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COMMON FIXED POINT THEOREMS IN PARTIAL IDEMPOTENT-VALUED METRIC SPACES

M. IRANMANESH*, S. RADENOVIĆ** AND F. SOLEIMANY***

*Department of Mathematical Sciences, Shahrood University of Technology,
Shahrood, Iran
E-mail: m.iranmanesh2012@gmail.com

**Faculty of Mechanical Engineering, University of Belgrade, Serbia E-mail: radens@beotel.rs

***Department of Mathematical Sciences, Shahrood University of Technology,
Shahrood, Iran
E-mail:enfazh.bmaam@gmail.com

Abstract. In this article, we consider complete partial idempotent-valued metric spaces and prove some common fixed point theorems in the setting of cone metric spaces over idempotent space.
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1. Introduction

Matthews (1994) introduced the notion of partial metric spaces. Recently, the authors (see [3, 2, 4, 5, 10]) have studied on this subject and have generalized some fixed point theorems in the setting of spaces. Huang and Zhang [6] defined the cone metric spaces. In setting, type metric spaces, the set of real numbers replacing by an ordered Banach space. After the definition of the concept of cone metric space in fixed point theory on these spaces has been developing (see [1, 8, 9]). In [10] the authors studied the operator-valued metric spaces and gave some fixed point theorems on the spaces.

In this paper, we introduce a new type of partial cone metric spaces with idempotent spaces, replacing Banach spaces and give some fixed and common fixed point theorems for partial valued metric spaces. We first introduce a concept of partial idempotent-valued metric space. Then the common fixed point results are established for this new class of self-maps are weakly compatible and the operators need not commute with each other.

2. Preliminaries

In this section, we shall define the partial idempotent valued metric space and give some properties.

Definition 2.1 A idempotent space is a vector space K over filed $\mathbb{R}$ in which a two fold $(K, \oplus)$ satisfies the following conditions:

(i) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for $a, b, c \in K$.
(ii) $a \oplus a = a$, for all $a \in K$.

A idempotent space is commutative if $a \oplus b = b \oplus a$ for $a, b \in K$.

Definition 2.2 Let $(K, \oplus)$ be a idempotent space, we shall employ the canonical order relation $\leq_\oplus$ on K defined by

$$a \leq_\oplus b \Leftrightarrow a \oplus b = b.$$

We shall also write $b \geq_\oplus a$ instead of $a \leq_\oplus b$.

Example 2.3 Let $K = \mathbb{R}$ with $a \oplus b := \max\{a, b\}$ or $a \oplus b := \min\{a, b\}$ for $a, b \in \mathbb{R}$ is idempotent space.

Example 2.4 Consider matrices having entries in $(\mathbb{R}, \oplus)$. For conforming matrices $A = (a_{ij}), B = (b_{ij})$ matrix addition together with multiplication by a scalar $\lambda \in \mathbb{R}$ follow the conventional rules

$$\{A + B\}_{ij} = a_{ij} \oplus b_{ij}, \text{ and } \{\lambda A\} = \lambda a_{ij}.$$

is idempotent space.

Definition 2.5 Let $(K, \leq)$ be a partially ordered set and $P \subseteq K$, such that $P \neq \emptyset$. An element $a \in P$ is called a maximal (resp. minimal) of P, if there exists no $a \neq x \in P$ such that $a < x$ (resp. $x < a$). The set of all maximal (resp. minimal) elements of P is denoted by $\max(P)$ (resp. $\min(P)$). If for any finite subset P, $\max(P)$ and $\min(P)$ always exists and unique then K is called a totally lattice.

Example 2.6 The space $\mathbb{R}$ is totally lattice respect to order $\leq_\oplus$ which is defined in Example 2.3.

Example 2.7 Let $S = \{a\}$, and $X = P(S) = \{\emptyset, S\}$ with the inclusion relation $\subseteq$ is a totally lattice. But if $S = \{a, b\}$ then X is not totally lattice.

Example 2.8 Let $X = \mathbb{R}^2$ where $(a, b) \leq (c, d)$ if and only if either $a < c$ or $a = c$ and $b \leq d$ is totally lattice.

Definition 2.9 Suppose that $(K, \leq)$ partial ordered set for $a, b \in K$ we define

$$EMax\{a, b\} := \begin{cases} a & b \leq a \\ b & a < b \\ 0_K & o.w \end{cases}$$

and for $a_1, a_2, ...a_n$ define:

$$EMax\{a_1, a_2, ...a_n\} := EMax\{EMax\{a_1, ...a_{n-1}\}, a_n\}.$$

Remark 2.10 It is trivial when K is a totally ordered set we have

$$EMax\{a,b\} = \max\{a,b\}.$$

Definition 2.11 Let $(K, \leq)$ be a partial ordered vector space. Let $\{x_n\}$ be a sequence in K and $x \in K$. If for every $0_K < c$, there is n_0 such that for all $n > n_0$, $x_n - x < c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or, $x_n \to x$ as $n \to \infty$.

Definition 2.12 Let $(K, \leq)$ be a partial order vector space, we say the order relation on K has positive cone ordering property if vector $0_K \leq a \leq b$ and scalar inequalities $0 \leq r \leq c$ imply that the inequalities

$$0_K \leq ra \leq rb, \qquad rx \leq cx.$$

for all $0_K \leq x \in K$.

Definition 2.13 Let $(K, \leq)$ be a partial order vector space. We say K is a normal space if the order relation on K has positive cone ordering property.

Example 2.14 Let $K = \mathbb{R}$ with $a \oplus b := \max\{a,b\}$ for $a, b \in \mathbb{R}$. It is trivial that K is a normal space.

Throughout this paper, $(K, \oplus)$ is denotes a commutative idempotent space. Now with the help of this $\leq_\oplus$ one can give the definition of a partial idempotent-valued metric space.

For later applications, it will be convenient to use the notation

$$K^+ := \{a \in K : \ a \geq_\oplus 0_K\}.$$

Definition 2.15 Let X be a non-empty set. Consider the mapping $d : X \times X \to K^+$ satisfies:

(i) $d(x,x) \leq_\oplus d(x,y)$ for all $x, y \in X$.
(ii) $x = y \Leftrightarrow d(x,x) = d(x,y) = d(y,y)$.
(iii) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(v) $d(x,y) \leq_\oplus d(x,z) \oplus d(y,z)$ for all $x, y, z \in X$.

Then d is called a *partial idempotent-valued metric* on X and (X, K, d) is called a *partial idempotent-valued metric space.*

Example 2.16 Let $X = [0, \infty)$, $K = \mathbb{R}$ with the operations $a \oplus_{\max} b := \max\{a,b\}$. Define the metric $d : X \times X \to K^+$ by

$$d(a,b) := a \oplus b.$$

Then X is a partial idempotent-valued metric space.

Example 2.17 Let M be a nonempty set and $X = B(M, \mathbb{R}^+)$ be the set of bounded mappings(mappings with order-bounded range). Let $K = B(X, (\mathbb{R}, \oplus_{\max}))$, with the point-wise generalized addition $(h \oplus g)(a) = h(a) \oplus g(a)$ on X. Define the metric mapping $d := X \times X \to K^+$ by

$$d(f,g)(a) := \max\{f(a), g(a)\}.$$

Then (X, d) is a partial idempotent-valued metric space.

Definition 2.18 Let (X, K, d) be a partial idempotent-valued metric space.

(i) a sequence $\{x_n\} \subseteq X$ converges to $x \in X$ if and only if
$$d(x, x) = \lim_{n\to\infty} d(x, x_n) = \lim_{n\to\infty} d(x_n, x);$$

(ii) a sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if and only if for $n, m \in \mathbb{N}$, $\lim_{n,m\to\infty} d(x_n, x_m)$ exists.

iii) the (X, d) is said to be complete if every Cauchy sequence $\{x_n\} \subseteq X$ converges, to a point $x \in X$ such that
$$d(x, x) = \lim_{n,m\to\infty} d(x_m, x_n) = \lim_{n,m\to\infty} d(x_n, x_m).$$

Definition 2.19 Suppose that (X, K, d) be a partial idempotent-valued metric space. $T : X \to X$ is called a continuous function at x if for any $x_n \to x$ implies that $Tx_n \to Tx$.

Definition 2.20 [7] Let X be a nonempty set and $f, g : X \to X$. The mappings f, g are said to be weakly compatible if they commute at their coincidence points. A point $y \in X$ is called a point of coincidence of f and g if there exists a point $x \in X$ such that $y = fx = gx$.

3. Common fixed point theorem in partial idempotent-valued metric spaces

In this section, we prove some common fixed point theorem for self mapping satisfying type idempotent-valued contractive mapping.

Theorem 3.1 *Let (X, A, d) be partial idempotent-valued metric spaces, and let $f, g : X \to X$ be two mappings such that $f(X) \subseteq g(X)$ and one of these subsets of X is complete. Suppose that there exists such that for all $x, y \in X$;*
$$d(f(x), f(y)) \leq_\oplus \psi(EMax\{d(g(x), g(y)), d(g(x), f(x)), d(g(y), f(y))\}), \tag{3.1}$$
where $\psi : K^+ \to K^+$ is a continuous, nondecreasing function such that
$$\lim_{n\to\infty} \psi^n(a) = 0_K$$
and $\psi(a) <_\oplus a$ for $a \in K^+$. Then f and g have a unique point of coincidence. Moreover, if $f$and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and, using condition $f(X) \subseteq g(X)$, construct a sequence $\{y_n\}$ satisfying $y_n = f(x_{n-1}) = g(x_n)$, $n = 1, 2, \cdots$. Suppose that $d(y_n, y_{n+1}) >_\oplus 0_K$ for each n otherwise the conclusion follows easily. Using (3.1) we conclude that
$$\begin{aligned} d(y_n, y_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\leq_\oplus \psi(Emax\{d(g(x_{n-1}), g(x_n)), d(g(x_n), g(x_{n+1}))\}) \\ &= \psi(EMax\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}), \quad (*). \end{aligned}$$
Now, if $EMax\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \neq 0$ for some n and
$$EMax\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_n, y_{n+1}),$$

then from (*) we have

$$d(y_n, y_{n+1}) \leq_{\oplus} \psi(d(y_{n+1}, y_n)) <_{\oplus} d(y_{n+1}, y_n).$$

which is a contradiction. Thus for all $n > 1$

$$EMax\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_n, y_{n-1}) \text{ or } 0_K.$$

As $0_K \leq_{\oplus} d(y_n, y_{n-1})$ therefore for all $n > 1$ we have

$$EMax\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \leq_{\oplus} d(y_n, y_{n-1}).$$

Then

$$d(y_n, y_{n+1}) \leq_{\oplus} \psi(d(y_{n-1}, y_n)),$$

and so

$$d(y_n, y_{n+1}) \leq_{\oplus} \psi^n(d(y_1, y_2)).$$

Since $\psi^n(a)$ is convergent to 0_K, this shows that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0_K$. Now, for $n, p \in \mathbb{N}$ we have

$$\begin{aligned}
d(y_{n+p}, y_n) &\leq_{\oplus} \psi^n(d(y_1, y_2)) \oplus \psi^{n+1}(d(y_1, y_2)) \oplus \cdots \oplus \psi^{n+p-1}(d(y_1, y_2)) \\
&\leq_{\oplus} \psi^n(d(y_1, y_2)) \oplus \psi^n(d(y_1, y_2)) \oplus \cdots \oplus \psi^{n+p-2}(d(y_1, y_2)) \\
&. \\
&. \\
&. \\
&\leq_{\oplus} \psi^n(d(y_1, y_2)) \oplus \psi^n(d(y_1, y_2)) \oplus \cdots \oplus \psi^n(d(y_1, y_2)) \\
&= \psi^n(d(y_1, y_2)). \quad \text{(From part (ii) of Definition 2.1).}
\end{aligned}$$

Since $\psi^n(a)$ is convergent to 0_K for each $a \in K^+$, $d(y_{n+p}, y_n) \to 0_K$ as $n \to \infty$. Put $m = n + p$, $m \to \infty$ as $n \to \infty$. Therefore

$$\lim_{n,m\to\infty} d(x_m, x_n) = \lim_{n\to\infty} d(y_{n+p}, y_n) = 0_K.$$

Then $\{y_n\}$ is a Cauchy sequence in the metric space $f(X)$. Since $f(X)$ (otherwise $g(X)$) is complete, then $\{y_n\}$ is convergent to p. Therefore,

$$\lim_{n\to\infty} d(g(x_n), p) = \lim_{n\to\infty} d(y_n, p) = \lim_{n,m\to\infty} d(y_m, y_n) = 0_K. \tag{3.2}$$

Consequently, we can find x in X such that $g(x) = p$. We claim that $f(x) = g(x)$. We show that $d(g(x), f(x)) = 0_K$. Assume this is not true. From we obtain

$$\begin{aligned}
d(f(x_n), f(x)) &\leq_{\oplus} \psi(EMax\{d(g(x), g(x_n)), d(g(x), f(x)), d(g(x_n), f(x_n))\}) \\
&= \psi(Emax\{d(p, g(x_n)), d(g(x), f(x)), d(g(x_n), g(x_{n-1}))\}) \\
&= \psi(d(g(x), f(x)). \ (**)
\end{aligned}$$

letting $n \to \infty$, from (3.2) it is obvious that $d(f(x_n), f(x)) \to d(g(x), f(x))$.
From $(**)$ we have

$$d(g(x), f(x)) \leq_{\oplus} \psi(d(g(x), f(x)) <_{\oplus} d(g(x), f(x)).$$

which is a contradiction. Thus $d(g(x), f(x)) = 0_K$. Since

$$0_K \leq_{\oplus} d(f(x), f(x)) \leq_{\oplus} d(g(x), f(x)) = 0_K.$$

Thus $d(f(x), f(x)) = d(g(x), f(x)) = 0_K$.
Similar to we can show that $d(g(x), g(x)) = d(g(x), f(x)) = 0_K$ and so $g(x) = f(x)$. If f and g are weakly compatible, then

$$g(p) = gf(x) = fg(x) = f(p),$$

therefore,

$$\begin{aligned} d(f(x), f(p)) &= \psi(EMax\{d(g(x), g(p)), d(g(x), f(x)), d(g(p), f(p))\}) \\ &= \psi(d(g(x), g(p)) \\ &<_\oplus d(g(x), g(p)). \end{aligned}$$

which implies $f(p) = f(x) = p$.

Example 3.2 Let $K = \mathbb{R}$ with $\oplus := \min$ and $X = (-1, 0]$ be endowed with the partially idempotent-valued metric

$$d(x, y) = x \oplus y = \min\{x, y\}.$$

Let $f, g : X \to X$ be defined by $f(x) = \frac{x}{2}$, $g(x) = x$ and $\psi : K^+ = (-\infty, 0] \to K^+$, $\psi(t) = \frac{t}{2}$, then f and g satisfy the condition (3.1). Without loss of generality, assume $x \le y$, respect to ordered by the standard on real numbers, therefore $\min\{x, y\} = x$, as $\min\{x, \frac{x}{2}\} = x$ for $x \in [-1, 0]$ we have

$$\frac{1}{2}\min\{\min\{x, y\}, \min\{x, \frac{x}{2}\}, \min\{y, \frac{y}{2}\}\} \le \min\{\frac{x}{2}, \frac{y}{2}\} = \frac{x}{2}.$$

i.e,

$$d(f(x), f(y)) \le_\oplus \psi(\max\{d(g(x), g(y)), d(g(x), f(x)), d(g(y), f(y))\}).$$

Using Theorem 3.1, we deduce that f and g have a common fixed point.

Corollary 3.3 *Let (X, A, d) be a partial idempotent-valued metric space and K be a normal space. Let f, g be two mappings such that $f(X) \subseteq g(X)$. Assume that*

$$d(f(x), f(y)) \le_\oplus r\ Emax\{d(g(x), f(x)), d(g(y), f(y))\}, \tag{3.3}$$

for all x, y, where $0 \le r < 1$. If $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. By setting $\psi(k) = rk$ where $k \in K^+$, it is a Consequence of Theorem 3.1.

Corollary 3.4 *Let X be a partial idempotent-valued metric space and K be a normal space. Let f, g two mappings such that $f(X) \subseteq g(X)$. Assume that*

$$d(f(x), f(y)) \le_\oplus rd(g(x), g(y)), \tag{3.4}$$

for all $x, y \in X$, where $0 \le r < 1$. If $f(X)$ or $g(X)$ be a complete subspace of X, then and have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. It is a Consequence of Theorem 3.1.

Corollary 3.5 *Let X be complete partially idempotent-valued metric space and $f : X \to X$ is a mapping such that*

$$d(fx, fy) \le_\oplus \psi(EMax\{d(x, fx), d(x, y), d(y, fy)\}). \tag{3.5}$$

for all $x, y \in X$, where $\psi : K^+ \to K^+$ is a continuous, nondecreasing function such that

$$\lim_{n\to\infty} \psi^n(a) = 0_K$$

and $\psi(a) <_\oplus a$ for $a \in K^+$. Then there exists unique $x \in X$ such that $x = f(x)$.

Proof. By setting $g = I_X$, we obtain Corollary 3.5.

Example 3.6 Let $X = \mathbb{R}^+$, $K = \mathbb{R}$ and $d(x, y) = x \oplus y = \max\{x, y\}$. Let

$$f(x) = \frac{x^2}{1+x} \text{ and } \psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \; \psi(t) = \frac{t}{1+t}.$$

The map ψ is continuous and nondecreasing. Then for all $x, y \in X$ with $x \geq_\oplus y$ we have

$$\begin{aligned} d(f(x), f(y)) &= \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} \\ &= \frac{x^2}{1+x} \\ &\leq_\oplus \psi(\max\{d(x, f(x)), d(x, y), d(y, f(y))\}). \end{aligned}$$

This shows that all conditions of Corollary 3.5 are satisfied and so f has a fixed point in X.

In the sequel, we give the first results about a common fixed point theorem which the operators need not commute with each other.

Theorem 3.7 *Let (X, A, d) be complete partial idempotent-valued metric spaces, and let $f, g : X \to X$ be two mappings such that $f(X) \subseteq g(X)$ and one of these subsets of X is complete. Suppose that there exists such that for all $x, y \in X$;*

$$d(f(x), g(y)) \leq_\oplus \psi(Emax\{d(f(x), y), d(y, g(y)), d(x, y)\}), \tag{3.6}$$

where $\psi : K^+ \to K^+$ is a continuous, nondecreasing function such that

$$\lim_{n\to\infty} \psi^n(a) = 0_K$$

and $\psi(a) <_\oplus a$ for $a \in K^+$. Then f and g have a unique point of coincidence common fixed point.

Proof. Let $x_0 \in X$. Define the sequence x_n in a way that $x_2 = f(x_1)$ and $x_1 = g(x_0)$ and inductively

$$x_{2k} = f(x_{2k+1}), \;\; x_{2k+1} = g(x_{2k}), \;\; for \;\; k = 0, 1, 2, ...$$

If there exists a positive integer N such that $x_{2N} = x_{2N+1}$, then x_{2N} is a fixed point of f and hence a fixed point of g. Indeed, since $x_{2N} = x_{2N+1} = gx_{2N}$, then

$$gx_{2N} = gx_{2N+1} = ggx_{2N}.$$

Also, due to (3.6) we have

$$\begin{aligned} d(x_{2N+2}, x_{2N+1}) &\leq_\oplus \psi(EMax\{d(f(x_{2N+1}), x_{2N+1}), d(x_{2N+1}, x_{2N})\}) \\ &<_\oplus d(x_{2N+2}, x_{2N+1}). \end{aligned}$$

This implies that $d(x_{2N+2}, x_{2N+1}) = 0_K$ which yields that

$$f(x_{2N+1}) = x_{2N+2} = x_{2N+1}.$$

Notice that $x_{2N+1} = x_{2N}$ is the fixed point of g. As a result, $x_{2N+1} = x_{2N}$ is the common fixed point of g and f. A similar conclusion holds if $x_{2N+1} = x_{2N+2}$ for some positive integer N. Therefore, we may assume that $x_k \neq x_{k+1}$ for all k. If k is odd, due to (3.6), we have

$$\begin{aligned} d(x_{k+2}, x_{k+1}) &\leq_{\oplus} \psi(EMax\{d(x_{k+1}), x_k), d(x_{k+2}, x_{k+1}), d(x_k), x_{k+1})\}). \\ &= \psi(EMax\{d(x_{k+1}, x_k), d(x_{k+2}, x_{k+1})\}), (*). \end{aligned}$$

Now, if

$$EMax\{d(x_{k+1}), x_k), d(x_{k+2}, x_{k+1})\} = d(x_{k+2}, x_{k+1}) \neq 0_K,$$

for some n, then from (*) we have

$$d(x_{k+2}, x_{k+1}) \leq_{\oplus} \psi(d(x_{k+2}, x_{k+1})) <_{\oplus} d(x_{k+2}, x_{k+1}).$$

which is a contradiction. Thus

$$EMax\{d(x_{k+1}, x_k), d(x_{k+2}, x_{k+1})\} \leq_{\oplus} d(x_{k+1}, x_k).$$

Therefore, we have

$$d(x_{k+1}, x_k) \leq_{\oplus} \psi(d(x_k, x_{k-1})),$$

If k is even, analogously, can be obtained the same inequality. And so for $k = 1, 2, ...$ can observe that

$$d(x_{k+1}, x_k) \leq_{\oplus} \psi^k(d(x_1, x_0)).$$

Now, for $n, p \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq_{\oplus} \psi^n(d(x_1, x_0)) \oplus \psi^{n+1}(d(x_1, x_0)) \oplus \cdots \oplus \psi^{n+p-1}(d(x_1, x_2)) \\ &\leq_{\oplus} \psi^n(d(x_1, x_0)) \oplus \psi^n(d(x_1, x_0)) \oplus \cdots \oplus \psi^{n+p-2}(d(x_1, x_0)) \\ &. \\ &. \\ &. \\ &\leq_{\oplus} \psi^n(d(x_1, x_0)) \oplus \psi^n(d(x_1, x_2)) \oplus \cdots \oplus \psi^n(d(x_1, x_2)) \\ &= \psi^n(d(x_1, x_0)). \quad \text{(From part (ii) of Definition 2.1).} \end{aligned}$$

Since $\psi^n(a)$ is convergent to 0_K for each $a \in K^+$, $\{x_k\}$ is a Cauchy sequence in the metric space X. Since X is complete, then $\{x_k\}$ is convergent to x. Therefore,

$$d(x, x) = \lim_{k \to \infty} d(x_k, x) = \lim_{k,m \to \infty} d(x_m, x_k) = 0_K.$$

Now, we claim that $f(x) = x$. Suppose $d(x, f(x)) >_{\oplus} 0_K$. Let $\{x_{2k(i)}\}$ be a subsequence of $\{x_{2k}\}$ and hence of $\{x_k\}$. Due to (3.6) we have

$$\begin{aligned} d(gx_{2k(i)}, f(x)) &\leq_{\oplus} \psi(EMax\{d(x_{2k(i)}, x_{2k(i)+1}), d(f(x), x), d(x_{2k(i)}, x)\}). \\ &= \psi(d(x, f(x)) \\ &<_{\oplus} d(x, f(x)). \end{aligned}$$

which is a contradiction thus $d(x, f(x) = 0_K$, therefore $x = f(x)$. Analogously, if we choose a subsequence $\{x_{2k(i)+1}\}$ of $\{x_{2k(i)+1}\}$, we obtain $g(x) = x$. Hence

$$g(x) = f(x) = x.$$

Example 3.8 Let $X = [-1, 0]$ and $K = \mathbb{R}$ with the operations $a \oplus b := \min\{a, b\}$. And the metric $d : X \times X \to K$ defined by

$$d(x, y) = x \oplus y, \quad for\ x, y \in X.$$

Now define the mapping $g, f : X \to X$ by $T(x) = g(x) = \frac{x}{4}$. Without loss of generality, assume $x \geq_{\oplus} y$. Then

$$d(T(x), T(y)) \leq_{\oplus} \frac{1}{2} d(x, y).$$

Thus, all conditions of Theorem are satisfied, and 0 is the common fixed point of f.

4. Conclusion

In this paper, we introduce an partial metric space over idempotent-valued. Also we prove the existence and uniqueness of some common fixed point theorems for mappings respect to metric space were not contractive but are idempotent-valued contractive mapping.

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