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IMPLICIT CONTRACTIONS FOR A SEQUENCE OF MULTI-VALUED MAPPINGS

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Abstract. In the present paper, we define a new class of implicit contractions for a sequence of multi-valued mappings on a metric space endowed with a graph to obtain sufficient conditions for existence of common fixed points for this sequence. This will enable us to obtain a simultaneous generalization of various types of fixed point theorems for a sequence of multi-valued mappings. Moreover, multi-fractal operators related to these contractive mappings are considered.
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1. Introduction

The Banach contraction principle [3], which appeared first in Banach's thesis, is one of the most important results of mathematics. It also has numerous applications in different branches of sciences and plays an influential role in the modeling of physical processes. Due to its large area of influence and its significant impact, this result has been extended by some mathematicians either by weakening the contractive condition or imposing some additional conditions on the space [4, 6, 19]. One of these generalizations was taken in metric spaces endowed with a partial order. Ran and Reurings [18] pioneered the field by exploiting the existence of fixed points for continuous monotone contractive mappings. Later, Nieto and Rodríguez-López [12, 13] extended the result appeared in [18] and used this extension to solve some differential equations. Jachymski [6] obtained a more general unified version of these results by considering a directed graph instead of a partial order [1, 8].

The multi-valued version of the Banach's theorem was innovated by Nadler [11]. Following this extension, many mathematicians focused their attention on the study of the fixed points of multi-valued mappings [9, 16, 17, 20].

In 1997, Popa [14] initiated the study of fixed points for mappings which satisfy an implicit relation [5, 15]. In this paper, we use this approach to define a class of implicit relations for a sequence of multi-valued mappings to obtain new conditions for a sequence of multi-valued mappings in complete metric spaces endowed with a graph, which guarantees the existence of its common fixed points. Our general result demonstrates a concurrent generalization of various types fixed point theorems for a sequence of multi-valued mappings. Through applications of our results, we obtain a new fixed point theorem for multi-fractals.

2. Results

We start this section by recalling some definitions.

Definition 2.1. Let (X, d) be a metric space and let $CB(X)$ denote the family of all nonempty closed bounded subsets of X. The Hausdorff distance between two elements A and B in $CB(X)$ is defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

Definition 2.2. Let $T : X \to CB(X)$ be a multi-valued operator. An element $x \in X$ is called a fixed point for T if $x \in Tx$. We denote by $Fix(T)$ the fixed point set of T.

Hereafter, we will assume that (X, d) is a metric space endowed with a directed graph G and $\{T_n : X \to CB(X)\}_{n\geq 1}$ is a sequence of multi-valued mappings such that $T_n = T_1$ for infinity many $n \in \mathbb{N}$ and $T_{n+1} = T_2$ whenever $T_n = T_1$.

Definition 2.3. Let $\mathcal{R}$ denote the class of all mappings $g : [0,\infty)^5 \to [0,\infty)$ with the following properties:

(*i*) g is homogeneous, that is,

$$g(\lambda x_1, \dots, \lambda x_5) = \lambda g(x_1, \dots, x_5),\ (x_1, \dots, x_5, \lambda \geq 0).$$

(*ii*) If $x_i \leq y_i$, for $1 \leq i \leq 5$, then $g(x_1, \dots, x_5) \leq g(y_1, \dots, y_5)$.

(*iii*) $g(1, 1, 1, 2, 0) < 1$.

Let $g \in \mathcal{R}$ and G be a graph in X. The sequence $\{T_n\}_{n\geq 1}$ is called a (g, G)-contraction if for each $i \geq 1$, there corresponds a subgraph G_i of G such that $(x_0, x_1) \in E(G_i)$ implies that

$$\mathcal{H}(T_i x_0, T_{i+1}x_1) \leq g(d(x_0, x_1), d(x_0, T_i x_0), d(x_1, T_{i+1}x_1), d(x_0, T_{i+1}x_1), d(x_1, T_i x_0)). \tag{2.1}$$

The following result shows that the sets of fixed points of a sequence of (g, G)-contractive mappings are equal.

Lemma 2.4. *Let $\{T_n\}$ be a (g, G)-contraction sequence for some $g \in \mathcal{R}$. Then $Fix(T_i) = Fix(T_j)$ for each $i, j \in \mathbb{N}$ provided that $(x, x) \in E(G_i)$ for each $i \in \mathbb{N}$ and each $x \in Fix(T_i)$.*

Proof. Let $x \in Fix(T_i)$ for some $i \in \mathbb{N}$, then $(x, x) \in E(G_i)$. Therefore

$$\begin{aligned} d(x, T_{i+1}x) \leq \mathcal{H}(T_i x, T_{i+1}x) &\leq g(d(x,x), d(x,T_i x), d(x,T_{i+1}x), d(x,T_{i+1}x), d(x,T_i x)) \\ &\leq g(0,0,d(x,T_{i+1}x), d(x,T_{i+1}x), 0) \\ &\leq g(0,0,1,1,0) d(x,T_{i+1}x) \\ &\leq g(1,1,1,2,0) d(x,T_{i+1}x). \end{aligned}$$

Since $g(1,1,1,2,0) < 1$, we have $d(x, T_{i+1}x) = 0$. Thus $x \in Fix(T_{i+1})$. Hence $Fix(T_i) \subseteq Fix(T_{i+1})$ for each $i \geq 1$. Therefore $Fix(T_1) \subseteq Fix(T_2) \subseteq \cdots$. Since $T_n = T_1$ for infinity many n, we get to the desired result. □

Definition 2.5. For each $T : X \to CB(X)$, the graph of T is defined by

$$Gr(T) := \{(x,y);\ x \in X \ \textit{and} \ y \in Tx\}.$$

Also for each $t \in (0,1)$, define

$$Gr_t(T) := \{(x,y);\ x \in X,\ y \in Tx \ \textit{and} \ td(x,y) \leq d(x,Tx)\}.$$

Trivially for each $t \in (0,1)$, $Gr_t(T) \subseteq Gr(T)$.

We also need the following result.

Lemma 2.6. *Let (X,d) be a metric space, $g \in \mathcal{R}$ and $\{T_n\}$ a $(g,G)-$contraction sequence. Assume that there exists $t \in (0,1)$ with $t > g(1,1,1,2,0)$ such that $E(G_i)$ contains $Gr_t(T_i)$ for each $i \in \mathbb{N}$. Then there is $q \in [0,1)$ and a sequence $\{x_n\}$ with $(x_n, x_{n+1}) \in E(G_{n+1})$ and $x_n \in T_n x_{n-1}$ such that for each $n \in \mathbb{N}$*

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n). \tag{2.2}$$

In particular, if X is complete, then $\{x_n\}$ is convergent.

Proof. Let t be a real number with $g(1,1,1,2,0) < t < 1$ and $Gr_t(T_i) \subseteq E(G_i)$ for each $i \geq 1$. Put $q = \frac{g(1,1,1,2,0)}{t}$, then $0 \leq q < 1$. Take some arbitrary point $x_0 \in X$. Choose $x_1 \in T_1 x_0$ with $td(x_0, x_1) \leq d(x_0, T_1 x_0)$. Then $(x_0, x_1) \in Gr_t(T_1) \subseteq E(G_1)$. Thus

$$\mathcal{H}(T_1 x_0, T_2 x_1) \leq g(d(x_0,x_1), d(x_0,T_1x_0), d(x_1,T_2x_1), d(x_0,T_2x_1), d(x_1,T_1x_0)). \tag{2.3}$$

Take some $x_2 \in T_2 x_1$ with $td(x_1, x_2) \leq d(x_1, T_2 x_1)$. Then

$$(x_1, x_2) \in Gr_t(T_2) \subseteq E(G_2).$$

By (2.3) we get

$$\begin{aligned} td(x_1,x_2) &\leq g(d(x_0,x_1), d(x_0,T_1x_0), d(x_1,T_2x_1), d(x_0,T_2x_1), d(x_1,T_1x_0)) \\ &\leq g(d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), d(x_0,x_1) + d(x_1,x_2), 0) \\ &\leq \max\{d(x_0,x_1), d(x_1,x_2)\} g(1,1,1,2,0). \end{aligned}$$

If $\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_1,x_2)$, then we get $t \leq g(1,1,1,2,0)$, which is a contradiction. So $\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_0,x_1)$. Thus we have $d(x_1,x_2) \leq qd(x_0,x_1)$. This proves (2.2) for $n = 1$. Suppose that (2.2) is true for some $n \geq 1$.

Then $x_{n+1} \in T_{n+1}x_n$ and $(x_n, x_{n+1}) \in Gr_t(T_{n+1}) \subseteq E(G_{n+1})$.
Choose $x_{n+2} \in T_{n+2}x_{n+1}$ with $td(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, T_{n+2}x_{n+1})$. Then

$$(x_{n+1}, x_{n+2}) \in Gr_t(T_{n+2}) \subseteq E(G_{n+2}).$$

Hence

$$\begin{aligned}
&td(x_{n+1}, x_{n+2}) \leq \mathcal{H}(T_{n+1}x_n, T_{n+2}x_{n+1}) \\
&\leq g(d(x_n, x_{n+1}), d(x_n, T_{n+1}x_n), d(x_{n+1}, T_{n+2}x_{n+1}), d(x_n, T_{n+2}x_{n+1}), d(x_{n+1}, T_{n+1}x_n)) \\
&\leq g(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0) \\
&\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} g(1, 1, 1, 2, 0).
\end{aligned}$$

The above inequality together with the fact that $g(1, 1, 1, 2, 0) < t$ proves (2.2) when n is replaced by $n + 1$. The inequality (2.2) implies that

$$d(x_{n+1}, x_n) \leq qd(x_n, x_{n-1}) \leq \cdots \leq q^n d(x_1, x_0) \quad (n \geq 1).$$

Therefore for each $n, p \in \mathbb{N}$,

$$\begin{aligned}
d(x_{n+p}, x_n) &\leq d(x_{n+1}, x_n) + \cdots + d(x_{n+p}, x_{n+p-1}) \\
&\leq \sum_{k=n}^{n+p-1} q^k d(x_0, x_1).
\end{aligned}$$

Thus $\{x_n\}$ is Cauchy and converges to some $x \in X$ if X is complete. □

Now, we are ready to state the main result of this paper.

Theorem 2.7. *Let (X, d) be a complete metric space endowed with a graph G, $g \in \mathcal{R}$ be continuous and $t \in (0, 1)$ with $t > g(1, 1, 1, 2, 0)$. Let $\{T_n\}$ be a (g, G)-contraction sequence and for each $n \in \mathbb{N}$, $E(G_n)$ contains $Gr_t(T_n)$. Assume that $\{x_n\}$ be the Cauchy sequence defined in proof of Lemma 2.6 and $x \in X$ be its limit. Then x is a common fixed point of $\{T_n\}$ provided that there exists a subsequence $\{x_{n_k}\}$ and $i \geq 1$ such that $(x_{n_k}, x) \in E(G_i)$ and $x_{n_k+1} \in T_i x_{n_k}$ for each $k \in \mathbb{N}$.*

Proof. By the definition for each $k \geq 1$, we have

$$\begin{aligned}
d(x_{n_k+1}, T_{i+1}x) &\leq \mathcal{H}(T_i x_{n_k}, T_{i+1}x) \\
&\leq g(d(x_{n_k}, x), d(x_{n_k}, T_i x_{n_k}), d(x, T_{i+1}x), d(x_{n_k}, T_{i+1}x), d(x, T_i x_{n_k})) \\
&\leq g(d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, T_{i+1}x), d(x_{n_k}, T_{i+1}x), d(x, x_{n_k+1})).
\end{aligned}$$

Letting $k \to \infty$, by continuity of g, we have

$$d(x, T_{i+1}x) \leq g(0, 0, d(x, T_{i+1}x), d(x, T_{i+1}x), 0) \leq g(0, 0, 1, 1, 0) d(x, T_{i+1}x).$$

Since $g(0, 0, 1, 1, 0) < 1$, the above inequality implies that $x \in T_{i+1}x$. In view of Lemma 2.4, x is a common fixed point of $\{T_n\}$. □

Remark 2.8. Let (X, d) be a complete metric space endowed with a graph G and $g \in \mathcal{R}$ be continuous. Assume that $\{T_n\}$ be a (g, G)-contraction sequence and $Gr(T_i) \subseteq E(G_i)$, for each $i \geq 1$. Then for each $t \in (0, 1)$ with $t > g(1, 1, 1, 2, 0)$ and for each $i \geq 1$ we have $Gr_t(T_i) \subseteq E(G_i)$. Thus Lemma 2.6 and Theorem 2.7 also hold if we replace condition $Gr_t(T_i) \subseteq E(G_i)$ for each $i \geq 1$ by stronger condition $Gr(T_i) \subseteq E(G_i)$ for each $i \geq 1$.

Example 2.9. Let (X, d) be a metric space such that $X = \{\frac{1}{2^n};\ n \geq 1\} \cup \{0, 1\}$ and

$$d(x,y) = \begin{cases} 0 & \textit{if } x = y \\ \max\{x,y\} & \textit{otherwise} \end{cases} \tag{2.4}$$

Then (X, d) is a complete metric space. Define $T_1, T_2 : X \to CB(X)$ by

$$T_1(x) = \begin{cases} \{0\} & \textit{if } x = 0 \\ \{\frac{1}{2^{n+1}}, 0\} & \textit{if } x = \frac{1}{2^n},\ n \geq 1 \\ \{\frac{1}{2}\} & \textit{if } x = 1 \end{cases},$$

and

$$T_2(x) = \begin{cases} \{0\} & \textit{if } x = 0 \\ \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\} & \textit{if } x = \frac{1}{2^n},\ n \geq 1 \\ \{0\} & \textit{if } x = 1 \end{cases}.$$

For each $i \in \{1, 2\}$ define graph G_i by $V(G_i) = X$,

$$E(G_1) = Gr(T_1) = \left\{\left(\frac{1}{2^n}, 0\right) :\ n \geq 1\right\} \cup \left\{\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) :\ n \geq 1\right\} \cup \left\{(0,0), \left(1, \frac{1}{2}\right)\right\},$$

and

$$E(G_2) = Gr(T_2) = \left\{\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) :\ n \geq 1\right\} \cup \left\{\left(\frac{1}{2^n}, \frac{1}{2^{n+2}}\right) :\ n \geq 1\right\} \cup \{(0,0), (1,0)\}.$$

Let the implicit function g be defined by

$$g(x_1, \ldots, x_5) = \frac{1}{2}\max\{x_1, x_2\} \quad \text{where } x_1, \ldots, x_5 \geq 0.$$

We will show that $\{T_n\}$ is a $(g, G)-$ contraction sequence where $T_{2n-1} = T_1$ and $T_{2n} = T_2$ for each $n \geq 1$. Since

$$\begin{aligned}
\mathcal{H}(T_1 0, T_2 0) = 0 &\leq \frac{1}{2}\max\{d(0,0), d(0,0)\} = 0, \\
\mathcal{H}(T_1 \frac{1}{2^n}, T_2 0) = \frac{1}{2^{n+1}} &\leq \frac{1}{2}\max\{d(0, \frac{1}{2^n}), d(\frac{1}{2^n}, \{\frac{1}{2^{n+1}}, 0\})\} = \frac{1}{2^n}, \\
\mathcal{H}(T_1 \frac{1}{2^n}, T_2 \frac{1}{2^{n+1}}) = \frac{1}{2^{n+1}} &\leq \frac{1}{2}\max\{\frac{1}{2^n}, \frac{1}{2^n}\} = \frac{1}{2^{n+1}}, \\
\mathcal{H}(T_1 1, T_2 \frac{1}{2}) = \frac{1}{2} &\leq \frac{1}{2}\max\{1, 1\} = \frac{1}{2}, \\
\mathcal{H}(T_2 0, T_1 0) = 0 &\leq \frac{1}{2}\max\{d(0,0), d(0,0)\} = 0, \\
\mathcal{H}(T_2 \frac{1}{2^n}, T_1 \frac{1}{2^{n+1}}) = \frac{1}{2^{n+1}} &\leq \frac{1}{2}\max\{\frac{1}{2^n}, \frac{1}{2^n}\} = \frac{1}{2^{n+1}}, \\
\mathcal{H}(T_2 \frac{1}{2^n}, T_1 \frac{1}{2^{n+2}}) = \frac{1}{2^{n+1}} &\leq \frac{1}{2}\max\{\frac{1}{2^n}, \frac{1}{2^n}\} = \frac{1}{2^{n+1}}, \\
\mathcal{H}(T_2 1, T_1 0) = 0 &\leq \frac{1}{2}\max\{1, 1\} = \frac{1}{2},
\end{aligned}$$

$\{T_n\}$ is $(g, G)-$ contraction. One can easily check that other conditions of Theorem 2.7 are satisfied. Therefore T_1 and T_2 have a common fixed point that is $x = 0$.

3. Some applications

In this section, we apply our main result to improve some Nadler's type fixed point theorems. In 2008, Kikkawa and Suzuki improved Nadler's fixed point theorem as follows.

Theorem 3.1. [7, Theorem 2] *Define a strictly decreasing function* η *from* $[0, 1)$ *onto* $(\frac{1}{2}, 1]$ *by*

$$\eta(r) = \frac{1}{1+r}.$$

Let (X, d) *be a complete metric space and* T *be a mapping from* X *into* $CB(X)$ *and there exists* $r \in [0, 1)$ *such that*

$$\eta(r)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq rd(x, y),$$

for all $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Mot and Petruşel extended the above result for generalized multi-valued contractions.

Theorem 3.2. [10, Theorem 6.6] *Let* (X, d) *ba a complete metric space and* T *be a multi-valued function from* X *into nonempty closed subsets of* X. *Assume that for some* $r_1, r_2, r_3 \in [0, 1)$ *with* $r_1 + r_2 + r_3 < 1$, $\frac{1-r_2-r_3}{1+r_1}d(x, Tx) \leq d(x, y)$ *implies that*

$$\mathcal{H}(Tx, Ty) \leq r_1 d(x, y) + r_2 d(x, Tx) + r_3 d(y, Ty)$$

for all $x, y \in X$. *Then there is some* $x \in X$ *such that* $x \in Tx$.

Since $\frac{1-r_2-r_3}{1+r_1} < \frac{1}{1+r_1+r_2+r_3}$, the following result can be considered as a refinement of Theorem 3.2 when $r_4 = 0$.

Theorem 3.3. *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$. *Assume that there exist* $0 \leq r_1, r_2, r_3, r_4 < 1$ *with* $r = r_1 + r_2 + r_3 + 2r_4 < 1$ *and* $s \in (r, 1)$ *such that* $\frac{1}{1+r}d(x, Tx) \leq d(x, y) \leq \frac{1}{1-s}d(x, Tx)$ *implies that*

$$\mathcal{H}(Tx, Ty) \leq r_1 d(x, y) + r_2 d(x, Tx) + r_3 d(y, Ty) + r_4(d(x, Ty) + d(y, Tx)).$$

Then T *has a fixed point.*

Proof. Define graph G by $V(G) = X$ and

$$E(G) = \{(x, y) : \frac{1}{1+r}d(x, Tx) \leq d(x, y) \leq \frac{1}{1-s}d(x, Tx)\}.$$

Assume that $g : [0, \infty)^5 \to [0, \infty)$ is defined by

$$g(a_1, \ldots, a_5) = r_1 a_1 + r_2 a_2 + r_3 a_3 + r_4(a_4 + a_5).$$

Then $g(1, 1, 1, 2, 0) = r_1 + r_2 + r_3 + 2r_4 = r < 1$. Therefore $g \in \mathcal{R}$.
Let $\{T_n\}$ be a sequence in which $T_n = T$ for each $n \in \mathbb{N}$. we show that $\{T_n\} = T$ is a $(g, G)-$contraction. Let $(x_0, x_1) \in E(G)$. Thus we have

$$\frac{1}{1+r}d(x_0, Tx_0) \leq d(x_0, x_1) \leq \frac{1}{1-s}d(x_0, Tx_0).$$

By our hypothesis

$$\mathcal{H}(Tx_0, Tx_1) \leq r_1 d(x_0, x_1) + r_2 d(x_0, Tx_0) + r_3 d(x_1, Tx_1) + r_4 (d(x_0, Tx_1) + d(x_1, Tx_0))$$
$$= g(d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), d(x_0, Tx_1), d(x_1, Tx_0)).$$

Hence T is a $(g, G)-$ contraction.
Since $0 \leq r < s < 1$, we have $r < \frac{r}{1+r-s} < s$. Choose $s_1 \in \left(\frac{r}{1+r-s}, s\right)$. Put $t = \frac{1-s}{1-s_1}$.
Then we get

$$\frac{r}{1+r-s} < s_1 \Rightarrow r < s_1 + rs_1 - s_1 s \Rightarrow r - rs_1 < s_1 - s_1 s \Rightarrow \frac{r(1-s_1)}{1-s} < s_1.$$

Hence $r < \frac{r}{s_1} < \frac{1-s}{1-s_1} = t$. Thus $r = g(1, 1, 1, 2, 0) < t$.
We show that $Gr_t(T) \subseteq E(G)$. Let $(x_0, x_1) \in Gr_t(T)$. Then $x_1 \in Tx_0$ and

$$\frac{1-s}{1-s_1} d(x_0, x_1) = td(x_0, x_1) \leq d(x_0, Tx_0).$$

We have

$$\frac{1}{1+r} d(x_0, Tx_0) \leq d(x_0, Tx_0) \leq d(x_0, x_1) \leq \frac{1-s_1}{1-s} d(x_0, Tx_0) \leq \frac{1}{1-s} d(x_0, Tx_0).$$

Hence $(x_0, x_1) \in E(G)$. Therefore $Gr_t(T) \subseteq E(G)$. Lemma 2.6 implies that there exists a Cauchy sequence $\{x_n\}$ such that for each $n \geq 1$, $x_n \in Tx_{n-1}$, $(x_{n-1}, x_n) \in E(G)$ and $d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n)$ where $q = \frac{g(1,1,1,2,0)}{t} = \frac{r}{t}$.
Since X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. We show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$, for each $k \geq 0$.
For each $n \geq 0$ and each $p \geq 1$ we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &= d(x_n, x_{n+1})(1 + q + q^2 + \cdots + q^{p-1}) \\ &= \frac{1-q^p}{1-q} d(x_n, x_{n+1}). \end{aligned}$$

Tending $p \to \infty$ we get $d(x_n, x) \leq \frac{1}{1-q} d(x_n, x_{n+1})$. Since $\frac{r}{s_1} < t$ and $q = \frac{r}{t}$ we get $q < s_1$. So $d(x_n, x) \leq \frac{1}{1-q} d(x_n, x_{n+1}) \leq \frac{1}{1-s_1} d(x_n, x_{n+1})$, for each $n \geq 0$.
Thus for each $n \geq 0$ we get

$$\begin{aligned} d(x_n, x) &\leq \frac{1}{1-s_1} d(x_n, x_{n+1}) \\ &\leq \frac{1}{1-s_1} \frac{1}{t} d(x_n, Tx_n) \leq \frac{1}{1-s_1} \frac{1-s_1}{1-s} d(x_n, Tx_n) \\ &= \frac{1}{1-s} d(x_n, Tx_n). \end{aligned}$$

On the other hand, for each $n \geq 0$ we have $(x_n, x_{n+1}) \in E(G)$. Since T is a $(g,G)-$contraction, for each $n \geq 0$ we get

$$\begin{aligned} & d(x_{n+1}, Tx_{n+1}) \leq \mathcal{H}(Tx_n, Tx_{n+1}) \\ \leq\ & r_1 d(x_n, x_{n+1}) + r_2 d(x_n, Tx_n) + r_3 d(x_{n+1}, Tx_{n+1}) \\ & + r_4(d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)) \\ \leq\ & r_1 d(x_n, x_{n+1}) + r_2 d(x_n, Tx_n) + r_3 d(x_{n+1}, Tx_{n+1}) + r_4 d(x_n, Tx_{n+1}). \end{aligned}$$

Thus for each $n \geq 0$,

$$\begin{aligned} & d(x_{n+1}, Tx_{n+1}) - r_3 d(x_{n+1}, Tx_{n+1}) - r_4 d(x_{n+1}, Tx_{n+1}) \\ & \leq (r_1 + r_4) d(x_n, x_{n+1}) + r_2 d(x_n, Tx_n), \end{aligned}$$

and so

$$(1 - r_3 - r_4) d(x_{n+1}, Tx_{n+1}) \leq (r_1 + r_4) d(x_n, x_{n+1}) + r_2 d(x_n, Tx_n). \tag{3.1}$$

Now, suppose in contrary that there exists $N \in \mathbb{N}$ such that $d(x, x_n) < \frac{1}{1+r} d(x_n, Tx_n)$ for each $n \geq N$. By (3.1), for each $n \geq N$ we get

$$\begin{aligned} d(x_n, x_{n+1}) \leq d(x, x_n) + d(x, x_{n+1}) &< \frac{1}{1+r}[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \\ &\leq \frac{1}{1+r}[d(x_n, Tx_n) + \frac{r_1 + r_4}{1 - r_3 - r_4} d(x_n, x_{n+1}) + \frac{r_2}{1 - r_3 - r_4} d(x_n, Tx_n)]. \end{aligned}$$

Hence

$$\begin{aligned} & d(x_n, x_{n+1}) - \frac{1}{(1+r)} \frac{r_1 + r_4}{1 - r_3 - r_4} d(x_n, x_{n+1}) \\ & < \frac{1}{1+r} d(x_n, Tx_n) + \frac{1}{1+r} \frac{r_2}{1 - r_3 - r_4} d(x_n, Tx_n). \end{aligned}$$

So

$$(1+r) d(x_n, x_{n+1}) - \frac{r_1 + r_4}{1 - r_3 - r_4} d(x_n, x_{n+1}) < d(x_n, Tx_n) + \frac{r_2}{1 - r_3 - r_4} d(x_n, Tx_n).$$

Thus we get

$$\frac{(1+r)(1 - r_3 - r_4) - (r_1 + r_4)}{1 - r_3 - r_4} d(x_n, x_{n+1}) < \frac{1 - r_3 - r_4 + r_2}{1 - r_3 - r_4} d(x_n, Tx_n),$$

and so

$$(1+r)(1 - r_3 - r_4) - (r_1 + r_4) d(x_n, x_{n+1}) < (1 - r_3 - r_4 + r_2) d(x_n, Tx_n).$$

If $r_3 = r_4 = 0$ then

$$\frac{(1+r)(1 - r_3 - r_4) - (r_1 + r_4)}{1 - r_3 - r_4 + r_2} = \frac{1 + r_2}{1 + r_2} = 1.$$

Otherwise $r_3 > 0$ or $r_4 > 0$. We show that

$$\frac{(1+r)(1 - r_3 - r_4) - (r_1 + r_4)}{1 - r_3 - r_4 + r_2} > 1. \tag{3.2}$$

It is easy to see that (3.2) is equivalent to

$$\frac{r-(r_3+r_4)}{r-r(r_3+r_4)}<1$$

which is true in this case. Hence

$$\frac{(1+r)(1-r_3-r_4)-(r_1+r_4)}{1-r_3-r_4+r_2}\geq 1.$$

Thus for each $n \geq N$,

$$d(x_n,x_{n+1})\leq \frac{(1+r)(1-r_3-r_4)-(r_1+r_4)}{1-r_3-r_4+r_2}d(x_n,x_{n+1})<d(x_n,Tx_n),$$

which is a contradiction. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x,x_{n_k})\geq \frac{1}{1+r}d(x_{n_k},Tx_{n_k})$, for each $k\geq 0$.
Since for each $n\geq 0$, $d(x,x_n)\leq \frac{1}{1-s}d(x_n,Tx_n)$, we may consider $\{x_{n_k}\}$ as a sequence where $(x_{n_k},x)\in E(G)$ for each $k\geq 0$. Theorem 2.7 implies that x is a fixed point of T. □

Remark 3.4. By imitating the proof of Theorem 3.3 for implicit function

$$g(t_1,\ldots,t_5)=r\max\left\{t_1,t_2,t_3,\frac{t_4+t_5}{2}\right\},$$

one can prove the following result.

Theorem 3.5. [16, Theorem 2.7] *Let (X,d) be a complete metric space and $T:X\to CB(X)$. If $r,s\in[0,1)$, $s>r$ and $x,y\in X$ with*

$$\frac{1}{1+r}d(x,Tx)\leq d(x,y)\leq \frac{1}{1-s}d(x,Tx)$$

implies

$$\mathcal{H}(Tx,Ty)\leq r\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\},$$

then T has a fixed point.

Now, we give some results that ensure the existence of common fixed points for a sequence of multi-valued mappings. All of which are consequences of Theorem 2.7. The following result may be considered as a generalization of Amini Theorem [2].

Theorem 3.6. *Let (X,d) be a complete metric space and $T_1,T_2,\cdots:X\to CB(X)$ be a sequence of multi-valued mappings such that $T_n=T_1$ for infinity many $n\in\mathbb{N}$ and $T_{n+1}=T_2$ whenever $T_n=T_1$. Assume that $\frac{1}{1+r}d(x,T_ix)\leq d(x,y)$ implies that*

$$\mathcal{H}(T_ix,T_{i+1}y)\leq r\max\{d(x,y),d(x,T_ix),d(y,T_{i+1}y),d(x,T_{i+1}y),d(y,T_ix)\} \quad (3.3)$$

for each $x,y\in X$, each $i\in\mathbb{N}$ and for some $r\in[0,\frac{1}{2})$. Then $\{T_n\}$ has a common fixed point.

Proof. Define $g : [0,\infty)^5 \to [0,\infty)$ by $g(t_1,\dots,t_5) = r\max\{t_1,\dots,t_5\}$ for each $t_1,\dots,t_5 \in \mathbb{R}$. Then g is sub-homogenous and increasing with respect to each variable and $g(1,1,1,2,0) = 2r \in (0,1)$. Therefore $g \in \mathcal{R}$. For each $i \geq 1$ define G_i by $V(G_i) = X$ and

$$E(G_i) = \{(x,y);\ \frac{1}{1+r}d(x,T_ix) \leq d(x,y)\}.$$

Define graph G by

$$V(G) = X \text{ and } E(G) = \bigcup_{i\geq 1} E(G_i).$$

Then $E(G_i) \supseteq Gr(T_i)$ for each $i \geq 1$. For each $i \geq 1$ and $(x_0,x_1) \in E(G_i)$, we have

$$\begin{aligned}\mathcal{H}(T_ix_0,T_{i+1}x_1) &\leq r\max\{d(x_0,x_1),d(x_0,T_ix_0),d(x_1,T_{i+1}x_1),d(x_0,T_{i+1}x_1),d(x_1,T_ix_0)\}\\ &= g(d(x_0,x_1),d(x_0,T_ix_0),d(x_1,T_{i+1}x_1),d(x_0,T_{i+1}x_1),d(x_1,T_ix_0)).\end{aligned}$$

Thus $\{T_n\}$ is a $(g,G)-$contraction sequence. By Lemma 2.4,

$$Fix(T_i) = Fix(T_j) \text{ for each } i,j \in \mathbb{N}.$$

Lemma 2.6 implies that there exists a Cauchy sequence $\{x_n\}$ such that $x_n \in T_nx_{n-1}$ for each $n \geq 0$. Since X is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n,x) = 0.$$

Suppose that for some $n \geq 2$,

$$x_{n-1} \in T_1x_{n-2} \text{ and } x_n \in T_2x_{n-1}.$$

We claim that either

$$\frac{1}{1+r}d(x_{n-2},T_1x_{n-2}) \leq d(x_{n-2},x) \tag{3.4}$$

or

$$\frac{1}{1+r}d(x_{n-1},T_2x_{n-1}) \leq d(x_{n-1},x). \tag{3.5}$$

Suppose that neither (3.4) nor (3.5) holds. Then we get

$$\begin{aligned}(1+r)d(x_{n-2},x_{n-1}) &\leq (1+r)d(x_{n-2},x) + (1+r)d(x_{n-1},x)\\ &< d(x_{n-2},T_1x_{n-2}) + d(x_{n-1},T_2x_{n-1})\\ &\leq d(x_{n-2},x_{n-1}) + d(x_{n-1},x_n)\\ &\leq d(x_{n-2},x_{n-1}) + rd(x_{n-2},x_{n-1})\\ &= (1+r)d(x_{n-2},x_{n-1}),\end{aligned}$$

which is a contradiction. Thus either (3.4) or (3.5) holds. Hence either $(x_{n-1},x) \in E(G_1)$ or $(x_{n-1},x) \in E(G_2)$ for infinity many $n \in \mathbb{N}$. Theorem 2.7 implies that $\{T_n\}$ has a common fixed point. □

The following generalization of Amini-Harandi fixed point theorem follows immediately from Theorem 3.6.

Corollary 3.7. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mappings. Assume that $\frac{1}{1+r}d(x,Tx) \le d(x,y)$ implies that*

$$\mathcal{H}(Tx,Ty) \le r\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{3.6}$$

for some $r \in [0,\frac{1}{2})$. Then T has a fixed point.

By imitating the proof of Theorem 3.6 for the implicit function

$$g(t_1,\dots,t_5) = r\max\left\{t_1,t_2,t_3,\frac{t_4+t_5}{2}\right\},$$

one can prove the following result.

Theorem 3.8. *Let (X,d) be a complete metric space and $\{T_n\}$ be a sequence of multi-valued mappings of X to $CB(X)$ such that $T_n = T_1$ for infinity many $n \in \mathbb{N}$ and $T_{n+1} = T_2$ whenever $T_n = T_1$. Assume that $\frac{1}{1+r}d(x,T_ix) \le d(x,y)$ implies that*

$$\mathcal{H}(T_ix,T_{i+1}y) \le r\max\left\{d(x,y), d(x,T_ix), d(y,T_{i+1}y), \frac{d(x,T_{i+1}y)+d(y,T_ix)}{2}\right\} \tag{3.7}$$

for each $x,y \in X$, each $i \in \mathbb{N}$ and for some $r \in [0,1)$. Then $\{T_n\}$ has a common fixed point.

The next result is another extension of Theorem 3.2 when $\{T_n\}$ is a sequence of multi-valued mappings.

Theorem 3.9. *Let (X,d) ba a complete metric space and $\{T_n\}$ be a sequence of multi-valued mappings such that $T_n = T_1$ for infinity many $n \in \mathbb{N}$ and $T_{n+1} = T_2$ whenever $T_n = T_1$. Assume that $\frac{1-b-c}{1+a}d(x,T_ix) \le d(x,y)$ implies that*

$$\mathcal{H}(T_ix,T_{i+1}y) \le ad(x,y) + bd(x,T_ix) + cd(y,T_{i+1}y), \tag{3.8}$$

for some $a,b,c \in [0,1)$ with $a+b+c<1$ and for all $x,y \in X$. Then $\{T_n\}$ has a common fixed point.

Proof. Let $g : [0,\infty)^5 \to [0,\infty)$ be defined by $g(t_1,\dots,t_5) = at_1 + bt_2 + ct_3$, for each $t_1,\dots,t_5 \in [0,\infty)$, then $g \in \mathcal{R}$. Let $V(G_i) = X$ and

$$E(G_i) = \{(x_0,x_1);\ \frac{1-b-c}{1+a}d(x_0,T_ix_0) \le d(x_0,x_1)\},$$

for $i \in \mathbb{N}$. Define the graph of G by

$$V(G) = X \text{ and } E(G) = \bigcup_{i\ge 1} E(G_i).$$

Let $(x_0,x_1) \in E(G_i)$ for some $i \in \mathbb{N}$, then $\frac{1-b-c}{1+a}d(x_0,T_ix_0) \le d(x_0,x_1)$. Therefore

$$\begin{aligned}\mathcal{H}(T_ix_0,T_{i+1}x_1) &\le ad(x_0,x_1) + bd(x_0,T_ix_0) + cd(x_1,T_{i+1}x_1))\\ &= g(d(x_0,x_1), d(x_0,T_ix_0), d(x_1,T_{i+1}x_1), d(x_0,T_{i+1}x_1), d(x_1,T_ix_0)).\end{aligned}$$

Hence $\{T_n\}$ is a (G,g)-contraction sequence. By Lemma 2.4,

$$Fix(T_i) = Fix(T_j) \text{ for each } i,j \in \mathbb{N}.$$

Since $E(G_i)$ contains $Gr(T_i)$ for each $i \geq 1$, we can use Lemma 2.6 to obtain a Cauchy sequence $\{x_n\}$ such that $x_n \in T_n\ x_{n-1}$ for each $n > 0$. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. Let for some $n \geq 2$,

$$x_{n-1} \in T_1 x_{n-2} \textit{ and } x_n \in T_2 x_{n-1}.$$

We claim that either

$$\frac{1-b-c}{1+a} d(x_{n-2}, T_1 x_{n-2}) \leq d(x_{n-2}, x) \text{ or} \tag{3.9}$$

$$\frac{1-b-c}{1+a} d(x_{n-1}, T_2 x_{n-1}) \leq d(x_{n-1}, x). \tag{3.10}$$

Suppose that neither (3.9) nor (3.10) holds. Then we have

$$\begin{aligned}
(1+a)d(x_{n-2}, x_{n-1}) &\leq (1+a)d(x_{n-2}, x) + (1+a)d(x_{n-1}, x) \\
&< (1-b-c)\left(d(x_{n-2}, T_1x_{n-2}) + d(x_{n-1}, T_2x_{n-1})\right) \\
&\leq (1-b-c)\left(d(x_{n-2}, x_{n-1}) + \mathcal{H}(T_1x_{n-2}, T_2x_{n-1})\right) \\
&\leq (1-b-c)\big(d(x_{n-2}, x_{n-1}) + ad(x_{n-2}, x_{n-1}) \\
&+ bd(x_{n-2}, T_1x_{n-2}) + cd(x_{n-1}, T_2x_{n-1})\big) \\
&\leq (1-b-c)\big(d(x_{n-2}, x_{n-1}) + ad(x_{n-2}, x_{n-1}) \\
&+ bd(x_{n-2}, T_1x_{n-2}) + cd(x_{n-1}, x_n)\big) \\
&\leq (1-b-c)(1+a+b+c)d(x_{n-2}, x_{n-1}).
\end{aligned}$$

Since $(1-b-c)(1+a+b+c) = 1+a-(b+c)(a+b+c)$, the above inequality implies that $-(b+c)(a+b+c) > 0$, which is a contradiction. Thus either (3.9) or (3.10) holds for infinity many $n \in \mathbb{N}$. So either $(x_n, x) \in E(G_1)$ or $(x_n, x) \in E(G_2)$ for infinity many n. By Theorem 2.7, $\{T_n\}$ has a common fixed point. ☐

4. Multi-fractal mappings

Let (X, d) be a metric space and $T : X \to CB(X)$. Define

$$CB_{cp}(X) = \{A \in CB(X) \textit{ and } A \textit{ is compact}\}.$$

The mapping $\widehat{T} : CB_{cp}(X) \to CB_{cp}(X)$ defined by $\widehat{T}(A) = \bigcup_{a\in A} Ta$ is said to be the multi-fractal operator generated by T.

In this section, we give an application of our results to finding unique fixed point of a multi-fractal. In order to achieve this goal, we Consider a subset $\mathcal{R}'$ of $\mathcal{R}$ such that the condition (iii) in Definition 2.3 is replaced by following property:

$(iii)'$ $g(1, 1, 1, 2, 1) < 1$.

Trivially $\mathcal{R}' \subseteq \mathcal{R}$.

Regarding $\{T_n\}$ as a sequence of single valued mappings and $g \in \mathcal{R}'$, we get the following result.

Theorem 4.1. *Let (X, d) be a complete metric space endowed with a graph G, $g \in \mathcal{R}'$ be continuous. Let $\{T_n\}$ be single valued (g, G)-contraction sequence and $(x, T_nx) \in E(G_n)$ for each $x \in X$ and each $n \in \mathbb{N}$. Let $\{x_n\}$ be the Cauchy sequence defined in proof of Lemma* 2.6 *and $x \in X$ be its limit. Then x is a common fixed point of $\{T_n\}$*

provided that there exists a subsequence $\{x_{n_k}\}$ *and* $i \geq 1$ *such that* $(x_{n_k}, x) \in E(G_i)$ *and* $x_{n_k+1} = T_i x_{n_k}$ *for each* $k \in \mathbb{N}$. *Further* x *is a unique common fixed point of* $\{T_n\}$ *if* $y, z \in Fix(T_i)$ *for some* $i \geq 1$ *implies that there exists* $j \geq 1$ *such that* $(y, z) \in E(G_j)$.

Proof. By our hypothesis, for each $n \in \mathbb{N}$ and each $z \in Fix(T_n)$ we have $(z, z) = (z, T_n z) \in E(G_n)$. Therefore the first part follows by Theorem 2.7. Let $y, z \in Fix(T_i)$ for some $i \geq 1$. Lemma 2.4 implies that $y, z \in Fix(T_i)$ for each $i \geq 1$. Assume that $(y, z) \in E(G_j)$ for some $j \geq 1$. Then we get

$$\begin{aligned} d(y,z) &\leq d(T_j y, T_{j+1} z) \\ &\leq g(d(y,z), d(y,T_j y), d(z,T_{j+1} z), d(y,T_{j+1} z), d(z,T_j y)) \\ &= g(d(y,z), 0, 0, d(y,z), d(z,y)) \\ &= g(1,0,0,1,1)d(y,z) \leq g(1,1,1,2,1)d(y,z), \end{aligned}$$

Since $g \in \mathcal{R}'$, $g(1,1,1,2,1) < 1$. Therefore $d(y, z) = 0$. Thus $y = z$. □

The following result gives conditions under which a multi-fractal has a unique fixed point.

Corollary 4.2. *Let* (X, d) *be a metric space and* $g \in \mathcal{R}'$. *Let* $T : X \to CB(X)$ *be a upper semi continuous* $(g, G)-$*contraction where* $V(G) = X$ *and* $E(G) = X \times X$. *Then the multi-fractal* $\widehat{T}$ *generated by* T *has a unique fixed point.*

Proof. We show that the single-valued mapping $\widehat{T} : CB_{cp}(X) \to CB_{cp}(X)$ satisfies in conditions of Theorem 4.1. Define graph $\widehat{G}$ by

$$V(\widehat{G}) = CB_{cp}(X) \text{ and } E(\widehat{G}) = CB_{cp}(X) \times CB_{cp}(X).$$

Let $(A, B) \in E(\widehat{G})$. Then

$$\begin{aligned} \sup_{a\in A} \inf_{b\in B} \mathcal{H}(Ta, Tb) &\leq \sup_{a\in A} \inf_{b\in B} g(d(a,b), d(a,Ta), d(b,Tb), d(a,Tb), d(b,Ta)) \\ &\leq \sup_{a\in A} g(d(a,B), d(a,Ta), \mathcal{H}(B,TB), d(a,TB), d(B,Ta)) \\ &\leq g(\mathcal{H}(A,B), \mathcal{H}(A,TA), \mathcal{H}(B,TB), \mathcal{H}(A,TB), \mathcal{H}(B,TA)). \end{aligned}$$

Similarly one can show that

$$\sup_{b\in B} \inf_{a\in A} \mathcal{H}(Ta, Tb) \leq g(\mathcal{H}(A,B), \mathcal{H}(A,TA), \mathcal{H}(B,TB), \mathcal{H}(A,TB), \mathcal{H}(B,TA)).$$

Hence for each $(A, B) \in E(\widehat{G})$,

$$\mathcal{H}(TA, TB) \leq g(\mathcal{H}(A,B), \mathcal{H}(A,TA), \mathcal{H}(B,TB), \mathcal{H}(A,TB), \mathcal{H}(B,TA)).$$

Thus $\widehat{T}$ is a $(g, \widehat{G})$-contraction. Also for each $A \in CB_{cp}(X)$, $(A, \widehat{T}(A)) \in E(\widehat{G})$. Thus conditions of Theorem 4.1 hold. Hence $\widehat{T}$ has a unique fixed point. □

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