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THE CAUCHY PROBLEM IN SCALE OF BANACH SPACES WITH DEVIATING VARIABLES

NGUYEN BICH HUY* AND PHAM VAN HIEN**

*Ho Chi Minh City University of Education, Vietnam
E-mail: huynb@hcmue.edu.vn

**Faculty of Applied Sciences, HCMC University of Technology and Education, Vietnam
E-mail: hienpv@hcmute.edu.vn

Abstract. In this paper, we first prove the existence and uniqueness results for the Cauchy problems in a scale of Banach spaces with deviating variables of the form $u'(t) = F[t, A(t, u(t)), B(u(h(t)))]$. We then apply it to study a Cauchy problem for PDEs in a Gevrey class with deviation at the derivatives. This extends some known results.
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1. Introduction

The Cauchy problem

$$\frac{du}{dt} = F(t, u),\ t \in (0, T),\ u(0) = u_0, \tag{1.1}$$

with operator F acting in a scale of Banach spaces $(X_s, |.|_s),\ s \in [a, b]$, is the abstract version of the Cauchy–Kovalevskaya–Nagumo partial differential equation

$$\partial_t u = F(t, x, u, \nabla u).$$

The existence and uniqueness results of the problem (1.1) (also called the abstract Cauchy–Kovalevskaya theorems) in the Lipschitz case of F,

$$|F(t, u) - F(t, v)|_s \leq \frac{C}{r - s}|u - v|_r,\ s < r,$$

were first proven by T. Yamanaka, and V. Ovsyannikov [20, 21, 27]. They were further generalized and simplified by F. Treves, L. Nirenberg, T. Nishida, Baouendi–Goulaouic, K. Asano, and others, (see [1, 3, 17, 18, 23, 25] and the references therein). When F satisfies certain conditions concerning compactness, the problem (1.1) has been investigated by H. Begehr, K. Deimling, M. Ghisi, N.B. Huy and W. Tutschke in [4, 6, 8, 11, 12, 26].

Abstract results of the problem (1.1) can be applied to equations that involve non-local operators, such as the water wave equation [20], the Boltzmann equation in the fluid dynamic limit [19], the incompressible fluid equations in the zero-viscosity limit [15, 24], and the vortex sheet equations [7]. New applications of the abstract Cauchy problems in a scale of Banach spaces were recently discovered in the integrable Camassa–Holm type equation [5], the Navier–Stokes equations for viscous incompressible flows [15, 16], the Hele-Shaw flows in the plane [22], birth-and-death stochastic dynamics in the continuum [9, 10], and fractional differential equations [4].

The Cauchy-Kovalevskaya-type theorems for some classes of differential equations with deviating variables have been proven in [2, 13, 14, 28]. However, to the best of our knowledge, the abstract version has not yet been considered. In this paper, we study two Cauchy problems with deviating variables in the scale of Banach spaces. The first problem is

$$\frac{du}{dt} = F(t, u(t), u(h(t))),\ t \in (0,1),\ u(0) = u_0, \tag{1.2}$$

where the function $h : [0,1) \to [0,1)$ is continuous, and satisfies $h(t) < t^{1/p},\ t \in (0,1)$ for some $p \in (0,1)$, and the operator F satisfies the combination of the Lipschitz and Holder conditions as follows:

$$|F(t,u_1,v_1) - F(t,u_2,v_2)|_s \leq \frac{C}{r-s}\big(|u_1 - u_2|_r + |v_1 - v_2|_r^p\big),\ s < r.$$

To the best of our knowledge, such a condition for the Cauchy problems in a scale of Banach spaces has not been considered yet. Our second problem has the form

$$\frac{du}{dt} = F(t, A(t,u(t)), B(u(h(t)))),\ t \in (0,T_0),\ u(0) = u_0, \tag{1.3}$$

where the operator F acts in each space of the scale but singularities are contained in operators A and B. In the study of problem (1.2), we used the iterative method, whereas to treat problem (1.3), we applied a special norm. We proved the existence and uniqueness results for problems (1.2) and (1.3) in Section 2 of the paper.
General results on problem (1.3) will be then applied to solve the following equation:

$$\partial_1 u(t,x) = f[t, x, \partial_2^{(l_1)} u(t, \sigma(t)x), \partial_2^{(l_2)} u(h(t), x)] \tag{1.4}$$

in the class of Gevrey functions. The problem (1.4) was considered in [14, 28] without being reduced to an abstract form, with the following restricted conditions on functions $\sigma(t)$ and $h(t)$:

$$0 \leq \sigma(t) \leq m,\ 0 \leq h(t) \leq mt, \quad \text{for some } 0 < m < 1. \tag{1.5}$$

In investigating (1.4) we separated the singular parts and obtained the abstract form (1.3) of the problem. In turn, applying the general results of (1.3) to treat (1.4) makes the study clearer and easier to follow, and allowed us to extend condition (1.5). This is detailed in Section 3 of the paper.

2. Abstract results

In this section, we proved the existence and uniqueness results for the Cauchy problems (1.2) and (1.3) in the scale of Banach spaces $(X_s, |.|_s),\ s \in [a, b]$, i.e.,

$$X_r \subset X_s,\ |u|_s \leq |u|_r,\ \text{ if } s, r \in [a, b],\ s < r.$$

Theorem 2.1 *Assume that $u_0 \in X_b$ and*

(1) *the function $h : [0, 1) \to [0, 1)$ is continuous and increasing, and there exists a number $p \in (0, 1)$ such that $h(t) < t^{1/p},\ \forall\, t \in (0, 1)$; and*
(2) *there exists a constant $C > 0$ such that for $s < r$, the operator F is continuous from $[0, 1) \times X_r \times X_r$ into X_s and satisfies*

$$|F(t, u_1, v_1) - F(t, u_2, v_2)|_s \leq \frac{C}{r - s}\left(|u_1 - u_2|_r + |v_1 - v_2|_r^p\right)$$

for all $u_1, u_2, v_1, v_2 \in X_r$.

Then, for $s \in (a, b)$ such that $(b - s)/(2Ce) < 1$, the problem (1.2) has a unique solution $u : [0, T_s] \to X_s$ where $T_s < (b - s)/(2Ce)$.

Proof. Fix $s \in (a, b)$ and T_s so that $T_s < (b - s)/(2Ce) < 1$, and choose $s' \in (s, b)$ such that $T_s < (s' - s)/(2Ce)$. By the Stirling formula,

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n C^n (1 + n)^n}{(s' - s)^n (n + 1)!}} = \frac{2Ce}{s' - s} > 1.$$

Thus, we can choose a number M with the following properties:

$$\frac{M(2C)^n (n + 1)^n}{(s' - s)^n (n + 1)!} > 1,\ \forall\, n \in \mathbb{N}^* \text{ and } M \geq \sup_{t\in[0,T_s]} |F(t, u_0, u_0)|_{s'}. \tag{2.1}$$

Clearly, the differential equation (1.2) is equivalent to find solutions $u \in C([0, T_s], X_s)$ of the following integral equation:

$$u(t) = u_0 + \int_0^t F[\tau, u(\tau), u(h(\tau))]d\tau. \tag{2.2}$$

To solve (2.2), we construct a successive sequence $\{u_n\}$ by

$$u_0(t) \equiv u_0,\ u_{n+1}(t) = u_0 + \int_0^t F[\tau, u_n(\tau), u_n(h(\tau))]d\tau.$$

We prove by induction that

$$|u_{n+1}(t) - u_n(t)|_r \leq \frac{M(2C)^n (n + 1)^n t^{n+1}}{(s' - r)^n (n + 1)!},\ t \in [0, T_s],\ r \in [s, s'). \tag{2.3}$$

Indeed, as

$$|u_1(t) - u_0|_r \leq \int_0^t |F(\tau, u_0, u_0)|_{s'} d\tau \leq Mt,$$

we see that (2.3) holds for $n = 0$. Let (2.3) be established for $n = k$. For $r \in [s, s')$, we set $\varepsilon = (s' - r)/(k + 2)$ and apply (2.3) with $n = k$ and $r + \varepsilon$ in place of r. We obtain

$$|u_{k+1}(t) - u_k(t)|_{r+\varepsilon} \le \frac{M(2C)^k(k+1)^k t^{k+1}}{(s'-r-\varepsilon)^k(k+1)!},\ t \in [0, T_s],$$

and

$$\begin{aligned}|u_{k+2}(t)-u_{k+1}(t)|_r &\le \frac{C}{\varepsilon}\int_0^t \Big[|u_{k+1}(\tau)-u_k(\tau)|_{r+\varepsilon} + |u_{k+1}(h(\tau))-u_k(h(\tau))|^p_{r+\varepsilon}\Big]d\tau \\ &\le \frac{C}{\varepsilon}\int_0^t \Big[\frac{M(2C)^k(k+1)^k\tau^{k+1}}{(s'-r-\varepsilon)^k(k+1)!} + \Big(\frac{M(2C)^k(k+1)^k(h(\tau))^{k+1}}{(s'-r-\varepsilon)^k(k+1)!}\Big)^p\Big]d\tau \\ &\le \frac{C}{\varepsilon}\frac{M(2C)^k(k+1)^k}{(s'-r-\varepsilon)^k(k+1)!}\int_0^t \big[\tau^{k+1} + (\tau^{1/p})^{p(k+1)}\big]d\tau \\ &\le \frac{M(2C)^{k+1}(k+2)^{k+1}t^{k+2}}{(s'-r)^{k+1}(k+2)!},\end{aligned}$$

which proves (2.3) with $n = k+1$. The induction is complete. From (2.3) and

$$\lim_{n\to\infty}\left(\frac{M(2C)^n(n+1)^nT_s^{n+1}}{(s'-s)^n(n+1)!}\right)^{\frac{1}{n}} = \frac{2CeT_s}{s'-s} < 1,$$

it follows that the sequence $\{u_n\}$ uniformly converges in X_s to a function $u \in C([0,T_s], X_s)$. As

$$\lim_{n\to\infty}\int_0^t F[\tau, u_n(\tau), u_n(h(\tau))]d\tau = \int_0^t F[\tau, u(\tau), u(h(\tau))]d\tau$$

in $X_{s''}$, with $s'' < s$, we conclude that u is an X_s-valued solution of (2.2) on $[0, T_s]$.
We now prove the uniqueness. Let $u, v : [0,T] \to X_s$ be solutions of (1.2) with $T < 1$. We choose $l \in (a, s)$ satisfying $s - l < 2Ce$ and $N > 1$ such that

$$N \ge \sup_{t\in[0,T]} |u(t)-v(t)|_s,\ N\frac{(2C)^n n^n}{(s-l)^n n!} > 1.$$

From this,

$$|u(t)-v(t)|_r \le \frac{C}{\varepsilon}\int_0^t \Big[|u(\tau)-v(\tau)|_{r+\varepsilon} + |u(h(\tau))-v(h(\tau))|^p_{r+\varepsilon}\Big]d\tau,$$

which gives

$$|u(t)-v(t)|_r \le \frac{N2Ct}{s-r} \text{ when } \varepsilon = s-r.$$

Applying the arguments used in the proof of existence, we deduce that

$$|u(t)-v(t)|_r \le \frac{N(2C)^n n^n t^n}{(s-r)^n n!},\ \forall\, t \in [0,T],\ r \in [l,s),\ \forall\, n \in \mathbb{N}^*$$

This clearly forces $u(t) = v(t)$ into an interval $[0, t_1]$ with $t_1 < \min\{T, (s-l)/(2Ce)\}$.
Now, we define

$$T_1 = \sup\{t_1 \in [0,T] : u(t) = v(t) \text{ in } [0,t_1]\}$$

and prove $T_1 = T$.
Assume $T_1 < T$; there exists $T_2 \in (T_1, T]$ satisfying $u(h(t)) = v(h(t)),\ \forall\, t \in [T_1, T_2]$.
Indeed, if $h(T) \le T_1$, we take $T_2 = T$. In case $T_1 < h(T)$, there exists $T_2 \in (T_1, T)$

such that $h(T_2) = T_1$. By monotonicity of h, $h(t) \le T_1$ and $u(h(t)) = v(h(t))$ for all $t \in [T_1, T_2]$. By the properties of T_1, T_2, we get

$$|u(t) - v(t)|_r \le \frac{C}{\varepsilon} \int_{T_1}^{t} |u(\tau) - v(\tau)|_{r+\varepsilon} d\tau, \ \forall\, r, r+\varepsilon \in (l,s), \ t \in [T_1, T_2].$$

Then, we can prove by induction that

$$|u(t) - v(t)|_r \le \frac{NC^n n^n (t-T_1)^n}{(s-r)^n n!}, \ \forall\, t \in [T_1, T_2], \ r \in (l,s).$$

This implies that $u(t) = v(t)$ in an interval $[T_1, T_1 + \varepsilon)$, a contradiction to the choice of T_1. The proof is complete.

We return to the Cauchy problem (1.3). We denoted by $B_s(u_0, R)$ the closed ball in X_s, centered at u_0 with radius R. Clearly, the null of space X_b was also that of all X_s. We denote it by θ.

Theorem 2.2 *Assume that $u_0 \in X_b$ and*

(1) there exist positive numbers C, R such that

i) $F([0,T_0) \times B_s(u_0,R) \times B_s(u_0,R)) \subset X_s$, and

$$|F(t,u_1,v_1) - F(t,u_2,v_2)|_s \le C(|u_1 - u_2|_s + |v_1 - v_2|_s)$$

for all $u_1, u_2, v_1, v_2 \in B_s(u_0,R)$ and $s \in [a,b]$;

ii) for $s < r$, F is continuous from $[0,T_0) \times B_r(u_0,R) \times B_r(u_0,R)$ into X_s;

(2) for $s < r$, $A(t,\cdot) \in L(X_r, X_s)$, $t \in [0,T_0)$ and the function $t \mapsto A(t,\cdot)$ is continuous in the operator-norm; moreover, there exists a function $\alpha(t) \in L^1(0,T_0)$ such that

$$\|A(t,\cdot)\|_{L(X_r,X_s)} \le \alpha(t), \ \forall\, t \in (0,T_0), \ r,s \in [a,b], \ s < r;$$

(3) there are positive constants C_1, q such that

$$B \in L(X_r,X_s), \ \|B\|_{L(X_r,X_s)} \le \frac{C_1}{(r-s)^q}, \ \forall\, r,s \in [a,b], \ s<r;$$

(4) the function $h : [0,T_0) \to [0,T_0)$ is continuous, $h(t) < t$ for all $t \in (0,T_0)$, and there exists a function $S : \Delta = \{(t,T) : 0 \le t \le T < T_0\} \to [a,b]$ satisfying: S is continuous on Δ, decreasing with respect to the variable t; and

$$\lim_{T\to 0} \int_0^T \frac{dt}{[S(h(t),T) - S(t,T)]^q} = 0.$$

Then, there exists a number $T < T_0$ such that the problem (1.3) *has a unique solution $u \in C([0,T], X_a)$ satisfying*

$$u(t) \in X_{S(t,T)}, \ |u(t)|_{S(t,T)} \le R, \ \forall\, t \in [0,T].$$

Proof. Let E be the Banach space of functions $u \in C([0,T], X_a)$ such that

$$u(t) \in X_{S(t,T)} \text{ for all } t \in [0,T], \ \|u\| = \sup_{t\in[0,T]} |u(t)|_{S(t,T)} < \infty,$$

where the number $T < T_0$ is specified later. In E, we denote by u_0 the function $u_0(t) = u_0$ and by $B(u_0, R)$ the closed ball centered at u_0 with radius R. We reduce the problem (1.3) to the equivalent equation of finding $u \in E$ satisfying

$$u(t) = u_0 + \int_0^t F[\tau, A(\tau, u(\tau)), B(u(h(\tau)))]d\tau := Gu(t). \tag{2.4}$$

The main idea of the proof is to show that $G(B(u_0, R)) \subset B(u_0, R)$ and that G is contractive on $B(u_0, R)$. The fixed point of G is a solution to problem (1.3).
For $u \in B(u_0, R)$ due to properties of function F, S

$$\begin{aligned}
|Gu(t) - u_0|_{S(t,T)} &\le \int_0^t |F(\tau, A(\tau, u(\tau)), B(u(h(\tau)))|_{S(t,T)} d\tau \\
&\le \int_0^t \Big[|F(\tau, \theta, \theta)|_b + C[|A(\tau, u(\tau))|_{S(t,T)} + |B(u(h(\tau)))|_{S(t,T)}]\Big] d\tau \\
&\le \int_0^T |F(t, \theta, \theta)|_b dt + C \int_0^T \left[\alpha(\tau)|u(\tau)|_{S(\tau,T)} + \frac{C_1|u(h(\tau))|_{S(h(\tau),T)}}{[S(h(\tau), T) - S(\tau, T)]^q}\right] d\tau \\
&\le \int_0^T |F(t, \theta, \theta)|_b dt + C\|u\| \int_0^T \left[\alpha(\tau) + \frac{C_1}{[S(h(\tau), T) - S(\tau, T)]^q}\right] d\tau.
\end{aligned}$$

Therefore, we can choose T to be sufficiently small for the right-hand side to be less than R for all $u \in B(u_0, R),\ t \in [0, T]$. This proves $Gu \in B(u_0, R)$.
Similarly, for $u, v \in B(u_0, R)$

$$\begin{aligned}
|Gu(t) - Gv(t)|_{S(t,T)} \le C \int_0^T \Big[&\alpha(\tau)|u(\tau) - v(\tau)|_{S(\tau,T)} \\
&+ \frac{C_1|u(h(\tau)) - v(h(\tau))|_{S(h(\tau),T)}}{[S(h(\tau), T) - S(\tau, T)]^q}\Big] d\tau,
\end{aligned}$$

which gives

$$\|Gu - Gv\| \le C\|u - v\| \int_0^T \left[\alpha(\tau) + \frac{C_1}{[S(h(\tau), T) - S(\tau, T)]^q}\right] d\tau.$$

From this and hypothesis 4. we see that G is contractive if T is sufficiently small. The proof is complete.

Example. Given $q > 0$, let $\alpha > 0$ be such that $\alpha q < 1$ and

$$S(t, T) = b - \frac{(b - a)t^\alpha}{T^\alpha},\ 0 \le t \le T < T_0.$$

(1) If there exists $m \in (0, 1)$ such that $0 \le h(t) \le mt,\ \forall\, t \in [0, T_0)$, then

$$\frac{1}{[S(h(t), T) - S(t, T)]^q} \le \frac{T^{\alpha q}}{(1 - m^\alpha)^q (b - a)^q} \frac{1}{t^{\alpha q}},\ 0 < t \le T < T_0.$$

(2) Assume that $0 < h(t) < t^{1/p},\ \forall\, t \in (0, 1)$ with $p \in (0, 1)$. We have

$$\frac{1}{[S(h(t), T) - S(t, T)]^q} \le \frac{T^{\alpha q}}{(b - a)^q} \frac{1}{t^{\alpha q}[1 - t^{\alpha((1/p)-1)}]^q},\ 0 < t \le T < 1.$$

Thus, hypothesis 4 of Theorem 2.2 holds in both cases.

3. Application to a Cauchy problem in the Gevrey class, with deviating variables

Following Yamanaka and Kawaghisi [14, 28] we consider the problem

$$\begin{cases}\partial_1 u(t,x) = f[t,x,\partial_2^{(l_1)}u(t,\sigma(t)x),\partial_2^{(l_2)}u(h(t),x)],\ t\in(0,T_0),\ x\in\Omega,\\ u(0,x)=0,\ x\in\Omega,\end{cases} \tag{3.1}$$

where $\Omega=(-R_0,R_0)$, $u(t,x)$ is an unknown function belonging to the Gevrey class in the second variable, $\partial_i,\ i=1,2$ denotes the derivative of the i-th variable of $u(t,x)$, $l_1,l_2\in\mathbb{N}^*$ and the functions $h,\sigma:[0,T_0)\to\mathbb{R}$. $f:[0,T_0)\times U\to\mathbb{R},\ U=\{(x,u,v)\in\mathbb{R}^3:|x|<R_0,|u|,|v|<R_1\}$ satisfy suitable conditions.

We begin by recalling the main definitions and notations. Let $\mathbb{Z}_+$ be the set of all non-negative integers and $\lambda>1$ be a constant. We fix such a λ throughout this section. A C^∞ function $u:\Omega\to\mathbb{R}$ is called a Gevrey function on Ω of order λ if there are two constants C,r such that

$$|u^{(k)}(x)|\le\frac{C(k!)^\lambda}{r^k},\ \forall\, k\in\mathbb{Z}_+,x\in\Omega,(u^{(0)}=u). \tag{3.2}$$

Define function $\Gamma:\mathbb{Z}_+\to\mathbb{R}$ by $\Gamma(k)=2^{-8}(k!)^\lambda k^{-2-\lambda}$ for $k\ge1$ and $\Gamma(0)=2^{-6}$. It is easily seen that the condition (3.2) is equivalent to the following one:

$$\exists C',s>0:\ |u^{(k)}(x)|\le\frac{C'\Gamma(k)}{s^k},\ \forall\, k\in\mathbb{Z}_+,x\in\Omega. \tag{3.3}$$

We denote by $X_s=\mathcal{G}_s(\Omega)$ the space of all Gevrey functions satisfying (3.3), endowed with the norm

$$|u|_s=\sup\left\{\frac{|u^{(k)}(x)|}{\Gamma(k)}s^k:\ x\in\Omega,k\in\mathbb{Z}_+\right\}.$$

The spaces $(X_s,|.|_s),\ s\in[a,b]\subset(0,\infty)$ form a scale of Banach spaces.

Let $V\subset\mathbb{R}^m$ be an open set. We write $f\in C[0,T_0)\otimes\mathcal{G}_s(V)$ if the function $f:[0,T_0)\times V\to\mathbb{R}$ has continuous partial derivative $\partial^k f$ in variable $y\in V$ for all $k\in\mathbb{Z}_+^m$, and

$$\exists C>0:\ |\partial^k f(t,y)|\le\frac{C\Gamma(k)}{s^{|k|}},\ \forall\,(t,y)\in[0,T_0)\times V,k\in\mathbb{Z}_+^m,$$

where, for $k=(k_1,...,k_m)\in\mathbb{Z}_+^m$, we define $\partial^k f=\partial_1^{k_1}...\partial_m^{k_m}f,\ k!=k_1!...k_m!,\ |k|=k_1+...+k_m$ and $\partial^k f=f$ if $k=(0,...,0)$.

Lemma 3.1 *Let $u\in X_r,\ r\in[a,b],\ l\in\mathbb{N}^*$.*

(1) If $s\in[a,b]$, $s<r$, then $u^{(l)}\in X_s$ and

$$|u^{(l)}|_s\le\frac{C(l)}{(r-s)^{\lambda l}}|u|_r,\quad \textit{where } C(l)=\max\left\{\frac{2^6\Gamma(l)(b-a)^{\lambda l}}{a^l},\left(\frac{b\lambda l}{e}\right)^{\lambda l}\frac{1}{a^l}\right\}.$$

(2) If $\sigma\in(0,1)$ and $v(x)=u^{(l)}(\sigma x)$, then $v\in X_r$ and

$$|v|_r\le\frac{C(l)}{[a(1-\sigma)]^{\lambda l}}|u|_r.$$

Proof. (1) With $w = u^{(l)}$, we have

$$\frac{|w(x)|}{\Gamma(0)} \le \frac{2^6 |u|_r \Gamma(l)}{r^l} \le \frac{2^6 \Gamma(l)(b-a)^{\lambda l}}{a^l} \frac{|u|_r}{(r-s)^{\lambda l}},$$

$$\begin{aligned}\frac{|w^{(k)}(x)|}{\Gamma(k)} s^k &= \frac{|u^{(k+l)}(x)|}{\Gamma(k)} s^k \le \frac{|u|_r \Gamma(k+l)}{r^{k+l}\Gamma(k)} s^k \\ &\le \left(\frac{k+l}{k}\right)^{-2-\lambda} \frac{|u|_r}{s^l} (k+l)^{\lambda l} \left(\frac{s}{r}\right)^{k+l} \\ &\le \frac{|u|_r}{s^l} \sup_{z \ge 0} z^{\lambda l} \left(\frac{s}{r}\right)^z = \frac{|u|_r}{s^l} \left(\frac{\lambda l}{e(\ln r - \ln s)}\right)^{\lambda l} \\ &\le \left(\frac{b\lambda l}{e}\right)^{\lambda l} \frac{1}{a^l} \frac{|u|_r}{(r-s)^{\lambda l}}, \ k \ge 1,\end{aligned}$$

which proves the assertion.

(2) The assertion follows from

$$\begin{aligned}|v|_r &= \sup_{x,k} \frac{|\left(u^{(l)}\right)^{(k)}(\sigma x)|}{\Gamma(k)} (\sigma r)^k \\ &\le |u^{(l)}|_{\sigma r} \le \frac{C(l)|u|_r}{(r-\sigma r)^{\lambda l}}.\end{aligned}$$

Lemma 3.2 *Let the function $\sigma : [0, T_0) \to \mathbb{R}$ be continuous and $0 \le \sigma(t) < 1, \ \forall t \in (0, T_0)$. Consider the operators $A(t, \cdot)$ and B defined on $X_r, \ r \in [a, b]$ by*

$$A(t, u)(x) = u^{(l_1)}(\sigma(t)x), \ Bu = u^{(l_2)}.$$

Then, for $s < r$

(1) $A(t, \cdot) \in L(X_r, X_s)$ and

$$\|A(t, \cdot)\|_{L(X_r, X_s)} \le \frac{C(l_1)}{[a(1-\sigma(t))]^{\lambda l_1}}, \ \forall t \in (0, T_0).$$

Moreover, the function $t \mapsto A(t)$ is continuous in the operator-norm of $L(X_r, X_s)$.

(2) $B \in L(X_r, X_s)$ and

$$\|B\|_{L(X_r, X_s)} \le \frac{C(l_2)}{(r-s)^{\lambda l_2}}.$$

Proof. The estimates for $A(t), B$ follow from Lemma 3.1.
Let $t, t_0 \in [0, T_0), u \in X_r$ and $v = A(t, u) - A(t_0, u)$. We have

$$\begin{aligned}v(x) = u^{(l_1)}(\sigma(t)x) - u^{(l_1)}(\sigma(t_0)x) &= \int_{\sigma(t_0)}^{\sigma(t)} [u^{(l_1)}(\tau x)]'_\tau d\tau \\ &= \int_{\sigma(t_0)}^{\sigma(t)} x u^{(l_1+1)}(\tau x) d\tau,\end{aligned}$$

$$\frac{v^{(k)}(x)}{\Gamma(k)}s^k = \int_{\sigma(t_0)}^{\sigma(t)} \frac{s^k}{\Gamma(k)} \Big[xu^{(l_1+1)}(\tau x)\Big]_x^{(k)} d\tau$$
$$= \int_{\sigma(t_0)}^{\sigma(t)} \frac{s^k}{\Gamma(k)} \Big[x\tau^k u^{(l_1+1+k)}(\tau x) + k\tau^{k-1}u^{(l_1+k)}(\tau x)\Big] d\tau. \qquad (3.4)$$

On the other hand by Lemma 3.1, we get

$$\frac{s^k}{\Gamma(k)}\Big|x\tau^k u^{(l_1+1+k)}(\tau x)\Big| \le R_0 \Big|\big[u^{(1+l_1)}\big]^{(k)}(\tau x)\Big| \frac{s^k}{\Gamma(k)} \le R_0|u^{(l_1+1)}|_s$$
$$\le \frac{R_0 C(1+l_1)}{(r-s)^{\lambda(1+l_1)}}|u|_r, \qquad (3.5)$$

$$\frac{s^k}{\Gamma(k)}\Big|u^{(l_1+k)}(\tau x)\Big| \le \Big|u^{(l_1)}\Big|_s \le \frac{C(l_1)}{(r-s)^{\lambda l_1}}|u|_r. \qquad (3.6)$$

From (13)-(15) we deduce that

$$|v|_s \le \left[\frac{R_0 C(1+l_1)}{(r-s)^{\lambda(1+l_1)}}|\sigma(t)-\sigma(t_0)| + \frac{C(l_1)}{(r-s)^{\lambda l_1}}|\sigma^k(t)-\sigma^k(t_0)|\right]|u|_r,$$

which proves $\|A(t)-A(t_0)\|_{L(X_r,X_s)} \to 0$ as $t \to t_0$. Thus, the function $t \mapsto A(t)$ is continuous at t_0.

Lemma 3.3 *Assume that $f \in C[0,T_0) \otimes \mathcal{G}_S(U)$ and $R \le \{2^5R_1, S\}$, $s \le S/4$. Then:*

(1) *There is a constant D depending only on f such that if $u, v \in B_s(\theta, R)$, the function $w(t,x) = f[t,x,u(x),v(x)]$ belongs to the class $C[0,T_0) \otimes \mathcal{G}_s(\Omega)$ and $|w(t,\cdot)|_s \le D$, $\forall t \in [0,T_0)$. Moreover, for $u_i, v_i \in B_s(\theta,R)$ the functions $w_i(t,x) = f[t,x,u_i(x),v_i(x)]$, $i=1,2$ satisfy the inequality*

$$|w_1(t,\cdot)-w_2(t,\cdot)|_s \le D\big(|u_1-u_2|_s + |v_1-v_2|_s\big),\ \forall t \in [0,T_0).$$

(2) *If $s < r \le S/4$, the operator F defined by*

$$F(t,u,v)(x) = f[t,x,u(x),v(x)]$$

is continuous from $[0,T_0) \times B_r(\theta,R) \times B_r(\theta,R)$ into X_s.

Proof. The first statement of the lemma was proved in [28]-propositions 8-9. This statement yields

$$|F(t',u',v') - F(t,u,v)|_s \le |F(t',u,v) - F(t,u,v)|_s + D\big(|u-u'|_r + |v-v'|_r\big)$$

for $u, u', v, v' \in B_r(\theta,R)$. Therefore, to prove the $X_r - X_s$ continuity of F, it is sufficient to show the continuity of the function $t \mapsto w(t) = F(t,u,v)$ in X_s-norm when $u, v \in B_r(\theta,R)$ are fixed. We have by Statement 1:

$$\frac{|w_x^{(k)}(t,x)|}{\Gamma(k)}r^k \le D,\ \forall (t,x) \in [0,T_0) \times (-R_0,R_0),\ k \in \mathbb{Z}_+.$$

Thus, given $\varepsilon > 0$ there exists an integer k_0 such that

$$\frac{|w_x^{(k)}(t,x) - w_x^{(k)}(t_0,x)|}{\Gamma(k)}s^k \le 2D\left(\frac{s}{r}\right)^k$$
$$\le \varepsilon,\ \forall k > k_0,\ (t,x) \in [0,T_0) \times (-R_0,R_0). \qquad (3.7)$$

We now estimate the left-hand side of (16) for $k \le k_0$. Let $0 < R_2 < R_0$; by uniform continuity of the functions $w_x^{(k)}(t,x), k \le k_0$ on a compact set $[t_1, t_2] \times [-R_2, R_2]$, there exists $\delta > 0$ such that

$$\frac{|w_x^{(k)}(t,x) - w_x^{(k)}(t_0,x)|}{\Gamma(k)} s^k \le \frac{\varepsilon}{2}, \tag{3.8}$$

$\forall\, x \in [-R_2, R_2],\ t \in (t_0 - \delta, t_0 + \delta) \cap [0, T_0),\ k \le k_0$.
For $x \in (-R_0, R_0) \setminus [-R_2, R_2]$, we can choose $y = R_2$ (or $y = -R_2$) such that $|x - y| \le R_0 - R_2$; by the Lagrange Theorem,

$$\begin{aligned}
&\frac{|w_x^{(k)}(t,x) - w_x^{(k)}(t_0,x)|}{\Gamma(k)} s^k \le |w_x^{(k)}(t,y) - w_x^{(k)}(t_0,y)| \frac{s^k}{\Gamma(k)} \\
&+ \frac{(R_0 - R_2)s^k}{\Gamma(k)} \left(|w_x^{(k+1)}(t,c_1)| + |w_x^{(k+1)}(t_0,c_2)| \right) \\
&\le \frac{\varepsilon}{2} + (R_0 - R_2) 2D \frac{\Gamma(k+1)}{s\Gamma(k)} \\
&\le \varepsilon,\ \forall\, k \le k_0,\ x \in (-R_0, R_0) \setminus [-R_2, R_2],\ t \in (t_0 - \delta, t_0 + \delta) \cap [0, T_0)
\end{aligned} \tag{3.9}$$

provided that $R_0 - R_2$ is sufficiently small. From (16)-(18) we obtain

$$|w(t) - w(t_0)|_s \le \varepsilon,\ t \in (t_0 - \delta, t_0 + \delta) \cap [0, T_0).$$

The lemma is thus proved.
From Theorem 2.2 and Lemmas 3.1-3.3, we obtain the following result:
Theorem 3.4 *Suppose the following hypotheses:*

(1) The function f is in class $C[0, T_0) \otimes \mathcal{G}_S(U)$.
(2) The functions σ, h are continuous on $[0, T_0)$ and satisfy

$$0 < \sigma(t) < 1,\ 0 < h(t) < t,\ \forall\, t \in (0, T_0),\ \int_0^{T_0} \frac{dt}{[1 - \sigma(t)]^{\lambda l_1}} < \infty.$$

(3) There exists a function $S : \Delta = \{(t,T) : 0 \le t \le T < T_0\} \to [a, S/4], (a > 0)$ such that:
S is continuous in (t,T), decreasing in the first variable, and

$$\lim_{T \to 0} \int_0^T \frac{dt}{[S(h(t),T) - S(t,T)]^{\lambda l_2}} = 0.$$

Let $R \le \min\{2^5 R_1, S\}$. Then, there exists a number $0 < T < T_0$ such that problem (3.1) *has a unique solution u in the Gevrey class of order λ and*

$$u(t) \in X_{S(t,T)},\ |u(t)|_{S(t,T)} \le R,\ \forall\, t \in [0, T].$$

Example. Let $\alpha > 0$ be such that $\alpha \lambda l_2 < 1$ and

$$S(t,T) = \frac{S}{4} - \left(\frac{S}{4} - a \right) \left(\frac{t}{T_0} \right)^{\alpha},\ 0 \le t \le T < T_0.$$

Assume that

(1) the function h is continuous on $[0, T_0)$ and satisfies at least one of the following conditions:

a) $0 < h(t) \leq mt,\ t \in (0, T_0)$ for some $m \in (0,1)$, and

b) $T_0 = 1$ and $0 < h(t) < t^{1/p},\ t \in (0,1)$ for some $p \in (0,1)$; and

(2) the function σ is continuous on $[0, T_0)$ and

$$0 < \sigma(t) \leq 1 - \left(\frac{t}{T_0}\right)^{\beta},\ t \in [0, T_0),\ \text{for some } \beta \in \left(0, \frac{1}{\lambda l_1}\right).$$

Then, hypotheses 2 and 3 of Theorem 3.4 hold.

Note that the function $\sigma(t) = 1 - \left(\frac{t}{T_0}\right)^{\beta}$ does not satisfy condition (1.5), which is the main condition in [14, 28].

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