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POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM VIA A MIXED MONOTONE OPERATOR

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Abstract. In this paper, by using a mixed monotone operator method we study the existence and uniqueness of positive solutions to the following nonlinear fractional boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) + f(t, u(t), (Hu)(t)) + g(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u'(1) = \gamma u'(\eta), \end{cases}$$

where $2 < \alpha \leq 3$, $\gamma, \eta \in (0, 1)$, ${}^{C}D^{\alpha}_{0^+}$ denotes de Caputo fractional derivative, $f : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ and $g : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous functions and H is an operator (not necessarily linear) applying $\mathcal{C}[0, 1]$ into itself. Moreover, in order to illustrate our results, we present some examples.

Key Words and Phrases: Fractional boundary value problem, positive solution, mixed monotone operator, fixed point.
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1. Introduction

The theory of fractional differential equations has received much attention over the past years due to its numerous applications in a great number of areas as physics, chemistry, economics, control theory, signal and image processing, etc. (see, [5, 6, 7, 8, 9, 10], for example and the references therein).

Many methods have been used to prove the existence of solutions to fractional boundary value problems with different boundary conditions such as fixed point theorems, upper and lower solution method, monotone iterative method, among others.

In this paper, we study the existence and uniqueness of positive solutions to the following nonlinear fractional boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) + f(t,u(t),(Hu)(t)) + g(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u'(1) = \gamma u'(\eta), \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $\gamma, \eta \in (0,1)$, ${}^{C}D^{\alpha}_{0^+}$ denotes the Caputo frational derivative $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ and $g : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous functions and $H : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is an operator (not necessarily linear) satisfying certain assumptions, by using a mixed monotone operator method. This technique has been used by other authors in the literature (see [1],[3, 4],[13] y [14] for example).

2. Preliminaries and previous results

This section is devoted to the background material needed to establish the main result of the paper.

Our starting point is to present some definitions and basic results about fractional calculus. This material can be found in [5].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$(I^{\alpha}_{0^+}f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined as

$$\left({}^{c}D^{\alpha}_{0^+}f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 < \alpha \leq n$, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3 *Let $n-1 < \alpha \leq n$ and $u \in \mathcal{C}^{(n)}[0,1]$. Then*

$$I^{\alpha}_{0^+}\, {}^{C}D^{\alpha}_{0^+}u(t) = u(t) - c_1 - c_2 t - \cdots - c_n t^{n-1},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \cdots, n)$.

Lemma 2.4 *The relation*

$$I^{\alpha}_{0^+} I^{\beta}_{0^+} u(t) = I^{\alpha+\beta}_{0^+} u(t)$$

holds if $Re(\alpha) > 0$, $Re(\alpha+\beta) > 0$ and $u \in L^1(0,\infty)$.

Next, we recall some basic concepts in ordered Banach spaces and some results about the mixed monotone operator theory.

In the sequel, $(E, \|\cdot\|)$ is a real Banach space.

A nonempty closed convex set $P \subset E$ is said to be a cone if it satisfies: (i) $x \in P$ and $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $-x, x \in P \Rightarrow x = \theta_E$, where θ_E denotes the zero element of E.

Suppose that P is a cone in the Banach space $(E, \|\cdot\|)$, then P induces a partial order in E given by, for $x, y \in E$,

$$x \leq y \iff x - y \in P.$$

By $x < y$, we mean that $x \leq y$ and $x \neq y$. If interior of P, $\mathring{P}$, is nonempty, the cone P is said to be solid. If there exists a constant $C > 0$ such that, for any $x, y \in E$, with $\theta_E \leq x \leq y$ implies $\|x\| \leq C\|y\|$ then we say that the cone P is normal. In this case, the smallest constant C satisfying the last inequality is called the normality constant of P.

For $x, y \in E$, by $x \sim y$ we mean the existence of constants $\lambda, \mu > 0$ such that

$$\lambda y \leq x \leq \mu y.$$

It is easily seen that $\sim$ is an equivalence relation.

For $\theta_E < h$, we denote by P_h the set given by

$$P_h = \{x \in E : x \sim h\}.$$

It is easily proved that $P_h \subset P$.

Definition 2.5 An operator $T : E \to E$ is said to be increasing (resp. decreasing) if, for any $x, y \in E$, $x \leq y$ then $Tx \leq Ty$ (resp. $Tx \geq Ty$).

Definition 2.6 An operator $A : P \times P \to P$ is called mixed monotone if $A(x, y)$ is increasing in x and decreasing in y, i.e., for any $(x, y), (u, v) \in P \times P$,

$$x \leq u \quad \text{and} \quad y \geq v \Rightarrow A(x, y) \leq A(u, v).$$

Definition 2.7 An operator $B : P \to P$ is called subhomogeneous if

$$B(\lambda x) \geq \lambda B(x), \quad \text{for any} \quad \lambda \in (0, 1) \quad \text{and} \quad x \in P.$$

The following result appears in [12] and it is the main tool used in our study.

Theorem 2.8 *Suppose that $\alpha \in (0, 1)$, $h \in E$ with $\theta_E < h$, and P is a normal cone in the Banach space $(E, \|\cdot\|)$. Let $A : P \times P \to P$ be a mixed monotone operator satisfying*

$$A(tx, t^{-1}y) \geq t^{\alpha} A(x, y), \quad \textit{for any } t \in (0, 1) \textit{ and } x, y \in P.$$

Let $B : P \longrightarrow P$ be an increasing subhomogeneous operator.
Assume that

(i) *There exists $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$,*
(ii) *There exists a constant $\delta_0 > 0$ such that*

$$A(x, y) \geq \delta_0 Bx, \quad \textit{for any } x, y \in P.$$

Then

(a) *$A : P_h \times P_h \longrightarrow P_h$ and $B : P_h \longrightarrow P_h$.*
(b) *There exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$ and*

$$u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0.$$

(c) *There exists a unique* $x^* \in P_h$ *such that*

$$x^* = A(x^*, x^*) + Bx^*.$$

(d) *For any initial values* $x_0, y_0 \in P_h$*, the sequences defined by*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1},$$
$$y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}$$

for $n = 1, 2, \cdots$*, satisfy*

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - x^*\| = 0.$$

3. Main results

We will work in the space $E = C[0,1] = \{x : [0,1] \to \mathbb{R},\ \text{continuous}\ \}$, with the classical norm given by

$$\|x\| = \max\{|x(t)| : t \in [0,1]\}.$$

In E, we consider the cone P defined by

$$P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\}.$$

It is easily seen that P is a normal cone with normality constant $C = 1$. In this case, the partial order in $C[0,1]$ induced by P is given by, for $x, y \in E$,

$$x \leq y \Leftrightarrow x(t) \leq y(t),\ \text{ for any }\ t \in [0,1].$$

Previously, we recall the following lemma appearing in [11], which was proved by using Lemma 2.3 and Lemma 2.4.

Lemma 3.1 *Suppose that* $g \in L^1[0,1]$*. Then the following fractional boundary value problem*

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + g(t) = 0, & 0 < t < 1 \\ u(0) = u''(0) = 0, & u'(1) = \gamma u'(\eta), \end{cases}$$

where $2 < \alpha \leq 3$ *and* $\gamma, \eta \in (0,1)$ *has as unique solution*

$$u(t) = \int_0^1 G_1(t,s) g(s) ds + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) g(s) ds,$$

where $G_1(t,s)$ *and* $G_2(\eta,s)$ *are the functions given by*

$$G_1(t,s) = \begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$G_2(\eta,s) = \begin{cases} \dfrac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq s \leq \eta \leq 1, \\ \dfrac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq 1. \end{cases}$$

Remark 3.2 *It is clear that the functions* $G_1(t,s)$ *and* $G_2(\eta,s)$ *are continuous on* $[0,1]\times[0,1]$.

In [11], the author proves that $G_1(t,s)\geq 0$ for any $t,s\in[0,1]$.
In order to that the paper is self-contained, we present a proof of this fact.

Lemma 3.3 *The functions* $G_1(t,s)$ *and* $G_2(t,s)$ *appearing in Lemma 3.1 are non-negative.*
Proof. Since $2<\alpha\leq 3$, it is clear that, for $0\leq t\leq s\leq 1$,

$$G_1(t,s)=\frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}\geq 0.$$

For $0\leq s\leq t\leq 1$, we have

$$\begin{aligned}G_1(t,s)&=\frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\geq\frac{t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\&\geq\frac{(t-s)(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\geq\frac{(t-s)(t-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\&=\frac{(t-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}=0.\end{aligned}$$

This proves that $G_1(t,s)\geq 0$, for any $t,s\in[0,1]$.

For $0\leq\eta\leq s\leq 1$, it is clear that $G_2(\eta,s)=\dfrac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)}\geq 0$.

In the case $0\leq s\leq\eta\leq 1$, we have

$$\begin{aligned}G_2(\eta,s)&=\frac{(\alpha-1)(1-s)^{\alpha-2}-(\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)}\\&\geq\frac{(\alpha-1)(\eta-s)^{\alpha-2}-(\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)}=0.\end{aligned}$$

Therefore, $G_2(\eta,s)\geq 0$, for any $\eta,s\in[0,1]$.

The following lemma provides a lower and upper estimates on $G_1(t,s)$ which are very interesting for our study, in order to apply Theorem 2.8.

Lemma 3.4 *For any* $t,s\in[0,1]$*, the following inequalities*

$$\frac{(\alpha-2)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}\leq G_1(t,s)\leq\frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}$$

hold.
Proof. The right inequality is clear from the expression of $G_1(t,s)$.
In order to prove the left inequality, we note that, for $0\leq t\leq s\leq 1$,

$$G_1(t,s)=\frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}\geq\frac{(\alpha-2)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}.$$

For the case $0\leq s\leq t\leq 1$, since

$$t-s\leq t-ts=t(1-s),$$

we have, $(t-s)^{\alpha-1} \leq t^{\alpha-1}(1-s)^{\alpha-1}$ and, as the function $y = a^x$ with $0 < a < 1$ is decreasing and $1 < \alpha - 1$ (because $2 < \alpha \leq 3$), $t^{\alpha-1}(1-s)^{\alpha-1} \leq t(1-s)^{\alpha-1}$. Therefore,

$$\begin{aligned} G_1(t,s) &= \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \geq \frac{(\alpha-1)t(1-s)^{\alpha-2} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\geq \frac{(\alpha-1)t(1-s)^{\alpha-2} - t(1-s)^{\alpha-2}}{\Gamma(\alpha)} = \frac{(\alpha-2)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}, \end{aligned}$$

where, we have used the fact $(1-s)^{\alpha-1} \leq (1-s)^{\alpha-2}$.
Therefore, in any case, we have

$$G_1(t,s) \geq \frac{(\alpha-2)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}.$$

This completes the proof.

Now, we are ready to present the main result of the paper.

Theorem 3.5 *Suppose the following assumptions:*

(*i*) $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *and* $g : [0,1] \times [0,\infty) \to [0,\infty)$ *are continuous functions. Moreover, there exists* $t_0 \in [0,1]$ *such that* $g(t_0, 0) > 0$.

(*ii*) *For fixed* $t \in [0,1]$, $f(t,x,y)$ *is increasing in* x *and decreasing in* y *and* $g(t,x)$ *is increasing in* x.

(*iii*) $g(t, \lambda x) \geq \lambda g(t,x)$, *for any* $\lambda \in (0,1)$, $t \in [0,1]$ *and* $x \in [0,\infty)$.

(*iv*) *There exists a constant* $\rho \in (0,1)$ *such that, for any* $\lambda \in (0,1)$, $t \in [0,1]$ *and* $x, y \in [0,\infty)$,

$$f(t, \lambda x, \lambda^{-1} y) \geq \lambda^{\rho} f(t,x,y).$$

(*v*) *There exists a constant* $\delta_0 > 0$ *such that, for any* $t \in [0,1]$ *and* $x, y \in [0,\infty)$,

$$f(t,x,y) \geq \delta_0 g(t,x).$$

(*vi*) $H : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ *and satisfies the following assumptions:*
  - (*a*) $Hu \in P$ *for any* $u \in P$.
  - (*b*) *For* $u, v \in P$, $u \leq v \Rightarrow Hu \leq Hv$.
  - (*c*) *For any* $\lambda \in (0,1)$ *and* $u \in P$, $H(\lambda u) \geq \lambda Hu$.

Then

(*a*) *There exists* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \leq u_0 \leq v_0$$

and, moreover,

$$\begin{aligned} u_0(t) \leq &\int_0^1 G_1(t,s) f(s, u_0(s), (Hv_0)(s))ds \\ &+ \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) f(s, u_0(s), (Hv_0)(s))ds \\ &+ \int_0^1 G_1(t,s) g(s, u_0)ds + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta,s) g(s, u_0(s))ds \end{aligned}$$

and

$$\begin{aligned} v_0(t) \geq & \int_0^1 G_1(t,s) f(s, v_0(s), Hu_0(s)) ds \\ & + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) f(s, v_0(s), (Hu_0)(s)) ds \\ & + \int_0^1 G_1(t,s) g(s, v_0(s)) ds + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) g(s, v_0(s)) ds, \end{aligned}$$

where $h(t) = t$ *for* $t \in [0,1]$.

(b) *Problem (1.1) has a unique positive solution* $x^* \in P_h$ *(by positive solution we mean that* $x^*(t) > 0$ *for any* $t \in (0,1)$*).*

(c) *For any* $x_0, y_0 \in P_h$, *the sequences inductively defined by*

$$\begin{aligned} x_n(t) = & \int_0^1 G_1(t,s) f(s, x_{n-1}(s), Hy_{n-1}(s)) ds \\ & + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) f(s, x_{n-1}(s), Hy_{n-1}(s)) ds \\ & + \int_0^1 G_1(t,s) g(s, x_{n-1}(s)) ds \\ & + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) g(s, x_{n-1}(s)) ds \end{aligned}$$

and

$$\begin{aligned} y_n(t) = & \int_0^1 G_1(t,s) f(s, y_{n-1}(s), Hx_{n-1}(s)) ds \\ & + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) f(s, y_{n-1}(s), Hx_{n-1}(s)) ds \\ & + \int_0^1 G_1(t,s) g(s, y_{n-1}(s)) ds \\ & + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) g(s, y_{n-1}(s)) ds \end{aligned}$$

for $n = 1, 2, \cdots$ *satisfy*

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - x^*\| = 0.$$

Proof. Taking into account Lemma 3.1, Problem (1.1) has the following formulation integral

$$\begin{aligned} u(t) = & \int_0^1 G_1(t,s)[f(s, u(s), Hu(s)) + g(s, u(s))] ds \\ & + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s)[f(s, u(s), (Hu)(s)) + g(s, u(s))] ds, \qquad t \in [0,1]. \end{aligned}$$

Now, if we consider the operator A defined on $P \times P$ by

$$A(u,v)(t) = \int_0^1 G_1(t,s) f(s,u(s),(Hu)(s))ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s) f(s,u(s),(Hu)(s))ds,$$

and the operator B defined on P by

$$(Bu)(t) = \int_0^1 G_1(t,s) g(s,u(s))ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s) g(s,u(s))ds,$$

then a solution u to Problem (1.1) is equivalent to find a function u satisfying

$$u = A(u,u) + Bu.$$

Notice that by assumption (i) and Lemma 3.3, we infer that $A : P \times P \to P$ and $B : P \to P$.
Next, we check that the conditions appearing in Theorem 2.8 are satisfied.
By assumption (ii), it is easily proved that A is a mixed monotone operator and B is increasing.
From assumption $(vi-(c))$, we notice that, for $\lambda \in (0,1)$ and $u \in P$,

$$H(u) = H(\lambda\lambda^{-1}u) \geq \lambda H(\lambda^{-1}u),$$

and, consequently, $H(\lambda^{-1}u) \leq \lambda^{-1}H(u)$.
Now, we will prove that, for $\lambda \in (0,1)$ and $u, v \in P$ the following inequality which is a condition in Theorem 2.8

$$A(\lambda u, \lambda^{-1}v) \geq \lambda^{\rho} A(u,v)$$

holds, where $\rho \in (0,1)$.
Taking into account the last inequality and assumption (ii) and (iv), for $\lambda \in (0,1)$, $u, v \in P$ and $t \in [0,1]$, we have

$$\begin{aligned}
A(\lambda u, \lambda^{-1}v)(t) &= \int_0^1 G_1(t,s) f(s,\lambda u(s), H(\lambda^{-1}v)(s))ds \\
&+ \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s) f(s,\lambda u(s), H(\lambda^{-1}v)(s))ds \\
&\geq \int_0^1 G_1(t,s) f(s,\lambda u(s), \lambda^{-1}H(v)(s))ds \\
&+ \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s) f(s,\lambda u(s), \lambda^{-1}Hv(s))ds \\
&\geq \lambda^{\rho}\int_0^1 G_1(t,s) f(s,u(s),(Hv)(s))ds \\
&+ \lambda^{\rho}\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s) f(s,u(s),(Hv)(s))ds \\
&= \lambda^{\rho} A(u,v)(t).
\end{aligned}$$

In the sequel, we will prove that B is a subhomogeneous operator.
In fact, for $\lambda \in (0,1)$, $u \in P$ and $t \in [0,1]$, we have

$$\begin{aligned} B(\lambda u)(t) &= \int_0^1 G_1(t,s)g(s,\lambda u(s))ds \\ &\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,\lambda u(s))ds \\ &\geq \lambda \int_0^1 G_1(t,s)g(s,u(s))ds \\ &\quad + \lambda\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,u(s))ds \\ &= \lambda(Bu)(t), \end{aligned}$$

where we have used assumption (iii). This proves the subhomogeneity of B.
Now, we consider the function defined by $h(t) = t$ for $t \in [0,1]$.
Since $0 \leq h(t) \leq 1$ for any $t \in [0,1]$, $h \in P$ and $\theta_E < h$.
Moreover, by assumption (vi), $0 \leq Hh \leq H1$.
Taking into account these facts, Lemma 3.4 and assumption (ii), for any $t \in [0,1]$, we infer

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G_1(t,s)f(s,h(s),Hh(s))ds \qquad (3.1) \\ &\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,h(s),Hh(s))ds \\ &\leq \frac{(\alpha-1)t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}f(s,h(s),Hh(s))ds \\ &\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,h(s),Hh(s))ds \\ &\leq \frac{(\alpha-1)t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}f(s,1,0)ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,1,0)ds \\ &= t\left[\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}f(s,1,0)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,1,0)ds\right] \\ &= h(t)\left[\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}f(s,1,0)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,1,0)ds\right]. \end{aligned}$$

Using a similar argument and taking into account Lemma 3.4, we get

$$\begin{aligned} A(h,h)(t) \geq h(t) &\left[\frac{(\alpha-2)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}f(s,0,H(1))ds\right. \\ &\left. + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,0,H(1))ds\right] \qquad (3.2) \end{aligned}$$

Put

$$\alpha_1=\frac{\alpha-2}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}f(s,0,(H1)(s))ds+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,0,(H1)(s))ds$$

and

$$\alpha_2=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}f(s,1,0)ds+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,1,0)ds.$$

Then, from (3.1) and (3.2), it follows

$$\alpha_1 h\le A(h,h)\le\alpha_2 h.$$

In order to prove that $A(h,h)\in P_h$, we need that $\alpha_i>0$ for $i=1,2$. For the proof of this fact, it is sufficient to prove that $\alpha_1>0$ since $\alpha_1\le\alpha_2$. Taking into account assumption (v), it follows

$$\begin{aligned}\alpha_1&=\frac{(\alpha-2)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}f(s,0,(H1)(s))ds\\&\quad+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,0,(H1)(s))ds\\&\ge\frac{(\alpha-2)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}\delta_0 g(s,0)ds+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\delta_0 g(s,0)ds.\end{aligned}$$

Now, by assumption (i), since $g(t_0,0)>0$ for certain $t_0\in[0,1]$ and the fact that g is a continuous function, we can find a subset $E\subset[0,1]$ such that $t_0\in E$, $\mu(E)>0$, where μ denotes the Lebesgue measure and $g(t,0)>0$ for any $t\in E$. From this, we deduce

$$\alpha_1\ge\frac{(\alpha-2)}{\Gamma(\alpha)}\int_E(1-s)^{\alpha-2}\delta_0 g(s,0)ds+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\delta_0 g(s,0)ds>0.$$

Therefore, $A(h,h)\in P_h$.

Next, we will prove that $Bh\in P_h$.

By Lemma 3.4 and assumption (ii), for any $t\in[0,1]$, we have

$$\begin{aligned}(Bh)(t)&=\int_0^1 G_1(t,s)g(s,h(s))ds+\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,h(s))ds\qquad(3.3)\\&\le\frac{(\alpha-1)t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}g(s,h(s))ds+\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,h(s))ds\\&\le\frac{(\alpha-1)}{\Gamma(\alpha)}t\int_0^1(1-s)^{\alpha-2}g(s,1)ds+\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,1)ds\\&=t\left[\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}g(s,1)ds+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,1))ds\right]\\&=h(t)\left[\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}g(s,1)ds+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,1)ds\right].\end{aligned}$$

Using a similar argument, we obtain

$$(Bh)(t) \geq h(t)\left[\frac{(\alpha-2)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}g(s,0)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,0)ds\right] \quad (3.4)$$

Putting

$$\beta_1 = \frac{\alpha-2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}g(s,0)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,0)ds$$

and

$$\beta_2 = \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}g(s,1)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,1)ds,$$

from (3.3) and (3.4),

$$\beta_1 h \leq Bh \leq \beta_2 h.$$

In order to prove that $Bh \in P_h$, it is sufficient to see that $\beta_1 > 0$. By a similar argument to the one used above, we deduce

$$\begin{aligned}\beta_1 &= \frac{\alpha-2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}g(s,0)ds + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)g(s,0)ds \\ &\geq \frac{\alpha-2}{\Gamma(\alpha)}\int_E (1-s)^{\alpha-2}g(s,0)ds + \frac{\gamma}{1-\gamma}\int_E G_2(\eta,s)g(s,0)ds > 0.\end{aligned}$$

Therefore, $Bh \in P_h$.
Finally, we have to check that assumption (ii) of Theorem 2.8 is satisfied. In fact, by assumption (v), for $u, v \in P$ and $t \in [0,1]$, we have

$$\begin{aligned}A(u,v)(t) &= \int_0^1 G_1(t,s)f(s,u(s),(Hv)(s))ds \\ &\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)f(s,u(s),((Hv)(s)))ds \\ &\geq \int_0^1 G_1(t,s)\delta_0 g(s,u(s))ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\delta_0 g(s,u(s))ds \\ &= \delta_0(Bu)(t).\end{aligned}$$

Now, by applying Theorem 2.8, we obtain the desired result. Notice that the solution x^* is positive since $x^* \in P_h$ and $0 < h(t) = t$ for $t \in (0,1)$.

In what follows, we present some examples of operator $H : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ which satisfy assumption (vi) of Theorem 3.5.

4. Examples

(a) Suppose that $\varphi : [0.1] \to [0,1]$ is a continuous function and consider the composition operator C_φ defined on $\mathcal{C}[0,1]$ by

$$(C_\varphi x)(t) = x(\varphi(t))$$

for any $x \in \mathcal{C}[0,1]$ and $t \in [0,1]$. The operator C_φ satisfies assumption (vi) of Theorem 3.5.

(b) Suppose that $\varphi : [0,1] \to [0,\infty)$ is a continuous function. Consider the multiplication operator M_φ defined by

$$(M_\varphi x)(t) = \varphi(t)x(t),$$

for any $x \in \mathcal{C}[0,1]$ and $t \in [0,1]$. It is easily proved that M_φ satisfies assumption (vi) of Theorem 3.5.

(c) Consider the integral operator $I : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ given by

$$(Ix)(t) = \int_0^t x(s)ds,$$

for $x \in \mathcal{C}[0,1]$ and $t \in [0,1]$. This operator satisfies also assumption (vi) of Theorem 3.5.

In the above mentioned examples, the operators are linear.

In the following two examples, the operators defined on $\mathcal{C}[0,1]$ satisfy assumption (vi) of Theorem 3.5 and they are nonlinear.

(d) In [2], the authors proved that the operator Q defined on $\mathcal{C}[0,1]$ by

$$(Qx)(t) = \max\{|x(\tau)| : 0 \le \tau \le t\}$$

applies $\mathcal{C}[0,1]$ into itself. It can be easily seen that Q satisfies our assumption.

(e) Consider the operator F defined on $\mathcal{C}[0,1]$ by

$$(Fx)(t) = x(t)^r, \quad \text{where} \quad r \in (0,1).$$

It is easily checked that F satisfies assumption (vi) of Theorem 3.5.

Notice that the operators in Examples (a), (b), (c) and (d) satisfy the condition $H(\lambda u) = \lambda Hu$ for any $u \in P$ and $\lambda \in (0,1)$ which is a stronger condition that the condition (c) of assumpiont (vi) of Theorem 3.5. While the operator F defined in Example (e) satisfies $H(\lambda u) > \lambda Hu$ for any $u \in P - \{\theta_E\}$ and $\lambda \in (0,1)$.

On the other hand, it is easily checked that the composition of operators satisfying assumption (vi) of Theorem 3.5 also satisfies this condition.

Now we present some numerical examples where we can apply our result.

Example 4.1 Consider the following nonlinear boundary value problem with maximum

$$\begin{cases} {}^C D_{0^+}^{5/2} u(t) = 3 + t^2 + t^3 + 2\sqrt[3]{u(t)} + \dfrac{1}{\sqrt[4]{\max\limits_{0\le\tau\le t} u(\tau) + 5}} = 0, \\ u(0) = u''(0) = 0, \qquad\qquad u'(1) = \frac{1}{7}u'(\frac{1}{2}). \end{cases} \tag{4.1}$$

Notice that Problem (4.1) is a particular case of Problem (1.1) with $\alpha = \frac{5}{2}$, $\gamma = \frac{1}{7}$, $\eta = \frac{1}{2}$,

$$f(t,u,v) = 3 + t^2 + \sqrt[3]{u} + \frac{1}{\sqrt[4]{v+5}},$$

$$g(t,u) = t^3 + \sqrt[3]{u}$$

and $(Hu)(t) = \max\limits_{0\le\tau\le t} u(\tau)$.

It is clear that f applies $[0,1] \times [0,\infty) \times [0,\infty)$ into $[0,\infty)$ and g applies $[0,1] \times [0,\infty)$ into $[0,\infty)$ and that the functions f and g are continuous and $g(1/2,0) > 0$.

This says us that assumption (i) of Theorem 3.5 is satisfied.
Moreover, it is also clear that f is increasing in u and decreasing in v and g is increasing in u. Therefore, assumption (ii) of Theorem 3.5 is also satisfied.
Notice that, for $t \in [0,1]$, $u \geq 0$ and $\lambda \in (0,1)$, we have

$$g(t, \lambda u) = t^3 + \sqrt[3]{\lambda u} > \sqrt[3]{\lambda}(t^3 + \sqrt[3]{u}) > \lambda(t^3 + \sqrt[3]{u}) = \lambda g(t,u),$$

where we have used the decreasing character of the function $y = \lambda^x$ with $\lambda \in (0,1)$. This means that assumption (iii) of Theorem 3.5 is satisfied.
Now, we take $u, v \geq 0$ and $\lambda \in (0,1)$ and we infer

$$\begin{aligned} f(t, \lambda u, \lambda^{-1} v) &= 3 + t^2 + \sqrt[3]{\lambda u} + \frac{1}{\sqrt[4]{\lambda^{-1} v + 5}} \\ &> 3\sqrt[3]{\lambda} + \sqrt[3]{\lambda} t^2 + \sqrt[3]{\lambda}\sqrt[3]{u} + \frac{\sqrt[3]{\lambda}}{\sqrt[4]{v + 5\lambda}} \\ &> \sqrt[3]{\lambda}(3 + t^2 + \sqrt[3]{u} + \frac{1}{\sqrt[4]{v+5}}) \\ &= \sqrt[3]{\lambda} f(t,u,v), \end{aligned}$$

where we have used that $\sqrt[4]{\lambda} > \sqrt[3]{\lambda}$ and $\dfrac{1}{\sqrt[4]{v+5\lambda}} > \dfrac{1}{\sqrt[4]{v+5}}$.
This proves that assumption (iv) of Theorem 3.5 is satisfied with $\rho = \frac{1}{3}$. Moreover, for $t \in [0,1]$ and $u, v \geq 0$, we have

$$f(t,u,v) = 3 + t^2 + \sqrt[3]{u} + \frac{1}{\sqrt[4]{v+5}} > t^3 + \sqrt[3]{u} = g(t,u)$$

and, therefore, assumption (v) of Theorem 3.5 is satisfied with $\delta_0 = 1$.
Since that the operator H is the one appearing in Example (e), it satisfies assumption (vi) of Theorem 3.5.
Finally, by this theorem, Problem (4.1) has a unique positive solution $u^* \in \mathcal{C}[0,1]$ with $u^* \in P_h$, where $h(t) = t$ for $t \in [0,1]$.

Notice that if we replace in Example 4.1 the operator H by another of the operators defined in Examples, then the same argument is valid in order to obtain the desired result. This makes that Theorem 3.5 is interesting and useful from a practical point of view.

Example 4.2 Consider the following nonlinear boundary value problem

$$\left\{ \begin{array}{ll} {}^C D_{0^+}^{9/4} u(t) = 5 + t^3 + 2u(t)^a + \dfrac{1}{1 + (Hu)(t)^b} = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u'(1) = \dfrac{1}{3} u'(\dfrac{1}{\sqrt{2}}), & \end{array} \right\} \tag{4.2}$$

where H is one of the operators given in Examples and $a, b \in (0.1)$.
Notice that Problem (4.2) is a particular case of Problem (1.1) with $\alpha = 9/4$, $\gamma = 1/3$, $\eta = 1/\sqrt{2}$,

$$f(t,u,v) = 5 + u^a + \frac{1}{1 + v^b}$$

and

$$g(t,u) = t^3 + u^a.$$

It is clear that f applies $[0,1]\times[0,\infty)\times[0,\infty)$ into $[0,\infty)$ and it is continuous and that g applies $[0,1]\times[0,\infty)$ into $[0,\infty)$ and it is also continuous. Moreover, $g(1/4,0) > 0$. Moreover, it is clear that f is increasing in u and decreasing in v and that g is increasing in u.

Notice that, for $t \in [0,1]$, $u \geq 0$ and $\lambda \in (0,1)$, we have

$$g(t,\lambda u) = t^3 + \lambda^a u^a > \lambda^a(t^3 + u^a) > \lambda(t^3 + u^a),$$

where we have used the fact that $a \in (0,1)$.

Now, for $u, v \geq 0$ and $\lambda \in (0,1)$, we have

$$\begin{aligned} f(t,\lambda u,\lambda^{-1}v) &= 5 + \lambda^a u^a + \frac{1}{1+\lambda^{-b}v^b} \\ &> 5 + \lambda^{\max(a,b)}u^a + \frac{\lambda^{\max(a,b)}}{\lambda^b + v^b} \\ &> \lambda^{\max(a,b)}[5 + u^a + \frac{1}{1+v^b}] \\ &= \lambda^{\max(a,b)} f(t,u,v). \end{aligned}$$

Therefore, assumption (iv) of Theorem 3.5 is satisfied with $\rho = \max(a,b)$.

For $t \in [0,1]$ and $u, v \geq 0$ we have

$$f(t,u,v) = 5 + u^a + \frac{1}{1+v^b} > u^a + 5 > u^a + t^3 = g(t,u).$$

This says us that assumption (v) of Theorem 3.5 is satisfied with $\delta_0 = 1$.

For $t \in [0,1]$, $u \geq 0$ and $\lambda \in (0,1)$, we infer

$$g(t,\lambda u) = t^3 + \lambda^a u^a > \lambda^a(t^3 + u^a) = \lambda^a g(t,u) > \lambda g(t,u),$$

where we have used that $\lambda^a > \lambda$ for $\lambda, a \in (0,1)$.

Since assumptions of Theorem 3.5 are satisfied, it follows that Problem (4.2) has a unique positive solution $u^* \in \mathcal{C}[0,1]$ with $u^* \in P_h$, where $h(t) = t$ for $t \in [0,1]$.

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