in \widetilde{C}_{\mu\Gamma}\},$$

$$\gamma(\mu) = \max\{|g_{1\Gamma}(s, \varphi)| : (s, \varphi) \in \widetilde{C}_{\mu\Gamma}\}.$$

Lemma 3.3. There exists an area-preserving homeomorphism $G : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$ having a Γ-lifting $\widetilde{G}_\Gamma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$ presented as

$$\widetilde{G}_\Gamma(s, \varphi) = (h_{G\Gamma}(s, \varphi), \varphi + g_{G\Gamma}(s, \varphi)),$$

where $g_{G\Gamma}(s,\varphi), h_{R\Gamma}(s,\varphi)$ are continuous functions, 2π-periodic in φ, and

$$g_{G\Gamma}(s,\varphi)=0,\ h_{G\Gamma}(s,\varphi)=s \quad \text{for } (s,\varphi)\in\Gamma^{-1}(\mathbb{R}^2\setminus D_1),$$

$$g_{G\Gamma}(s,\varphi)\geq 0 \quad for\ (s,\varphi)\in\Gamma^{-1}(\mathbb{R}\times\mathbb{R}_+),$$

$$g_{G\Gamma}(s,\varphi)\geq\gamma(\mu)+a \quad for\ (s,\varphi)\in\widetilde{C}_{\mu\Gamma},\ \mu\in(0,1-\varepsilon]$$

Proof. We construct a hamiltonian system with respect to (s,φ) and for $\widetilde{G}_\Gamma$ we take a time shift map along the trajectories of this hamiltonian system. Following Ding's idea [7], let us consider a function $F\in C^\infty(0,+\infty)$ such that $F(u)=0$ for $u\geq 1$, $F(u)>0$ for $u\in(1-\varepsilon,1)$, $F(u)\geq\frac{\beta(u)(\gamma(u)+a)}{u}$ for $u\in(0,1-\varepsilon]$. Define the following function (Hamiltonian)

$$H(s,\varphi)=\int_0^{\lambda(s,\varphi)}F(u)du,$$

and consider the Hamiltonian system

$$\dot{s}=-\frac{\partial H}{\partial\varphi},\quad \dot{\varphi}=\frac{\partial H}{\partial s},$$

where $\dot{\varphi}=\frac{d\varphi}{dt},\ \dot{s}=\frac{ds}{dt}$. Taking into account the form of $H(s,\varphi)$ we obtain

$$\dot{s}=F(\lambda(s,\varphi))\cdot\frac{s}{s_\varphi^2}\cdot\frac{\partial s_\varphi}{\partial\varphi},\quad \dot{\varphi}=F(\lambda(s,\varphi))\cdot\frac{1}{s_\varphi}. \tag{3.2}$$

It is not difficult to show that $\lambda(s,\varphi)$ is a first integral of this system. Really, $H(s,\varphi)$ is a first integral and F is nonnegative. Moreover, the set of stationary points of (3.2) is $\lambda^{-1}([1,\infty))$, and $\widetilde{C}_{\Gamma\mu}$ are the trajectories of (3.2). Denote by $(\bar{s}(t,s,\varphi),\bar{\varphi}(t,s,\varphi))$ a solution of (3.2) such that $\bar{s}(0,s,\varphi)=s,\ \bar{\varphi}(0,s,\varphi)=\varphi$ for any $(s,\varphi)\in\widetilde{D}_{1\Gamma}$. Consider a time $(t=1)$ shift mapping $\widetilde{G}_\Gamma$ along the trajectories of the system

$$\widetilde{G}_\Gamma(s,\varphi)=(\bar{s}(1,s,\varphi),\bar{\varphi}(1,s,\varphi)).$$

Since the system (3.2) is hamiltonian then $\widetilde{G}_\Gamma$ is area-preserving homeomorphism. Moreover, denote, for brevity, by $\widetilde{G}_\Gamma$ such map that $\widetilde{G}_\Gamma(s,\varphi)=(s,\varphi)$ for $(s,\varphi)\in\mathbb{R}^+\times\mathbb{R}\setminus\bar{\widetilde{D}}_1$. Represent $\widetilde{G}_\Gamma$ as

$$\widetilde{G}_\Gamma(t,\varphi)=(h_{G\Gamma}(s,\varphi),\varphi+g_{G\Gamma}(s,\varphi)),$$

where $g_{G\Gamma}(s,\varphi),\ h_{G\Gamma}(s,\varphi)$ are 2π-periodic in φ continuous functions.

The second equation of (3.2) implies that $g_{G\Gamma}(s,\varphi)>0$ for any $(s,\varphi)\in\widetilde{C}_{\mu\Gamma}$. Let $\bar{s}(t)=\bar{s}(t,s,\varphi),\ \bar{\varphi}(t)=\bar{\varphi}(t,s,\varphi)$ be the solution of (3.2) corresponding to the trajectory $\widetilde{C}_{\Gamma\mu}:\ \lambda(\bar{s}(t),\bar{\varphi}(t))=\mu$. Then $\frac{\bar{s}(t)}{s_{\bar{\varphi}(t)}}=\mu$ and from (3.2) we obtain

$$\dot{\bar{\varphi}}(t)=F(\mu)\cdot\frac{\mu}{\bar{s}(t)}\geq\frac{\mu\beta(\mu)(\gamma(\mu)+a)}{\bar{s}(t)\mu}\geq\gamma(\mu)+a.$$

Integrating on $[0,1]$, we obtain $g_{G\Gamma}(s,\varphi)=\bar{\varphi}(1)-\varphi\geq\gamma(\mu)+a$ for $(s,\varphi)\in\widetilde{C}_{\mu\Gamma}$, $\mu\in(0,1-\varepsilon]$. Taking in account that $F(u)=0$ for $u\geq 1$, we get $g_{G\Gamma}(s,\varphi))=0$, $h_{G\Gamma}(s,\varphi))=s$ for $(s,\varphi)\in\Gamma^{-1}(\mathbb{R}^2\setminus D_1)$. Thus, Lemma 3.3 is proved.

Let us return to the proof of Theorem 3.1. Below, we follow the Ding's arguments [7] modifying them. Denote by f_2 the composition: $f_2 = f_1 \circ G$. Hence, the $\Gamma-$lifting $\widetilde{f}_{2\Gamma}$ of f_2 is the composition: $\widetilde{f}_{2\Gamma} = \widetilde{f}_{1\Gamma} \circ \widetilde{G}_{\Gamma}$. We represent $\widetilde{f}_{2\Gamma}$ as

$$\widetilde{f}_{2\Gamma}(s,\varphi) = (h_{2\Gamma}(s,\varphi), \varphi + g_{2\Gamma}(s,\varphi)),$$

where $h_{2\Gamma},\ g_{2\Gamma}$ are continuous and $2\pi-$periodic in the second variable functions. The above notations imply that

$$h_{2\Gamma}(s,\varphi) = h_{1\Gamma}(h_{G\Gamma}(s,\varphi), \varphi + g_{G\Gamma}(s,\varphi)),$$

$$g_{2\Gamma}(s,\varphi) = g_{G\Gamma}(s,\varphi) + g_{1\Gamma}(h_{G\Gamma}(s,\varphi), \varphi + g_{G\Gamma}(s,\varphi))$$

From (3.1) and the assumptions of Lemma 3.3 it follows that

$$h_{2\Gamma}(s,\varphi) = h_{\Gamma}(s,\varphi), \quad g_{2\Gamma}(s,\varphi) = g_{\Gamma}(s,\varphi), \text{ for } (s,\varphi) \in \widetilde{A}_{\Gamma}.$$

Now, consider $(s,\varphi) \in \Gamma^{-1}(D_1 \setminus \{O\})$. For $(s,\varphi) \in \Gamma^{-1}(\lambda^{-1}((1-\varepsilon,1))) \subset \Gamma^{-1}(U)$ we have

$$g_{2\Gamma}(s,\varphi) \geq g_{1\Gamma}(h_{G\Gamma}(s,\varphi), \varphi + g_{2\Gamma}(s,\varphi)) \geq a > 0.$$

For $(s,\varphi) \in \Gamma^{-1}(\lambda^{-1}((0,1-\varepsilon)))$ we have $(s,\varphi) \in \widetilde{C}_{\mu\Gamma}$, where $\mu \in (0,1-\varepsilon)$, and therefore, taking into account the assumptions of Lemma 3.3, we have

$$g_{2\Gamma}(s,\varphi) \geq \gamma(\mu) + a + g_{1\Gamma}(h_{G\Gamma}(s,\varphi), \varphi + g_{G\Gamma}(s,\phi)) \geq a > 0.$$

Therefore, we have $g_{2\Gamma}(s,\varphi) \geq a > 0$ for $(s,\varphi) \in \Gamma^{-1}(D_1 \setminus \{O\})$. Thus, f_2 has no fixed points in $D_1 \setminus \{O\}$.

Next, we prove that $g_{\Gamma}(s,\varphi) > 0$ if and only if $g(\varrho,\theta) > 0$, for sufficiently small s and ϱ. Consider any Γ_φ, $\varphi \in [0,2\pi)$ and a point $P = \Gamma(s,\varphi) = \Pi(\varrho,\theta) \in \Gamma_\varphi$, $\theta \in [0,2\pi)$. Since $\varphi = Arg(\frac{d\Gamma_\varphi(s)}{ds}(0))$ then $\theta = \theta(s) \to \varphi$ with $s \to 0$. Thus, $\theta(s) = \varphi + \delta(s)$, where $\delta(s) \to 0$ with $s \to 0$. If $\varphi' > \varphi$ and $\theta'(s) \to \varphi'$, $\theta'(s) = \varphi + \delta'(s)$, where $\delta'(s) \to 0$ with $s \to 0$, then $\theta'(s) - \theta(s) = \varphi' - \varphi + \delta'(s) - \delta(s)$. Therefore, for sufficiently small s we have $\theta'(s) - \theta(s) > 0$ if and only if $\varphi' - \varphi > 0$. Thus, we have proved that $g_{\Gamma}(s,\varphi) > 0$ if and only if $g(\varrho,\theta) > 0$. Moreover, if $s > 0$ is so small that $\|\delta'(s) - \delta(s)\| < \frac{a}{2}$, then $\|\theta'(s) - \theta(s)\| > \frac{a}{2} > 0$.

Thus, we have proved that if a polar lifting $\widetilde{f}_2$ of f_2 is defined as

$$\widetilde{f}_2 = (h_2(\varrho,\theta), \varphi + g_2(\varrho,\theta))$$

then $\liminf_{\varrho \to 0} g_2(\varrho,\theta) > 0$. Applying Rebelo's Theorem 2.5 to f_2, we finish the proof of Theorem 3.1.

Remark 3.4. The map $\widetilde{f}_{2\Gamma}$ preserves the standard area measure $dsd\varphi$. Note that $dxdy = J_\Gamma dsd\varphi$, where J_Γ is the jacobian of the map Γ. In order to obtain the preserving of $dxdy$ measure, let us consider the homeomorphism $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ such that $\Phi(s,\varphi) = (s_1,\varphi_1)$, where

$$\varphi_1 = \varphi,\ s_1 = \int_0^s J_\Gamma(s,\varphi)ds.$$

Hence $J_\Gamma = J_\Phi$, where J_Φ is the jacobian of Φ, then the homeomorphism

$$\widetilde{f}^*_{2\Gamma} = \Phi \circ \widetilde{f}_{2\Gamma} \circ \Phi^{-1}$$

preserves the measure $ds_1 d\varphi_1 = J_\Phi ds d\varphi = J_\Gamma ds d\varphi = dxdy$. Moreover, if $\widetilde{P}$ is a fixed point of $\widetilde{f}^*_{2\Gamma}$ then $\Phi^{-1}(\widetilde{P})$ is a fixed point of $\widetilde{f}_{2\Gamma}$: $\widetilde{f}_{2\Gamma}(\Phi^{-1}(\widetilde{P})) = \Phi^{-1}(\widetilde{P})$.

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