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EXISTENCE OF THREE WEAK SOLUTIONS FOR A CLASS OF DISCRETE PROBLEMS DRIVEN BY p-LAPLACIAN OPERATOR

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Abstract. In this paper, by using a theorem based on variational method which was recently proved by Ricceri, we establish the existence of three weak solutions for a class of p-Laplacian discrete problems. Remarks and examples are provided to illustrate our result.
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1. Introduction

The discrete Laplace operator is determined similarly to the continuous Laplace operator, defined so that it has meaning on a graph or a discrete grid. Most phenomena on many cases are expressed by the discrete Laplacian, such as physics problems, computer science, elasticity, control systems, artificial or biological, neural networks and economics. Recently, a great attention has been focused on the study of existence and multiplicity of solutions for equations involving the discrete p-Laplacian operator with different methods such as various fixed point theorems, critical point theory, variational methods, Morse theory, the mountain-pass theorem and lower and upper solutions method (see [3, 4, 9, 10, 11, 12, 13, 14, 16, 18]). In the paper [2], Atici and Guseinov investigated the existence of positive periodic solutions for nonlinear difference equations with periodic coefficients by employing a fixed point theorem in a cone. By using the upper and lower solution method, in [1] Atici and Cabada considered the equation $\Delta^2(x_{k-1}) + q_k x_k + f(k, x_k) = 0, \quad k \in [1, N]$ with boundary value conditions $x_0 = x_N$ and $\Delta x_0 = \Delta x_N$ and obtained a new existence result for it. Based on fixed point theorem in a cone due to Krasnoselskii, in [6] Jiang with Chu characterized the eigenvalues and showed the existence of positive solutions and, in

[15] with Zhou, based on Ricceri's variational principle, established the existence of at least three solutions for the problem

$$\begin{cases} -\Delta(\phi_p(u(k-1))) = f(x,u), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\phi_p : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $\phi_p(s) := |s|^{p-2}s$, for every $s \in \mathbb{R}$, $f : \mathbb{Z}[1,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and $\Delta u(k-1) := u(k) - u(k-1)$ is a forward difference operator. Wang and Guan in [20], by using the five functionals fixed point theorem, obtained the existence criteria for three positive solutions of p-Laplacian difference equation

$$\begin{cases} \Delta(\phi_p(\Delta u(t-1))) + a(t)f(u(t)) = 0, & \forall t \in [1, T+1], \\ u(0) = u(T+1) = 0, \text{ or} \\ u(0) = \Delta u(T+1) = 0, \end{cases} \tag{1.2}$$

where $f : [0,\infty) \to [0,\infty)$ is a continuous function. Cabada et al. in [5], based on three critical points theorems, investigated different sets of assumptions which guarantee the existence and multiplicity of solutions for difference equations involving the discrete p-Laplacian operator. In [7], by considering variational methods for smooth functionals, the existence of at least three solutions was established for the following problem

$$\begin{cases} -\Delta(\alpha(k))|\Delta u(k-1)|^{p^(k-1)-2} = \lambda f(k, u(k)), & \forall k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases}$$

In the present paper, we obtain a multiplicity result for the following discrete problem

$$\begin{cases} -\Delta(\phi_p(u(k-1))) = \varepsilon f(x,u) - \lambda g(x,u) + \nu h(x,u), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.3}$$

where $p > 1$, $\phi_p : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $\phi_p(s) := |s|^{p-2}s$, for every $s \in \mathbb{R}$, $f, g, h : \mathbb{Z}[1,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, ε, λ and ν are real parameters and

$$\Delta u(k-1) := u(k) - u(k-1)$$

is a forward difference operator. In fact, as motivated in the work by Molica Bisci and Pansera [17], and applying the critical point theorem given by Ricceri in [19], we prove that exist at least three solutions for the problem (1.3) under some assumptions. Also, at the finally in this paper we are going to provide some examples which illustrate main result.

2. Preliminaries

In this section, firstly we present the following theorem which is an important tool to prove our main result. This, including our other presented tools, has been successfully applied to different problems in [17].

Let E be a non-empty set. We define

$$\beta(\mu I + \Psi, \Phi, r) := \sup_{u\in\Phi^{-1}(r,+\infty)} \frac{\mu I(u) + \Psi(u) - \inf_{u\in\Phi^{-1}(-\infty,r)}(\mu I + \Psi)}{r - \Phi(u)},$$

where $I, \Psi, \Phi : E \longrightarrow \mathbb{R}$, $\mu > 0$ and $r \in (\inf_E \Phi, \sup_E \Phi)$. Moreover, if the map $\Psi + \Phi$ is bounded from below, for each

$$r \in \left(\inf_E \Phi, \sup_E \Phi\right),$$

such that

$$\inf_{u\in\Phi^{-1}(-\infty,r)} I(u) \leq \inf_u \in \Phi^{-1}(r) I(u),$$

we put

$$\mu^*(I, \Psi, \Phi, r) := \inf \left\{ \frac{\Psi(u) - \gamma + r}{\eta_r - I(u)} : u \in E, \Phi(u) < r, I(u) < \eta_r \right\},$$

where $\gamma := \inf_E(\Psi(u) + \Phi(u))$ and $\eta_r := \inf_{u\in\Phi^{-1}(r)} I(u)$.

Theorem 2.1. ([19]) *Let* $(E, \|.\|)$ *be a reflexive Banach space;* $I : E \longrightarrow \mathbb{R}$ *a sequentially weakly lower semicontinuous, bounded on each bounded subset of* E*,* C^1*-functional whose derivative admits a continuous inverse on the topological dual* $E^*, \Phi, \Psi : X \longrightarrow \mathbb{R}$ *two* C^1*-functionals with compact derivative. Assume also that the functional* $\Psi + \lambda\Phi$ *is bounded below for all* $\lambda > 0$ *and*

$$\liminf_{\|u\|\longrightarrow+\infty} \frac{\Psi(u)}{I(u)} = -\infty.$$

Then for each $r > \sup_s \Phi$*, where* S *is the set of all global minima of* I*, for each* $\mu > \max\{0, \mu^*(I, \Psi, \Phi, r)\}$ *and for each compact interval* $[a, b] \subset (0, \beta(\mu I + \Psi, \Phi, r))$*, there exists a number* $\rho > 0$ *such that, for each* $\nu \in [0, \delta]$*, the equation*

$$\mu I^{'}(u) + \Psi^{'}(u) + \lambda\Phi^{'}(u) + \nu\Gamma^{'}(u) = 0$$

has at least three solutions in E *whose norms are less than* ρ*.*

In [8], Theorem 2.1 was successfully employed to the existence of least three weak solutions to a degenerate quasilinear elliptic system with three parameters and Dirichlet boundary conditions.

Let's give some notations and definitions that we use to through the following.
We define the T-dimensional space H by

$$H := \{u : \mathbb{Z}[0, T + 1] \longrightarrow \mathbb{R} \,|\, u(0) = u(T + 1) = 0\}.$$

The space H can be normed by

$$\|u\| := \left(\sum_{k=1}^{T+1} |\Delta u(k - 1)|^p\right)^{\frac{1}{p}}.$$

It is easy to see that $(H, \|.\|)$ is a Banach space. Let $f, g, h : \mathbb{Z}[1, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions. Furthermore, let the functions $S, J_F(u), J_G(u), J_H(u) : X \to \mathbb{R}$ be defined by

$$S(u) = \frac{1}{p} \sum_{k=1}^{T+1} |\Delta u(k - 1)|^p,$$

$$J_F(u) = \sum_{k=1}^{T+1} F(k, u(k))dx,$$

$$J_G(u) = \sum_{k=1}^{T+1} F(k, u(k))dx,$$

$$J_H(u) = \sum_{k=1}^{T+1} F(k, u(k))dx,$$

and define $J : H \longrightarrow \mathbb{R}$ by

$$J(u) := \frac{1}{p} \sum_{k=1}^{T+1} |\Delta u(k-1)|^p - \epsilon \sum_{k=1}^{T} F_k(u(k)) + \lambda \sum_{k=1}^{T} G_k(u(k)) + \nu \sum_{k=1}^{T} H_k(u(k)),$$

for every $u \in H$, $k \in \mathbb{Z}[0, 1]$ and $\xi \in \mathbb{R}$ where

$$F_k(\xi) := \int_0^\xi f(k, s)ds,$$

$$G_k(\xi) := \int_0^\xi g(k, s)ds$$

and

$$H_k(\xi) := \int_0^\xi h(k, s)ds.$$

We recall that a solution of the problem (1.3) is a function $u \in H$ such that

$$\sum_{k=1}^{T+1} \phi_p(\Delta u(k-1))\Delta v(k-1) = \epsilon \sum_{k=1}^{T} f(k, u(k))v(k) - \lambda \sum_{k=1}^{T} g(k, u(k))v(k) - \nu \sum_{k=1}^{T} h(k, u(k))v(k),$$

for every $v \in H$. There exists a variational structure of the problem (1.3) (see [15]), and so

$$\langle J^{'}(u), v\rangle = \sum_{k=1}^{T+1} \phi_p(\Delta u(k-1))\Delta v(k-1) - \epsilon \sum_{k=1}^{T} f(k, u(k))v(k) + \lambda \sum_{k=1}^{T} g(k, u(k))v(k) + \nu \sum_{k=1}^{T} h(k, u(k))v(k),$$

for every $v \in H$. Thus, critical points of J are solutions to the problem (1.3).

3. Main results

In this section, we want to use Theorem 2.1 to obtain the critical points of the problem (1.3). For this purpose, we will first introduce some of the symbols that will be used later.
Let $g : \mathbb{Z}[0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function. If G-F is bounded from below for each $r > 0$, we set

$$\tilde{\mu}(f,g,r) := 2\inf\left\{\frac{r-\tilde{\gamma}-J_f(u)}{\tilde{\eta_r}-\|u\|^p} : u \in H, J_g(u) < r, \|u\|^p < v\right\},$$

where

$$\tilde{\gamma} := \sum_{k=1}^{T}\Big(\inf_{\xi\in\mathbb{R}} G_k(\xi) - F_k(\xi)\Big)$$

and

$$\tilde{\eta_r} := \inf_{u\in J_g^{-1}(r)} \|u\|^p.$$

Also, we define

$$\beta(\varepsilon,f,g,r) = \sup_{u\in J_g^{-1}(r,+\infty)} \frac{\|u\|^2 - 2\varepsilon J_f(u) - \inf_{u\in J_g^{-1}(r,+\infty)}(\|u\|^2 - 2\varepsilon J_f(u))}{2(r - J_g(u))},$$

for each $\varepsilon \in \left(0, \dfrac{1}{\max(0,\tilde{\mu}(f,g,r))}\right)$.

We formulate our main result as follows:

Theorem 3.1. *Let $p > 1$, $\sigma > p$ and $f, g : \mathbb{Z}[1,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions such that*

$$\lim_{\xi\longrightarrow+\infty} \frac{\min_{k\in\mathbb{Z}[1,T]} F_k(\xi)}{\xi^p} = +\infty, \qquad \limsup_{|\xi|\longrightarrow+\infty} \frac{\min_{k\in\mathbb{Z}[1,T]} F_k(\xi)}{\xi^\sigma} < +\infty$$

and

$$\lim_{|\xi|\longrightarrow+\infty} \frac{\min_{k\in\mathbb{Z}[1,T]} G_k(\xi)}{|\xi|^\sigma} = +\infty.$$

Then for each $r > 0$, for each

$$\varepsilon \in \left(0, \frac{1}{\max(0,\tilde{\mu}(f,g,r))}\right),$$

and for each compact interval $[a,b] \subset (0, \beta(\varepsilon,f,g,r))$, there exist a number $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and a continuous function h, there exists a number $\delta > 0$ such that for every $\nu \in [0,\delta]$, the problem (1.3) has at least three solutions whose norms in H are less than ρ.

Proof. First, we must show that

$$\limsup_{\|u\|\longrightarrow+\infty} \frac{J_f(u)}{\|u\|^p} = +\infty. \tag{3.1}$$

We have the following characterization of the eigenvalue

$$\lambda_{1,p} = \min_{H\setminus\{0_H\}} \frac{\sum_{k=1}^{T+1} |\Delta u(k-1)|^p}{\sum_{k=1}^{T} |u(k)|^p}. \tag{3.2}$$

Let the first eigenfunction $\varphi_1 \in H$ be positive, it follows by (3.2) that

$$\|\varphi_1\|^p = \lambda_{1,p} \sum_{k=1}^{T} \varphi_1(k)^p.$$

To show (3.1), it is enough to show that

$$\lim_{s\longrightarrow\infty} \frac{J_f(s\varphi_1)}{\|s\varphi_1\|^p} = +\infty. \tag{3.3}$$

For this purpose, let us fix two M, M_1 such that $M < M_1/2$. Since

$$\lim_{\xi\longrightarrow+\infty} \frac{\inf_{k\in\mathbb{Z}[1,T]} F_k(\xi)}{\xi^p} = +\infty,$$

there exists $\eta \geq 0$ such that for all $(x,\xi) \in H \times (\eta, +\infty)$, we have

$$F_k(\xi) \geq \lambda_{1,p} M_1 \xi^p.$$

For all $1 \leq k \leq T$, we set

$$A_s := \left\{ k \in \mathbb{Z}[1,T] : \varphi_1(k) \geq \frac{\eta}{s} \right\}.$$

Therefore, for all $s \in \mathbb{N}$, we have $A_s \subseteq A_{s+1}$, and so the numerical sequence $\{\Sigma_{k\in A_s}\varphi_1(k)^p\}$ is nondecreasing i.e.

$$\sum_{k\in A_s} \varphi_1(k)^p \leq \sum_{k\in A_{s+1}} \varphi_1(k)^p,$$

then, we can write

$$\sum_{k\in A_s} \varphi_1(k)^p \longrightarrow \sum_{k=1}^{k=T} \varphi_1(k)^p,$$

as $s \longrightarrow \infty$. Fixed $\tilde{s} \in \mathbb{N}$, so that

$$\sum_{k\in A_{\tilde{s}}} \varphi_1(k)^p > \frac{2M}{M_1} \sum_{k=1}^{T} \varphi_1(k)^p.$$

Applying the continuity of the function f, we have

$$\max_{(x,\xi)\in\mathbb{Z}[1,T]\times\in[0,\eta]} |F_k(\xi)| < +\infty.$$

Therefore, for each $s \in \mathbb{N}$ satisfying

$$s > \max\left\{ \tilde{s}, \left(\frac{T \times \max_{(x,\xi)\in H\times[0,\eta]} |F_k(\xi)|}{M\|\varphi_1\|^p} \right)^{\frac{1}{p}} \right\},$$

we have

$$\begin{aligned}\frac{J_f(k\varphi_1)}{\|k\varphi_1\|^p} &= \frac{\sum_{k=1}^{T} F_k(k\varphi_1(k))}{k^p\|\varphi_1\|^p} \\ &= \frac{\sum_{k\in A_s} F_k(s\varphi_1(k))}{k^p\|\varphi_1\|^p} + \frac{\sum_{k\in\mathbb{Z}[0,T]-A_s} F_k(s\varphi_1(k))}{k^p\|\varphi_1\|^p} \\ &\geq \frac{\lambda_1 M_1 \sum_{A_s}\varphi_1(k)^p}{\|\varphi_1\|^p} + \frac{\sum_{k\in\mathbb{Z}[0,T]-A_s} F_k(s\varphi_1(k))}{k^p\|\varphi_1\|^p} \\ &\geq \frac{2\lambda_1 M \sum_{A_s}\varphi_1(k)^p}{\|\varphi_1\|^p} - \frac{T\times \sup_{(k,\xi)\in\mathbb{Z}[1,T]\times[0,\eta]} F_k(\xi)}{k^p\|\varphi_1\|^p} \\ &> 2M - M = M,\end{aligned}$$

which shows (3.3). On the other hand, since

$$\limsup_{|\xi|\longrightarrow+\infty} \frac{\inf_{k\in\mathbb{Z}[1,T]} F_k(\xi)}{\xi^p} < +\infty \qquad \text{and} \qquad \lim_{|\xi|\longrightarrow+\infty} \frac{\inf_{k\in\mathbb{Z}[1,T]} G_k(\xi)}{|\xi|^p} = +\infty,$$

there exists $a > 0$ such that

$$F_k(\xi) \leq a(|\xi|^\sigma + 1), \qquad \forall (k,\xi)\in\mathbb{Z}[1,T]\times\mathbb{R}, \tag{3.4}$$

and for each $b > 0$, there exists $c_b > 0$ such that

$$G_k(\xi) \geq b|\xi|^\sigma - c_b, \qquad \forall (k,\xi)\in\mathbb{Z}[1,T]\times\mathbb{R}. \tag{3.5}$$

Applying (3.4) and (3.5), we conclude that $G - \lambda F : \mathbb{R} \longrightarrow \mathbb{R}$ for each $\lambda > 0$ is bounded from below in $\mathbb{R}$. Furthermore, fixing $\lambda > 0$ with $b > \lambda a$, by using (3.4) and (3.5), for each $(k,\xi)\in\mathbb{Z}[1,T]\times\mathbb{R}$, we obtain that

$$\begin{aligned}(G_k - \lambda F_k)(\xi) &\geq b|\xi|^\sigma - c_b - \lambda a(|\xi|^p + 1) \\ &= (b-\lambda a)|\xi|^\sigma - (c_b - a\lambda) \\ &\geq -(c_b + a\lambda).\end{aligned}$$

Thus, we have

$$\sum_{k=1}^{T} (G_k - \lambda F_k)(u(k)) \geq -(c_b - a\lambda)\times T.$$

Therefore, $J_g - \lambda J_f$ is bounded from below in H. We put

$$I(u) := \frac{\|u\|^p}{p}, \qquad \psi(u) := -J_f(u), \qquad \Phi(u) = J_g(u) \qquad \text{and} \qquad \Gamma(u) = J_h(u),$$

for each $u \in H$. Applying Theorem 2.1, the problem (1.3) has at least three solutions whose norms in H are less than ρ. □

The following remarks allow us to know more about the solutions of the problem (1.3).

Remark 3.2. In Theorem 3.1, we have guaranteed the existence of at least three nontrivial solutions for (1.3). The nontriviality of these solutions achived by taking either $f(k,t) \neq 0$ for all $(k,t)\in[1,T]\times\mathbb{R}$ or $g(k,t)\neq 0$ for all $(k,t)\in[1,T]\times\mathbb{R}$, or $h(k,t)\neq 0$ for all $(k,t)\in[1,T]\times\mathbb{R}$, or two of them are true, or all three are true.

If one of these conditions does not hold, the second solution u_2 of the problem (1.3) may be trivial.

Remark 3.3. If in Theorem 3.1, $f(k,t) \neq 0$ for all $(k,t) \in [1,T] \times \mathbb{R}$ or $g(k,t) \neq 0$ for all $(k,t) \in [1,T] \times \mathbb{R}$, or both to be true, then the ensured solution is obviously non-trivial. On the other hand, the non-triviality of the solution can be achieved also in the case $f(k,0) = g(k,0) = 0$ for all $k \in [1,T]$ requiring the extra condition at zero, that is there are discrete intervals $[1,T_1] \subseteq [1,T]$ and $[1,T_2] \subset [1,T_1]$ where $T_1, T_2 \geq 2$, such that

$$\limsup_{\xi \to 0^+} \frac{\inf_{k\in[1,T_2]} F_k(\xi)}{|\xi|^p} = \limsup_{\xi \to 0^+} \frac{\inf_{k\in[1,T_2]} G_k(\xi)}{|\xi|^p} = +\infty,$$

and

$$\liminf_{\xi \to 0^+} \frac{\inf_{k\in[1,T_1]} F_k(\xi)}{|\xi|^p} > -\infty, \qquad \liminf_{\xi \to 0^+} \frac{\inf_{k\in[1,T_1]} G_k(\xi)}{|\xi|^p} > -\infty.$$

Indeed, let $0 < \bar{\lambda} < \lambda^*$ where

$$\lambda^* = \frac{1}{pT^p} \sup_{\gamma>0} \frac{\gamma^p}{\sum_{k=1}^{T} \max_{|t|\leq\gamma}(F_k + G_k)(t)}.$$

Then, there exists $\bar{\gamma} > 0$ such that

$$\bar{\lambda}\frac{1}{pT^p} < \frac{\bar{\gamma}^p}{\sum_{k=1}^{T} \max_{|t|\leq\bar{\gamma}}(F_k + G_k)(t)}.$$

Let T and J_f be as given in Theorem 3.1. Due to Theorem 3.1, for every $\lambda \in (0,\bar{\lambda})$ there exists a critical point of $I_\lambda = T - \lambda J_f + \lambda J_g$ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda = \frac{1}{T^p p}\bar{\gamma}^p$. In particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$. We will prove that the function u_λ can not be trivial.
Let us show that

$$\limsup_{\|u\| \to 0^+} \frac{(J_f + J_g)(u)}{T(u)} = +\infty. \tag{3.6}$$

Thanks to our assumptions at zero, we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants ε and κ (with $\varepsilon > 0$) such that for every $\xi \in [0,\varepsilon]$

$$\lim_{n\to+\infty} \frac{\inf_{k\in[1,T_2]}(F_k + G_k)(\xi_n)}{|\xi_n|^p} = +\infty,$$

and $\inf_{k\in[1,T_1]}(F_k + G_k)(\xi) \geq \kappa|\xi|^p$.
Now, let us consider a discrete interval $[1,T_3] \subset [1,T_2]$ where $T_3 \geq 2$.
Further, let $v \in E$ be a function

(k_1) $v(k) \in [0,1]$ for every $k \in [1,T]$,
(k_2) $v(k) = 1$ for every $k \in [1,T_3]$,
(k_3) $v(k) = 0$ for every $k \in [T_1 + 1, T]$.

Finally, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta\, T_3 + \kappa \sum_{k=T_3+1}^{T_1} |v(k)|^p}{\frac{2^p(T+1)}{p}(C_p\|v\|^p + 1)},$$

where

$$C_p = \frac{2}{p(2^{p-1}-1)}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \varepsilon$ and $\inf_{k\in[1,T_2]} F_k(\xi_n) \geq \eta|\xi_n|^{p^-}$ for every $n > n_0$. At this point, for every $n > n_0$, and bearing in mind the properties of the function v (that is $0 \leq \xi_n v(k) < \varepsilon$ for n large enough), we obtain

$$\begin{aligned}\frac{J_f(\xi_n v) + J_g(\xi_n v)}{T(\xi_n v)} &= \frac{\sum_{k=1}^{T_3}(F_k + G_k)F(\xi_n) + \sum_{k=T_3+1}^{T_1}(F_k + G_k)(\xi_n v(k))}{T(\xi_n v)} \\ &> \frac{\eta\ T_3 + \kappa \sum_{k=T_3+1}^{T_1} |v(k)|^p}{\frac{2^p(T+1)}{p}(C_p\|v\|^p + 1)} > M.\end{aligned}$$

Since M can be chosen arbitrarily large, it follows that

$$\lim_{n\to\infty} \frac{J_f(\xi_n v) + J_f(\xi_n v)}{T(\xi_n v)} = +\infty,$$

from which (3.6) clearly follows. Hence, there exists a sequence $\{w_n\} \subset X$ converging to zero such that, for n large enough, $w_n \in T^{-1}(-\infty, r)$ and $I_\lambda(w_n) < 0$. Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we conclude that $I_\lambda(u_\lambda) < 0$, so that u_λ is not trivial.

Moreover, we present some examples to clarify Theorem 3.1.

Example 3.1. Let $1 < p_1 < p < p_2 < 2$ and consider the problem

$$\begin{cases} -\Delta(\phi_p(u(k-1))) = \varepsilon|u|^{p_1-1}u + \lambda|u|^{p_2-1}u + \nu h(u), \quad \forall k \in \mathbb{Z}[1, 10], \\ u(0) = u(11) = 0. \end{cases} \tag{3.7}$$

Then for each $\varepsilon > 0$ small enough, there exists λ_ε such that, for every compact interval $[\bar{a}, \bar{b}] \subset (0, \lambda_\varepsilon)$ there exists $\rho > 0$ with the property that, for every $\lambda \in [\bar{a}, \bar{b}]$ and every continuous function $h : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for every $\nu \in [0, \delta]$, the problem (3.7) has at least three solutions.

Example 3.2. Consider the problem

$$\begin{cases} -\Delta(|u(k-1)|^3 u(k-1) = \varepsilon f(u) - \lambda g(u) + \nu h(u), \quad \forall k \in \mathbb{Z}[1, 10], \\ u(0) = u(11) = 0, \end{cases} \tag{3.8}$$

with

$$f(t) = 1 + \sqrt{t^9} \quad \text{and} \quad g(t) = 1 + t^8.$$

By setting $\sigma = 7$, we have $1 < \varrho < 5 = p$, $p = 5 < 7 = \sigma$,

$$\lim_{t\in\mathbb{R}} \frac{f(\xi)}{\xi^{p-1}} = \lim_{\xi\to+\infty} \frac{1 + \xi^{\frac{9}{2}}}{\xi^4} = +\infty,$$

$$\limsup_{|\xi|\to+\infty} \frac{f(\xi)}{|\xi|^{\sigma-1}} = \frac{1 + \xi^{\frac{9}{2}}}{|\xi|^6} = 0 < +\infty,$$

$$\lim_{|\xi|\to+\infty} \frac{g(\xi)}{|\xi|^{p-1}} = \frac{1 + t^8}{|\xi|^4} = +\infty.$$

Thus, for each $\varepsilon > 0$ small enough, there exists λ_ε such that, for every compact interval $[\bar{a}, \bar{b}] \subset (0, \lambda_\varepsilon)$ there exists $\rho > 0$ with the property that, for every $\lambda \in [\bar{a}, \bar{b}]$ and every continuous function $h : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for every $\nu \in [0, \delta]$, the problem (3.8) has at least three solutions.

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