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REGULARIZED PROJECTION METHOD OF SOLVING SPLIT SYSTEM OF FIXED POINT SET CONSTRAINT EQUILIBRIUM PROBLEMS IN REAL HILBERT SPACE

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Abstract. In this paper, we propose two algorithms which combines Mann iterative scheme, regularization technique and projection method for solving finite family of split equilibrium problems and split common fixed point problems: we call the problems split system of fixed point set constraint equilibrium problems (SSFPSCEPs). The weak and strong convergence theorems for iterative sequences generated by the algorithms are established under widely used assumptions for equilibrium bifunctions. To obtain the strong convergence, we combine the first algorithm with the shrinking projection method in the second algorithm. Finally, an application and one numerical experiment is given to demonstrate the efficiency of our algorithms.
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1. Introduction

Throughout the paper, unless otherwise stated, we assume that H_1 and H_2 are real Hilbert spaces, $\mathbb{R}$ denote the set of all real numbers and for a subset E of a Hilbert space, Id_E denotes the mapping from E onto E defined by $Id_E(x) = x, \ \forall x \in E$.

Suppose C and D be nonempty closed convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a nonzero bounded linear operator. Let $U_{i'} : C \to C$ and $V_{j'} : D \to D$ be a nonexpansive mappings for $i' \in I' = \{1, \ldots, N'\}$, $j' \in J' = \{1, \ldots, M'\}$. Given bifunctions $f_i : C \times C \to \mathbb{R}$ and $g_j : D \times D \to \mathbb{R}$ for $i \in I = \{1, \ldots, N\}$, $j \in J = \{1, \ldots, M\}$. The problem under consideration in this paper is *split system of* *fixed point set constraint equilibrium problem* (SSFPSCEP) given by:

$$\text{find } x^* \in C \text{ such that } \begin{cases} x^* \in FixU_{i'}, \quad \forall i' \in I', \\ f_i(x^*, y) \geq 0, \ \forall y \in C, \quad \forall i \in I, \\ u^* = Ax^* \in D, \quad u^* \in FixV_{j'}, \ \forall j' \in J', \\ g_j(u^*, u) \geq 0, \ \forall u \in D, \quad \forall j \in J. \end{cases} \tag{1.1}$$

If $H_1 = H_2$, $C = D$, $I = \{1\}$, $U_{i'} = Id_C$ for all $i' \in I'$, $V_{j'} = Id_D$ for all $j' \in J'$ and $g_j = 0$ for all $j \in J$, then SSFPSCEP (1.1) is well known as the equilibrium problem (Fan inequality [13]) for the bifunction f_1 on C, denoted by EP(f_1, C). The set of all solutions of EP(f_1, C) is denoted by SEP(f_1, C), i.e., SEP$(f_1, C) = \{z^* \in C : f_1(z^*, z) \geq 0, \ \forall z \in C\}$. Note that, for a closed convex subset C of a Hilbert space a mapping $U : C \to C$ is said to be nonexpansive if $\|U(x) - U(y)\| \leq \|x - y\|, \ \forall x, y \in C$. The set of fixed points of U is denoted by $FixU$ and is given by $FixU = \{x \in C : Ux = x\}$. For many years, equilibrium problems and fixed point problems become an attractive field for many researchers both theory and applications in electricity market, transportation, economics and network; for example, see, [4, 17, 22]. Due to importance of the solutions of such problems, many approaches of researchers are devoted in the study of this area in its more general and particular cases.

SSFPSCEP (1.1) is of the form Split Inverse Problem (SIP) stated in [2] which is more general form of split feasibility problem introduced by Censor and Elfving [1]. Mathematically, the SIP is stated as follows:

$$\begin{cases} \text{find } x^* \in X \text{ that solves IP1} \\ \text{such that} \\ y^* = Ax^* \in Y \text{ and solves IP2} \end{cases} \tag{1.2}$$

where A is a bounded linear operator from a space X to another space Y and IP1 and IP2 are two inverse problems installed in X and Y, respectively. In this framework, many authors studied when IP1 and IP2 being equilibrium problems in a Hilbertian framework, and so, called the split equilibrium problem (SEP) and split variational inequality problem (SVIP), see eg [2, 5, 6, 8, 10, 12, 15, 21, 19, 20]. Our problem SSFPSCEP (1.1) is SIP where IP1 and IP2 are system of fixed point set constraint equilibrium problems installed in real Hilbert spaces H_1 and H_2, respectively. Note that in SSFPSCEP (1.1) if $f_i(x, y) = \langle A_i(x), y - x\rangle$ and $g_j(x, y) = \langle B_j(u), v - u\rangle$ where $A_i : C \to H_1$ and $B_j : D \to H_2$, then SSFPSCEP (1.1) becomes the more general form of classical SVIP.

Many researchers have been proposed algorithms for finding solution of the problem (1.1) when $I = J = \{1\}$ or $U_{i'} = Id_C$ for all $i' \in I'$, $V_{j'} = Id_D$ for all $j' \in J'$, see, for example, [8, 10, 11, 21, 19, 20] and the references therein. If $U_{i'} = Id_C$ for all $i' \in I'$, $V_{j'} = Id_D$ for all $j' \in J'$, then SSFPSCEP (1.1) will be reduced to split system of equilibrium problem (SSEP) studied in [11]. In many research works on SEP as well as SSEP the condition imposed and approach used for each bifunction g_j defined for inverse problem IP2 installed in H_2 is the same. This approach is regularization technique (regularization equilibrium problem or resolvent operator) $T_r^{g_j} : H \to D$, defined by

$$z \mapsto \{v \in D : g_j(v,y) + \frac{1}{r}\langle y-v, v-z\rangle \geq 0,\ \forall y \in D\},\quad j \in J \tag{1.3}$$

where r is a suitable parameter, see, for example, [21, 10, 8, 19, 20, 11]. Note that the problem (1.3) is strongly monotone when the bifunction g_j is monotone. Thus, its solution exists and is unique under certain assumption of the continuity of the bifunction g_j. With regard to the bifunctions f_i defined for inverse problem IP1 installed in H_1, recently in [11] proximity operator approach is used for solving SSEP, i.e., for convex and lower semicontinuous function $h : C \to \mathbb{R}$ and $\lambda > 0$, the proximity operator of h is a single-valued operator $prox_h : C \to C$ defined by

$$x \mapsto \arg\min\{\lambda h(y) + \frac{1}{2}\|x-y\|^2 : y \in C\}. \tag{1.4}$$

Using the proximity operator (1.4), the author in [11] obtained weak and strong convergence results for solving SSEPs, structured based on extragradient method by solving two strongly convex programs:

$$\begin{cases} y_i^k = \arg\min\{\lambda_k f_i(x^k,y) + \frac{1}{2}\|x^k-y\|^2 : y \in C\},\quad i \in I, \\ z_i^k = \arg\min\{\lambda_k f_i(y_i^k,y) + \frac{1}{2}\|y_i^k-y\|^2 : y \in C\},\quad i \in I, \end{cases}$$

where λ_k is a suitable parameter and each f_i satisfy a certain Lipschitz-type condition. In practice, $prox_h$ can be computed easily by the Matlab Optimization Toolbox. As a result of this, two optimization programs in the extragradient method are easily numerically solved at each iteration. However, this might be costly and affects the efficiency of the used method if the structure of feasible set and equilibrium bifunction are complex. Moreover, Lipschitz-type condition depends on two positive parameters c_1 and c_2 which in some cases, they are unknown or difficult to approximate.

Inspired and motivated by the results in [10, 11, 14], we propose two algorithms for solving SSFPSCEP (1.1) using the well-known Mann iterative scheme for fixed point [14], projection method and regularization equilibrium problem (1.3) by imposing the same conditions on each f_i and g_j for all i and j. However, regularization equilibrium problem (1.3) is not easier in computation and if each bifunction is more general monotone, for instance, pseudomonotone then problem (1.3) in general is not strongly monotone. Despite this, our result has advantage as it considers wide class of problems SSFPSCEPs which more general form of SSEPs without Lipschitz-type condition of f_i.

The remainders of the paper is organized as follows: Section 2 reviews some preliminaries. Section 3 gives two algorithms and proves for the weak and strong convergence for the algorithms solving (1.1). In Section 4 we will present some applications and numerical examples. Finally, we give some conclusions.

2. Preliminary

We write $x_k \rightharpoonup x$ to indicate that the sequence $\{x_k\}$ converges weakly to x as $k \to \infty$, and $x_k \to x$ means that $\{x_k\}$ converges strongly to x. The metric projection on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y-x\| : y \in C\},\quad x \in H.$$

Lemma 2.1 [18] *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Definition 2.1 Let C be a subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ be a bifunction. Then, f is said to be

(1) *monotone* on C iff

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C,$$

(2) *pseudomonotone* on C with respect to $x \in C$ iff

$$f(x, y) \geq 0 \ \text{ implies } \ f(y, x) \leq 0, \ \forall y \in C.$$

It is clear that monotone bifunction is pseudomonotone.

Definition 2.2 Let C be a subset of a Hilbert space H. A mapping $A : C \to H$ is said to be

(1) monotone on C if

$$\langle A(x) - A(y), x - y\rangle \geq 0, \quad \forall x, y \in C,$$

(2) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq \alpha\|A(x) - A(y)\|^2, \quad \forall x, y \in C.$$

Lemma 2.3 *For a real Hilbert space H, we have*

(i) $\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2, \quad \forall x, y \in H.$
(ii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \quad \forall x, y \in H.$
(iii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \ \forall x, y \in H, \ \forall \lambda \in [0, 1].$

The next lemma will is very useful to prove the convergence of our algorithms to a solution.

Lemma 2.4 (Opials condition) *For any sequence $\{x^k\}$ in the Hilbert space H with $x^k \rightharpoonup x$, the inequality*

$$\liminf_{k\to+\infty} \|x^k - x\| < \liminf_{k\to+\infty} \|x^k - y\|$$

holds for each $y \in H$ with $y \neq x$.

Lemma 2.5 [9] *Suppose C is closed convex subset of a Hilbert space H and $U : C \to C$ be nonexpansive mapping. Then,*

(i) *If U has a fixed point, then $FixU$ is a closed convex subset of H.*
(ii) *$Id_C - U$ is demiclosed, i.e., whenever $\{x^k\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(Id_C - U)x^k\}$ strongly converges to some y, it follows that $(Id_C - U)x = y$.*

Lemma 2.6 [7] *Suppose C is closed convex subset of a Hilbert space H and $U_i : C \to C$ be nonexpansive mappings for each $i \in \{1, \dots, q\}$ such that $\bigcap_{i=1}^{q} FixU_i \neq \emptyset$. Let*

$U(x) := \sum_{i=1}^{q} \theta_i U_i(x)$ *with* $0 < \theta_i \leq 1$ *for every* $i \in \{1, \ldots, q\}$ *and* $\sum_{i=1}^{q} \theta_i = 1$. *Then* U *is nonexpansive and* $FixU = \bigcap_{i=1}^{q} FixU_i$.

Let D be a subset of a real Hilbert space H and $g : D \times D \to \mathbb{R}$ be a bifunction. Then we say that g satisfy Condition A on D if the following assumptions are satisfied;

(A1) $g(u, u) = 0$, for all $u \in D$;
(A2) g is monotone on D, i.e., $g(u, v) + g(v, u) \leq 0$, for all $u, v \in D$;
(A3) for each $u, v, w \in D$,

$$\limsup_{\alpha \downarrow 0} g(\alpha w + (1 - \alpha)u, v) \leq g(u, v);$$

(A4) $g(u, .)$ is convex and lower semicontinuous on D for each $u \in D$.

From [3], we have the following useful results.

Lemma 2.7 [3, Lemma 2.12] *Let* g *satisfies Condition* A *on* D. *Then, for each* $r > 0$ *and* $u \in H_2$, *there exists* $w \in D$ *such that*

$$g(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0, \ \forall v \in D.$$

Lemma 2.8 [3, Lemma 2.12] *Let* g *satisfies Condition* A *on* D. *Then, for each* $r > 0$ *and* $u \in H_2$, *define a mapping (called resolvant of* g*), given by*

$$T_r^g(u) = \{w \in D : g(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0, \ \forall v \in D\}.$$

Then, the followings holds:

(i) T_r^g *is single-valued;*
(ii) T_r^g *is a firmly nonexpansive, i.e., for all* $u, v \in H$,

$$\|T_r^g(u) - T_r^g(v)\|^2 \leq \langle T_r^g(u) - T_r^g(v), u - v\rangle;$$

(iii) *Fix*$(T_r^g) = SEP(g, D)$, *where Fix*(T_r^g) *is the fixed point set of* T_r^g*;*
(iv) $SEP(g, D)$ *is closed and convex.*

Lemma 2.9 [3, Lemma 2.12] *For* $r, s > 0$ *and* $u, v \in H_2$. *Under the assumptions of Lemma 2, then* $\|T_r^g(u) - T_s^g(v)\| \leq \|u - v\| + \frac{|s-r|}{s}\|T_s^g(v) - v\|$.

3. Main Result

Let Ω be the solution set of SSFPSCEP (1.1) and let

$$\Omega_1 = \Big[\bigcap_{i'=1}^{N'} FixU_{i'}\Big] \cap \Big[\bigcap_{i=1}^{N} SEP(f_i, C)\Big]$$

and

$$\Omega_2 = \Big[\bigcap_{j'=1}^{M'} FixV_{j'}\Big] \cap \Big[\bigcap_{j=1}^{M} SEP(g_j, D)\Big].$$

Therefore, $\Omega = \{x^* \in \Omega_1 : Ax^* \in \Omega_2\}$. In this section, we propose two algorithms for solving (1.1) and analyse the convergence of the iteration sequences generated by the algorithms by assuming that the solution set Ω is nonempty and that each f_i and g_j defined in (1.1) satisfy Condition A on C and D, respectively. In order to design the algorithms, we consider the real parameter sequences $\{r_k\}$, $\{\delta_k\}$, $\{\mu_k\}$, $\{\xi_i^k\}$ $(i \in I)$, $\{\theta_{j'}^k\}$ $(j' \in J')$ satisfying the following four conditions.

Condition 1

(C1) $0 < \sigma_1 \le \delta_k \le \sigma_2 < 1$.
(C2) $r_k \ge r > 0$, $0 < \gamma_1 \le \mu_k \le \gamma_2 < \frac{1}{\sigma^2}$ for some $\sigma \in [\|A\|, +\infty)$.
(C3) $0 < \xi \le \xi_i^k \le 1$, $(i \in I)$ such that $\sum_{i=1}^{N} \xi_i^k = 1$.
(C4) $0 < \theta \le \theta_{j'}^k \le 1$, $(j' \in J')$ such that $\sum_{j'=1}^{M'} \theta_{j'}^k = 1$.

In the formulation of the following algorithms we need a real number σ such that either $\sigma = \|A\|$ or at least $\sigma > \|A\|$ so that we can determine the nature of the sequence $\{\mu_k\}$. Hence, our algorithms require prior knowledge or at least estimated value of operator norm $\|A\|$.

3.1. **Weak convergence.** We obtained the weak convergence result for solving SSF-PSCEP (1.1) using Mann iterative scheme for fixed point [14] and two methods regularization technique and projection method.

Algorithm 3.1
Initialization: Choose $x^0 \in C$. Let $\{r_k\}$, $\{\delta_k\}$, $\{\mu_k\}$, $\{\xi_i^k\}$ $(i \in I)$, $\{\theta_{j'}^k\}$ $(j' \in J')$ be real sequences satisfying **Condition 1**.

Step 1: For each $i \in I$ find $y_i^k = T_{r_k}^{f_i}(x^k)$.
Step 2: Evaluate $y^k = \sum_{i=1}^{N} \xi_i^k y_i^k$.
Step 3: For each $i' \in I'$ find $t_{i'}^k = \delta_k y^k + (1 - \delta_k) U_{i'}(y^k)$.
Step 4: Find among $t_{i'}^k$, $i' \in I'$, the farthest element from x^k, i.e.,

$$t^k = \arg\max\{\|v - x^k\| : v \in \{t_{i'}^k : i' \in I'\}\}.$$

Step 5: For each $j \in J$ find $u_j^k = T_{r_k}^{g_j}(At^k)$.
Step 6: Find among u_j^k, $j \in J$, the farthest element from At^k, i.e.,

$$u^k = \arg\max\{\|v - At^k\| : v \in \{u_j^k : j \in J\}\}.$$

Step 7: Evaluate $x^{k+1} = P_C\Big(t^k + \mu_k A^*\Big(\sum_{j'=1}^{M'} \theta_{j'}^k V_{j'}(u^k) - At^k\Big)\Big)$.
Step 8: Set $k := k + 1$ and go to Step 1.

Remark 1. Since each f_i and g_j satisfy Condition A on C and D, respectively, by Combettes and Hirstoaga in [3], for each r_k the problems in Step 1 and Step 5 are uniquely solvable, and since C is nonempty closed convex set, the projection in Step

7 exists and is unique. Moreover, each steps in Algorithm 3.1 are defined with no ambiguity. Hence, Algorithm 3.1 is well defined.

Here is our main Theorem for weak convergence of Algorithm 3.1.

Theorem 3.1 *Let $\{y^k\}$, $\{t^k\}$, $\{u^k\}$ and $\{x^k\}$ be sequences generated by Algorithm 3.1. Then, $\{y^k\}$, $\{t^k\}$ and $\{x^k\}$ converge weakly to a point $p \in \Omega$ and $\{u^k\}$ converges weakly to a point $Ap \in \Omega_2$.*

Proof. Let $x^* \in \Omega$. Then, using Lemma 2.8 (ii) and (iii) and Lemma 2.3 (i), we get

$$\begin{aligned}\|y_i^k - Ax^*\|^2 &= \|T_{r_k}^{f_i} x^k - x^*\|^2 = \|T_{r_k}^{f_i} x^k - T_{r_k}^{f_i} x^*\|^2\\ &\le \langle T_{r_k}^{f_i} x^k - T_{r_k}^{f_i} x^*, x^k - x^*\rangle = \langle T_{r_k}^{f_i} x^k - x^*, x^k - x^*\rangle\\ &= \tfrac{1}{2}(\|T_{r_k}^{f_i} x^k - x^*\|^2 + \|x^k - x^*\|^2 - \|T_{r_k}^{f_i} x^k - x^k\|^2)\\ &= \tfrac{1}{2}(\|y_i^k - x^*\|^2 + \|x^k - x^*\|^2 - \|y_i^k - x^k\|^2).\end{aligned}$$

This yields,

$$\|y_i^k - x^*\|^2 \le \|x^k - x^*\|^2 - \|y_i^k - x^k\|^2. \tag{3.1}$$

Using definition of $\{y^k\}$ and using convexity $\|.\|^2$, we have

$$\|y^k - x^*\|^2 \le \sum_{i=1}^{N} \xi_i^k \|y_i^k - x^*\|^2. \tag{3.2}$$

From (3.1) and (3.2) above, we have

$$\|y^k - x^*\|^2 \le \|x^k - x^*\|^2 - \sum_{i=1}^{N} \xi_i^k \|y_i^k - x^k\|^2. \tag{3.3}$$

Suppose $i'_{n'} \in I'$ such that

$$t_{i'_{n'}}^k = t^k = \arg\max\{\|v - x^k\| : v \in \{t_{i'}^k : i' \in I'\}\}.$$

Then, by definition of t^k and using Lemma 2.3 (iii) together with (3.3), we have

$$\begin{aligned}&\|t^k - x^*\|^2\\ &= \|\delta_k(y^k - x^*) + (1-\delta_k)(U_{i'_{n'}}(y^k) - x^*)\|^2\\ &= \delta_k\|y^k - x^*\|^2 + (1-\delta_k)\|U_{i'_{n'}}(y^k) - x^*\|^2 - \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2\\ &= \delta_k\|y^k - x^*\|^2 + (1-\delta_k)\|U_{i'_{n'}}(y^k) - U_{i'_{n'}}(x^*)\|^2 -\\ &\qquad\qquad \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2\\ &\le \delta_k\|y^k - x^*\|^2 + (1-\delta_k)\|y^k - x^*\|^2 - \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2\\ &\le \|y^k - x^*\|^2 - \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2\\ &\le \|x^k - x^*\|^2 - \sum_{i=1}^{N} \xi_i^k \|y_i^k - x^k\|^2 - \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2.\end{aligned} \tag{3.4}$$

Suppose $j_m \in J$ such that $u_{j_m}^k = u^k = \arg\max\{\|u_j^k - At^k\|; j \in J\}$. By Lemma 2.8 (ii) and (iii) and Lemma 2.3 (i), we obtain

$$\begin{aligned}\|u^k - Ax^*\|^2 &= \|T_{r_k}^{g_{j_m}} At^k - Ax^*\|^2 = \|T_{r_k}^{g_{j_m}} At^k - T_{r_k}^{g_{j_m}} Ax^*\|^2\\ &\le \langle T_{r_k}^{g_{j_m}} At^k - T_{r_k}^{g_{j_m}} Ax^*, At^k - Ax^*\rangle\\ &= \langle T_{r_k}^{g_{j_m}} At^k - Ax^*, At^k - Ax^*\rangle\\ &= \tfrac{1}{2}(\|T_{r_k}^{g_{j_m}} At^k - Ax^*\|^2 + \|At^k - Ax^*\|^2 - \|T_{r_k}^{g_{j_m}} At^k - At^k\|^2)\\ &= \tfrac{1}{2}(\|u^k - Ax^*\|^2 + \|At^k - Ax^*\|^2 - \|u^k - At^k\|^2).\end{aligned}$$

This yields,

$$\|u^k - Ax^*\|^2 \leq \|At^k - Ax^*\|^2 - \|u^k - At^k\|^2. \tag{3.5}$$

Let $V = \sum\limits_{j'=1}^{M'} \theta_{j'}^k V_{j'}$. By Lemma 2.6, V is nonexpansive and $FixV = \bigcap_{j'=1}^{M'} FixV_i$. Thus, using (3.5) and the nonexpansive property of V, we have

$$\begin{aligned} \|V(u^k) - Ax^*\|^2 = \|VT_{r_k}^{g_{j_m}} At^k - VAx^*\|^2 \leq \|u^k - Ax^*\|^2 \\ \leq \|At^k - Ax^*\|^2 - \|u^k - At^k\|^2. \end{aligned} \tag{3.6}$$

Moreover, using Lemma 2.3 (i), we obtain

$$\begin{aligned} &\langle A(t^k - x^*), V(u^k) - At^k\rangle \\ &= \langle A(t^k - x^*) + V(u^k) - At^k - V(u^k) + At^k, V(u^k) - At^k\rangle \\ &= \langle V(u^k) - Ax^*, V(u^k) - At^k\rangle - \|V(u^k) - At^k\|^2 \\ &= \tfrac{1}{2}\big(\|V(u^k) - Ax^*\|^2 + \|V(u^k) - At^k\|^2 - \|At^k - Ax^*\|^2\big) - \|V(u^k) - At^k\|^2 \\ &= \tfrac{1}{2}\big(\|V(u^k) - Ax^*\|^2 - \|V(u^k) - At^k\|^2 - \|At^k - Ax^*\|^2\big). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\langle A(t^k - x^*), V(u^k) - At^k\rangle \leq -\tfrac{1}{2}(\|u^k - At^k\|^2 + \|V(u^k) - At^k\|^2). \tag{3.8}$$

Then, by definition of x^{k+1}, Lemma 2.3 (ii) and using (3.7), we get

$$\begin{aligned} &\|x^{k+1} - x^*\|^2 \\ &= \|P_C(t^k + \mu_k A^*(V(u^k) - At^k)) - P_C(x^*)\|^2 \\ &\leq \|(t^k - x^*) + \mu_k(V(u^k) - At^k)\|^2 \\ &= \|t^k - x^*\|^2 + \|\mu_k A^*(V(u^k) - At^k)\|^2 + 2\mu_k\langle t^k - x^*, A^*(V(u^k) - At^k)\rangle \\ &\leq \|t^k - x^*\|^2 + \mu_k^2\|A^*\|^2\|(V(u^k) - At^k)\|^2 + 2\mu_k\langle A(t^k - x^*), V(u^k) - At^k\rangle \\ &\leq \|t^k - x^*\|^2 + \mu_k^2\|A\|^2\|(V(u^k) - At^k)\|^2 - \mu_k(\|u^k - At^k\|^2 + \\ &\qquad \|Vu^k - At^k\|^2) \\ &= \|t^k - x^*\|^2 - \mu_k(1 - \mu_k\|A\|^2)\|V(u^k) - At^k\|^2 - \mu_k\|u^k - At^k\|^2 \\ &= \|t^k - x^*\|^2 - \mu_k(1 - \mu_k\sigma^2)\|V(u^k) - At^k\|^2 - \mu_k\|u^k - At^k\|^2. \end{aligned} \tag{3.9}$$

Thus, from (3.4) and (3.9), we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - L_k \tag{3.10}$$

where

$$L_k = \sum_{i=1}^{N} \xi_i^k \|y_i^k - x^k\|^2 + \delta_k(1 - \delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2 +$$

$$\mu_k(1 - \mu_k\|A\|^2)\|V(u^k) - At^k\|^2 + \mu_k\|u^k - At^k\|^2.$$

In view of condition 1 and results in (3.3), (3.4) and (3.9), we have

$$\|x^{k+1} - x^*\| \leq \|t^k - x^*\| \leq \|y^k - x^*\| \leq \|x^k - x^*\|. \tag{3.11}$$

Therefore, $\lim_{k\to\infty} \|x^k - x^*\|^2$ exists and this implies that the sequence $\{x^k\}$ is bounded. Thus, from (3.10), we have $L_k \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2$ and hence letting $k \to +\infty$ gives $L_k \to 0$. Consequently, from $L_k \to 0$ and Condition 1, we have

$$\begin{aligned}\lim_{k\to\infty} \|y_i^k - x^k\| = \lim_{k\to\infty} \|U_{i'_{n'}}(y^k) - y^k\| &= \lim_{k\to\infty} \|V(u^k) - At^k\| \\ &= \lim_{k\to\infty} \|u^k - At^k\| = 0.\end{aligned} \tag{3.12}$$

Combining results from (3.12), it is easy to see that

$$\lim_{k\to\infty} \|y^k - x^k\| = \lim_{k\to\infty} \|U_{i'_{n'}}(y^k) - x^k\| = \lim_{k\to\infty} \|V(u^k) - u^k\| = 0. \tag{3.13}$$

Using triangle inequality and definition of t^k, we get

$$\begin{aligned}\|t^k - x^k\| &\le \|t^k - y^k\| + \|y^k - x^k\| \\ &= \|\delta_k y^k + (1-\delta_k)U_{i'_{n'}}(y^k) - y^k)\| + \|y^k - x^k\| \\ &= (1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\| + \|y^k - x^k\|.\end{aligned} \tag{3.14}$$

From (3.13), (3.14) and (C1) of Condition 1, we conclude that

$$\lim_{k\to+\infty} \|t^k - x^k\| = 0. \tag{3.15}$$

From the definition of $t^k = t_{i'_{n'}}$ and using (3.15), we conclude that

$$\lim_{k\to+\infty} \|t_{i'}^k - x^k\| = 0. \tag{3.16}$$

The inequality $\|t_{i'}^k - y^k\| \le \|t_{i'}^k - x^k\| + \|x^k - y^k\|$ together with (3.16) and (3.13) gives

$$\lim_{k\to+\infty} \|t_{i'}^k - y^k\| = 0. \tag{3.17}$$

On the other hand, since $t_{i'}^k = \delta_k y^k + (1-\delta_k)U_{i'}(y^k)$, we have

$$\|U_{i'}(y^k) - y^k\| = \frac{1}{1-\delta_k}\|t_{i'}^k - y^k\|.$$

The last inequality together with (3.17) and (C1) of Condition 1 yields

$$\lim_{k\to+\infty} \|U_{i'}(y^k) - y^k\| = 0, \ \ \forall i' \in I'. \tag{3.18}$$

Claim 1: *Every weak-cluster point of the sequence $\{x^k\}$ is in Ω.*
Let $\bar{p}$ be a weak-cluster point of the sequence $\{x^k\}$. There exists a subsequence $\{x^m\}$ of $\{x^k\}$ such that $x^m \rightharpoonup \bar{p}$ as $m \to +\infty$. Since C is convex and closed subset of a real Hilbert space H_1, C is weakly closed and hence $\bar{p} \in C$ (noting that $\{x^k\}$ is in C).
We want to show that $\bar{p} \in \Omega$.
From (3.13) and

$$\langle y^m, h\rangle = \langle y^m - x^m, h\rangle + \langle x^m, h\rangle, \ \ \forall h \in H_1,$$

we get $y^m \rightharpoonup \bar{p}$ as $m \to +\infty$. Thus, from (3.18) and the demiclosedness of $Id_C - U_{i'}$, we have $\bar{p} \in FixU_{i'}$ for all $i' \in I'$. Thus, $\bar{p} \in \bigcap_{i'=1}^{N'} FixU_{i'}$.

Assume $\bar{p} \notin Fix(T_r^{f_{i_0}})$ for some $i_0 \in I$ and for some $r > 0$. Thus, $T_r^{f_{i_0}}(\bar{p}) \neq \bar{p}$. That is, $\bar{p} \notin \bigcap_{i=1}^{M} Fix(T_r^{f_i})$. Thus, using Opial's condition, Lemma 2.9 and (3.12), we get

$$\begin{aligned}
\liminf_{m\to+\infty} \|x^m - \bar{p}\| &< \liminf_{m\to+\infty} \|x^m - T_r^{f_{i_0}}(\bar{p})\| \\
&= \liminf_{m\to+\infty} \|x^m - y_{i_0}^m + y_{i_0}^m - T_r^{f_{i_0}}(\bar{p})\| \\
&\leq \liminf_{m\to+\infty} (\|x^m - y_{i_0}^m\| + \|y_{i_0}^m - T_r^{f_{i_0}}(\bar{p})\|) \\
&= \liminf_{m\to+\infty} \|y_{i_0}^m - T_{r_m}^{f_{i_0}}(\bar{p})\| \\
&= \liminf_{m\to+\infty} \|T_{r_m}^{f_{i_0}} x^m - T_r^{f_{i_0}}(\bar{p})\| \\
&\leq \liminf_{m\to+\infty} (\|x^m - \bar{p}\| + \tfrac{|r_m - r|}{r_m} \|T_{r_m}^{f_{i_0}}(x^m) - x^m\|) \\
&= \liminf_{m\to+\infty} (\|x^m - \bar{p}\| + \tfrac{|r_m - r|}{r_m} \|y_{i_0}^m - x^m\|) \\
&= \liminf_{m\to+\infty} \|x^m - \bar{p}\|
\end{aligned}$$

which is a contradiction. Hence, it must be the case that $\bar{p} \in Fix(T_r^{f_i})$ for all $i \in I$ and $r > 0$. By Lemma 2.8 (iii), $Fix(T_r^{f_i}) = SEP(f_i, C)$. Hence, $\bar{p} \in \bigcap_{i=1}^{M} SEP(f_i, C)$. Therefore,

$$\bar{p} \in \Omega_1. \tag{3.19}$$

It is also easy to see that $t^m \rightharpoonup \bar{p}$ and hence $At^m \rightharpoonup A\bar{p}$. Thus, using (3.12) and

$$\langle u^m, h\rangle = \langle u^m - At^m, h\rangle + \langle At^m, h\rangle, \ \ \forall h \in H_2,$$

we get $u^m \rightharpoonup A\bar{p}$. Assume $A\bar{p} \notin FixV$. That is, $V(A\bar{p}) \neq A\bar{p}$. Thus, using Opial's condition and (3.13) together with nonexpansiveness of V, we get

$$\begin{aligned}
\liminf_{m\to+\infty} \|u^m - A\bar{p}\| &< \liminf_{m\to+\infty} \|u^m - V(A\bar{p})\| \\
&= \liminf_{m\to+\infty} \|u^m - V(u^m) + V(u^m) - V(A\bar{p})\| \\
&\leq \liminf_{m\to+\infty} (\|u^m - V(u^m)\| + \|V(u^m) - V(A\bar{p})\|) \\
&= \liminf_{m\to+\infty} \|V(u^m) - V(A\bar{p})\| \leq \liminf_{m\to+\infty} \|u^m - A\bar{p}\|
\end{aligned}$$

which is a contradiction. It must be the case that $A\bar{p} \in FixV = \bigcap_{j'=1}^{M'} FixV_{j'}$.

Similarly, using Opial's condition, Lemma 2.8, Lemma 2.9 and (3.12) we can show that $A\bar{p} \in \bigcap_{i=1}^{N} SEP(g_j, D)$. Hence,

$$A\bar{p} \in \Omega_2. \tag{3.20}$$

Therefore, from (3.19) and (3.20), we have $\bar{p} \in \Omega$.

Claim 2: *The sequence $\{x^k\}$ converges weakly to some point in Ω.*

Since the sequence $\{x^k\}$ is bounded, there exists a subsequence $\{x^m\}$ of $\{x^k\}$ such that $x^m \rightharpoonup p$ as $m \to +\infty$ for some $p \in C$. Clearly, $p \in \Omega$ by Claim 1 above.

We want to show that $x^k \rightharpoonup p$. Suppose not. Thus, there exists a subsequence $\{x^l\}$ of $\{x^k\}$ such that $x^l \rightharpoonup q$ with $p \neq q$. By Claim 1 above $q \in \Omega$. Then applying Opials condition and noting that from (3.11) for all $x^* \in \Omega$ the sequence $\{\|x^k - x^*\|\}$ is decreasing (and $\lim_{k\to+\infty} \|x^k - x^*\|$ exists), we have

$$\begin{aligned}\liminf_{l\to+\infty} \|x^l - q\| &< \liminf_{l\to+\infty} \|x^l - p\| \\ &= \liminf_{m\to+\infty} \|x^m - p\| \\ &< \liminf_{m\to+\infty} \|x^m - q\| \\ &= \liminf_{l\to+\infty} \|x^l - q\|.\end{aligned}$$

This is a contradiction. Hence $\{x^k\}$ converges weakly to p.
Therefore, $x^k \rightharpoonup p$ together with (3.13) and (3.15) gives $y^k \rightharpoonup p$ and $t^k \rightharpoonup p$, so $At^k \rightharpoonup Ap$. Combining with (3.12), it is immediate that $u^k \rightharpoonup Ap$.
Hence, Theorem 3.1 is proved. □

The following Corollary is immediate consequence of Theorem 3.1 obtaind for solving SSEPs by setting $U_{i'} = Id_C$ for all $i' \in I'$ and $V_{j'} = Id_D$ for all $j' \in J'$ in Algorithm 3.1.

Corollary 3.1 *Let $\{y^k\}$, $\{u^k\}$, and $\{x^k\}$ be sequences generated by iterative algorithm*

$$\begin{cases} x^0 \in C, \\ y_i^k = T_{r_k}^{f_i}(x^k), \quad i \in I, \\ y^k = \sum_{i=1}^{N} \xi_i^k y_i^k, \\ u_j^k = T_{r_k}^{g_j}(Ay^k), \quad j \in J, \\ u^k = \arg\max\{\|v - Ay^k\| : v \in \{u_j^k : j \in J\}\}, \\ x^{k+1} = P_C(t^k + \mu_k A^*(u^k - At^k)). \end{cases}$$

where $\{r_k\}$, $\{\mu_k\}$ and $\{\xi_i^k\}$ ($i \in I$) be real sequences satisfying (C2) and (C3) of ***Condition 1****. Then, sequences $\{y^k\}$ and $\{x^k\}$ converges weakly to a point $p \in \{x^* \in \bigcap_{i=1}^{N} SEP(f_i, C) : Ax^* \in \bigcap_{i=1}^{M} SEP(g_j, D)\}$ and $\{u^k\}$ converges weakly to a point $Ap \in \bigcap_{i=1}^{M} SEP(g_j, D)$.*

3.2. **Strong Convergence.** To obtain the strong convergence, we perform shrinking projection as one appropriate additional step in Algorithm 3.1.

Algorithm 3.2
Initialization: Choose $x^0 \in C$ and $C = C_0$. Let $\{r_k\}$, $\{\delta_k\}$, $\{\mu_k\}$,$\{\xi_i^k\}$ ($i \in I$), $\{\theta_{j'}^k\}$ ($j' \in J'$) be real sequences satisfying **Condition 1**.

Step 1: For each $i \in I$ find $y_i^k = T_{r_k}^{f_i}(x^k)$.

Step 2: Evaluate $y^k = \sum_{i=1}^{N} \xi_i^k y_i^k$.

Step 3: For each $i' \in I'$ find $t_{i'}^k = \delta_k y^k + (1 - \delta_k)U_{i'}(y^k)$.

Step 4: Find among $t_{i'}^k$, $i' \in I'$, the farthest element from x^k, i.e.,

$$t^k = \arg\max\{\|v - x^k\| : v \in \{t_{i'}^k : i' \in I'\}\}.$$

Step 5: For each $j \in J$ find $u_j^k = T_{r_k}^{g_j}(At^k)$.

Step 6: Find among u_j^k, $j \in J$, the farthest element from At^k, i.e.,

$$u^k = \arg\max\{\|v - At^k\| : v \in \{u_j^k : j \in J\}\}.$$

Step 7: Evaluate $s^k = P_C\Big(t^k + \mu_k A^*\Big(\sum_{j'=1}^{M'} \theta_{j'}^k V_{j'}(u^k) - At^k\Big)\Big)$.

Step 8: Evaluate

$$x^{k+1} = P_{C_{k+1}}(x^0)$$

where

$$C_{k+1} = \{w \in C_k : \|s^k - w\| \leq \|t^k - w\| \leq \|y^k - w\| \leq \|x^k - w\|\}.$$

Step 9: Set $k := k+1$ and go to Step 1.

Lemma 3.1 *For each $k \geq 0$, the set C_k defined in Algorithm 3.2 is nonempty closed convex subset of H_1.*

Proof. Let $V = \sum_{j'=1}^{M'} \theta_{j'}^k V_{j'}$ and let $x^* \in \Omega$. Then, from (3.3), (3.4) and (3.9), we have

$$\|y^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \sum_{i=1}^{N} \xi_i^k \|y_i^k - x^k\|^2, \tag{3.21}$$

$$\|t^k - x^*\|^2 \leq \|y^k - x^*\|^2 - \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2, \tag{3.22}$$

$$\|s^k - x^*\|^2 \leq \|t^k - x^*\|^2 - \mu_k(1-\mu_k\sigma^2)\|V(u^k) - At^k\|^2 - \mu_k\|u^k - At^k\|^2. \tag{3.23}$$

From (3.21), (3.22), (3.23) and Condition 1, we have

$$\|s^k - x^*\| \leq \|t^k - x^*\| \leq \|y^k - x^*\| \leq \|x^k - x^*\|, \;\; \forall k \geq 0. \tag{3.24}$$

Since $x^* \in C_0 = C$, we get by induction that $x^* \in C_k$ for all $k \geq 0$, i.e., $\Omega \subset C_k$, so $C_k \neq \emptyset$ for all k. Let

$$\begin{aligned} E_k &= \{w \in H_1 : \|s^k - w\| \leq \|t^k - w\|\}, \;\; \forall k \geq 0, \\ F_k &= \{w \in H_1 : \|t^k - w\| \leq \|y^k - w\|\}, \;\; \forall k \geq 0, \\ G_k &= \{w \in H_1 : \|y^k - w\| \leq \|x^k - w\|\}, \;\; \forall k \geq 0. \end{aligned}$$

Thus, $C_{k+1} = C_k \cap E_k \cap F_k \cap G_k$. Note that E_k, F_k and G_k are either the halfspaces or the whole space H_1 for all $k \geq 0$. Hence, they are closed and convex. Since C_0, E_k, F_k and G_k are closed and convex for all $k \geq 0$, C_k is also closed convex for all $k \geq 0$. □

Remark 2. For each k, since C_k is nonempty closed and convex subset of H_1 (by Lemma 3.1), the projection in Step 8 exists and is unique. Hence, well definedness of Algorithm 3.2 is straight using Remark 1.

Now, we prove the strong convergence of Algorithm 3.2.

Theorem 3.2 *Let $\{y^k\}$, $\{t^k\}$, $\{u^k\}$ and $\{x^k\}$ be sequences generated by Algorithm 3.2. Then, $\{y^k\}$, $\{t^k\}$ and $\{x^k\}$ converge strongly to a point $p \in \Omega$ and $\{u^k\}$ converges strongly to a point $Ap \in \Omega_2$.*

Proof. Let $x^* \in \Omega$. By Lemma 3.1, C_k is a nonempty closed convex set for all $k \geq 0$ and by definition $x^{k+1} \in C_{k+1}$ and $x^k = P_{C_k}(x^0)$, so $\|x^k - x^0\| \leq \|x^{k+1} - x^0\|$, $\forall k$. In addition, $x^{k+1} = P_{C_{k+1}}(x^0)$ and $x^* \in C_{k+1}$, we have $\|x^{k+1}-x^0\| \leq \|x^*-x^0\|, \ \ \forall k \geq 0$. This results

$$\|x^k - x^0\| \leq \|x^{k+1} - x^0\| \leq \|x^* - x^0\|, \ \ \forall k \geq 0.$$

Therefore, $\lim_{k\to+\infty} \|x^k - x^0\|$ exists, consequently $\{x^k\}$ is bounded, and hence using (3.24), we see that $\{y^k\}$, $\{t^k\}$ and $\{s^k\}$ are also bounded.
For $m \geq n$, from the definition of C_k, we have $x^m \in C_m \subset C_n$. Thus,

$$\|x^m - x^n\|^2 \leq \|x^m - x^0\|^2 - \|x^n - x^0\|^2 \ \ \forall k. \tag{3.25}$$

Passing to the limit in the inequality (3.25) as $m, n \to +\infty$, we get $\|x^m - x^n\| \to 0$, implying that $\{x^k\}$ is a Cauchy sequence, and hence it converges to some point p in C. From the definition of C_{k+1} and x^{k+1} we get

$$\|s^k - x^{k+1}\| \leq \|t^k - x^{k+1}\| \leq \|y^k - x^{k+1}\| \leq \|x^k - x^{k+1}\|.$$

Thus, using the last result, we obtain

$$\|x^k - s^k\| \leq \|s^k - x^{k+1}\| + \|x^k - x^{k+1}\| \leq 2\|x^k - x^{k+1}\|, \tag{3.26}$$

$$\|x^k - t^k\| \leq \|t^k - x^{k+1}\| + \|x^k - x^{k+1}\| \leq 2\|x^k - x^{k+1}\|, \tag{3.27}$$

$$\|x^k - y^k\| \leq \|y^k - x^{k+1}\| + \|x^{k+1} - x^k\| \leq 2\|x^k - x^{k+1}\|. \tag{3.28}$$

Letting $k \to \infty$ in (3.26), (3.27) and (3.28) and from the convergence $\|x^k - x^{k+1}\|$ to 0, we deduce

$$\lim_{k\to+\infty} \|x^k - s^k\| = \lim_{k\to+\infty} \|x^k - t^k\| = \lim_{k\to+\infty} \|x^k - y^k\| = 0. \tag{3.29}$$

Using (3.29) and since $\{x^k\}$ converges to p in C, we have

$$\lim_{k\to+\infty} s^k = \lim_{k\to+\infty} t^k = \lim_{k\to+\infty} y^k = p \tag{3.30}$$

and so $\lim\limits_{k\to+\infty} At^k = Ap$. Combining (3.21), (3.22) and (3.23), we have

$$\|s^k - x^*\|^2 \leq \|x^k - x^*\|^2 - Q_k \tag{3.31}$$

where

$$Q_k = \sum_{i=1}^{N} \xi_i^k \|y_i^k - x^k\|^2 + \delta_k(1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\|^2 +$$
$$\mu_k(1-\mu_k\sigma^2)\|V(u^k) - At^k\|^2 + \mu_k\|u^k - At^k\|^2.$$

But,

$$\begin{aligned} Q_k &\leq \|x^k - x^*\|^2 - \|s^k - x^*\|^2 \\ &\leq (\|x^k - x^*\| - \|s^k - x^*\|)(\|x^k - x^*\| + \|s^k - x^*\|) \\ &\leq \|x^k - s^k\|(\|x^k - x^*\| + \|s^k - x^*\|). \end{aligned} \tag{3.32}$$

Hence, $Q_k \to 0$ as $\|x^k - s^k\|(\|x^k - x^*\| + \|s^k - x^*\|) \to 0$. In view of $Q_k \to 0$ and Condition 1, it follows that

$$\begin{aligned} \lim_{k\to+\infty} \|y_i^k - x^k\| = \lim_{k\to+\infty} \|U_{i'_{n'}}(y^k) - y^k\| &= \lim_{k\to+\infty} \|V(u^k) - At^k\| \\ &= \lim_{k\to+\infty} \|u^k - At^k\| = 0. \end{aligned} \tag{3.33}$$

From (3.33), we can have

$$\lim_{k\to+\infty}\|y^k - x^k\| = \lim_{k\to+\infty}\|U_{i'_{n'}}(y^k) - x^k\| = \lim_{k\to+\infty}\|V(u^k) - u^k\| = 0. \tag{3.34}$$

Using triangle inequality and definition of t^k, we get

$$\|t^k - x^k\| \leq (1-\delta_k)\|U_{i'_{n'}}(y^k) - y^k\| + \|y^k - x^k\|. \tag{3.35}$$

From (3.33), (3.34), (3.35) and (C1) of Condition 1, we conclude that

$$\lim_{k\to+\infty}\|t^k - x^k\| = 0. \tag{3.36}$$

Hence, by definition of t^k and from (3.36), we have $\lim_{k\to+\infty}\|t^k_{i'} - x^k\| = 0$. This together with (3.34) results

$$\lim_{k\to+\infty}\|t^k_{i'} - y^k\| = 0. \tag{3.37}$$

Since $t^k_{i'} = \delta_k x^k + (1-\delta_k)U_{i'}(y^k)$, we have $\|U_{i'}(y^k) - y^k\| = \frac{1}{1-\delta_k}\|t^k_{i'} - y^k\|$ and this gives together with (3.37) gives

$$\lim_{k\to+\infty}\|U_{i'}(y^k) - y^k\| = 0, \quad \forall i' \in I'. \tag{3.38}$$

Claim: $p \in \Omega$.
Let $r > 0$. For each $i \in I$, using Lemma 2.9, we have

$$\begin{aligned}\|T_r^{f_i}(p) - p\| &\leq \|T_r^{f_i}(p) - T_{r_k}^{f_i}(x^k)\| + \|T_{r_k}^{f_i}(x^k) - x^k\| + \|x^k - p\| \\ &\leq \|p - x^k\| + \tfrac{|r-r_k|}{r_k}\|T_{r_k}^{f_i}(x^k) - p\| + \|T_{r_k}^{f_i}(x^k) - x^k\| + \|x^k - p\| \\ &= 2\|x^k - p\| + \tfrac{|r-r_k|}{r_k}\|y_i^k - p\| + \|y_i^k - x^k\|.\end{aligned} \tag{3.39}$$

In view of $\|y_i^k - p\| \leq \|y_i^k - x^k\| + \|x^k - p\|$, $x^k \to p$ and (3.33), we have $\|y_i^k - p\| \to 0$. Hence, $\|y_i^k - p\| \to 0$, (3.33) and since $x^k \to p$, (3.39) results $\|T_r^{f_i}(p) - p\| = 0$ for each $i \in I$, i.e., $p \in FixT_r^{f_i}$ for all $i \in I$. Therefore, by Lemma 2.8, we have $p \in \bigcap_{i=1}^{N} SEP(f_i, C)$. In view of

$$\begin{aligned}\|U_{i'}(p) - p\| &\leq \|U_{i'}(p) - U_{i'}(y^k)\| + \|U_{i'}(y^k) - y^k\| + \|y^k - p\| \\ &\leq \|y^k - p\| + \|U_{i'}(y^k) - y^k\| + \|y^k - p\| \\ &= 2\|y^k - p\| + \|U_{i'}(y^k) - y^k\|\end{aligned} \tag{3.40}$$

together with (3.34), we have $\|U_{i'}(p) - p\| = 0$ for all $i' \in I'$. Thus, $p \in FixU_{i'}$ for all $i' \in I'$, i.e., $p \in \bigcap_{i'=1}^{N'} FixU_{i'}$. Therefore,

$$p \in \Omega_1. \tag{3.41}$$

Since $t^k \to p$, $At^k \to Ap$ and hence $\|u^k - Ap\| \leq \|u^k - At^k\| + \|At^k - Ap\|$ together with (3.33) yields

$$\lim_{k\to+\infty}\|u^k - Ap\| = 0, \tag{3.42}$$

i.e., $u^k \to Ap$. Combining (3.34), (3.42) together with

$$\begin{aligned}\|V(Ap) - Ap\| &\leq \|V(Ap) - V(u^k)\| + \|V(u^k) - u^k\| + \|u^k - Ap\| \\ &\leq \|Ap - u^k\| + \|V(u^k) - u^k\| + \|u^k - Ap\| \\ &= 2\|u^k - Ap\| + \|V(u^k) - u^k\|\end{aligned}$$

gives $\|V(Ap) - Ap\| = 0$. That is $V(Ap) = Ap$ and hence $Ap \in \bigcap_{j'=1}^{M'} FixV_{j'}$.

From (3.33) and definition of u^k, we have

$$\lim_{k\to+\infty} \|u_j^k - At^k\| = 0, \quad \forall j \in J = \{1, 2, \dots, M\}. \tag{3.43}$$

Using Lemma 2.9, we have

$$\begin{aligned}\|T_r^{g_j}(Ap) - Ap\| &\leq \|T_r^{g_j}(Ap) - T_{r_k}^{g_j}(At^k)\| + \|T_{r_k}^{g_j}(t^k) - At^k\| + \|At^k - Ap\| \\ &= \|T_r^{g_j}(Ap) - T_{r_k}^{g_j}(At^k)\| + \|u_j^k - At^k\| + \|At^k - Ap\| \\ &\leq 2\|At^k - Ap\| + \tfrac{|r_k - r|}{r_k}\|u_j^k - Ap\| + \|u_j^k - At^k\|.\end{aligned} \tag{3.44}$$

From (3.43) and since $At^k \to Ap$, we have $\|u_j^k - Ap\| \to 0$. Thus, applying (3.43), $At^k \to Ap$ and $\|u_j^k - Ap\| \to 0$ in (3.44), we get $T_r^{g_j}(Ap) = Ap$ for all $j \in J$, which implies that $Ap \in Fix(T_r^{g_j}) = SEP(g_j, D),\ \forall j \in J$. Hence, $Ap \in \bigcap_{i=1}^{N} SEP(g_i, D)$. Thus, $Ap \in \Omega_2$. Therefore, $Ap \in \Omega_2$ and (3.41) gives $p \in \Omega$. This completes the proof. □

Corollary 3.2 *Let $\{y^k\}$, $\{u^k\}$ and $\{x^k\}$ be sequences generated by iterative algorithm*

$$\begin{cases} x^0 \in C = C_0, \\ y_i^k = T_{r_k}^{f_i}(x^k), \quad i \in I, \\ y^k = \sum_{i=1}^{N} \xi_i^k y_i^k, \\ u_j^k = T_{r_k}^{g_j}(Ay^k), \quad j \in J, \\ u^k = \arg\max\{\|v - Ay^k\| : v \in \{u_j^k : j \in J\}\}, \\ s^k = P_C(t^k + \mu_k A^*(u^k - At^k)), \\ C_{k+1} = \{w \in C_k : \|s^k - w\| \leq \|y^k - w\| \leq \|x^k - w\|\}, \\ x^{k+1} = P_{C_{k+1}}(x^0), \end{cases}$$

*where $\{r_k\}$, $\{\mu_k\}$ and $\{\xi_i^k\}$ $(i \in I)$ are real sequences satisfying (C2) and (C3) of **Condition 1**. Then, sequences $\{y^k\}$ and $\{x^k\}$ converge strongly to a point*

$$p \in \left\{ x^* \in \bigcap_{i=1}^{N} SEP(f_i, C) : Ax^* \in \bigcap_{i=1}^{M} SEP(g_j, D) \right\}$$

and $\{u^k\}$ converges strongly to a point $Ap \in \bigcap_{i=1}^{M} SEP(g_j, D)$.

4. Application

4.1. **Application to split system of fixed point set constraint variational inequality problems.** Consider problem (1.1) for $f_i(x, y) = \langle A_i(x), y - x\rangle$ for $i \in I$, and $g_j(u, v) = \langle B_j(u), v - u\rangle$ for $j \in J$ where $A_i : C \to H_1$ and $B_j : D \to H_2$ are nonlinear operators. This gives *split system of fixed point set constraint variational inequality problem* (SSFPSCVIP):

$$\text{find } x^* \in C \text{ such that } \begin{cases} x^* \in FixU_{i'}, \ \ \forall i' \in I', \\ \langle A_i(x^*), y - x^* \rangle \geq 0, \ \forall y \in C, \ \ \forall i \in I, \\ u^* = Ax^* \in D, \ \ \ u^* \in FixV_{j'}, \ \ \forall j' \in J', \\ \langle B_j(u^*), u - u^* \rangle \geq 0, \ \forall u \in D, \ \ \forall j \in J. \end{cases} \tag{4.1}$$

Let VI(A_i, C) and VI(B_j, D) stands for the variational inequality problems which is:

$$\text{find } x^* \in C \text{ such that } \langle A_i(x^*), y - x^* \rangle \geq 0, \ \ \forall y \in C$$

and

$$\text{find } u^* \in D \text{ such that } \langle B_j(u^*), v - u^* \rangle \geq 0, \ \ \forall v \in D,$$

respectively. Let Φ be solution set of (4.1) and let $SVI(A_i, C)$ and $SVI(B_j, D)$ represent solution of $VI(A_i, C)$ and $VI(B_j, D)$, respectively.

Lemma 4.1 *For each $r > 0$, we have*

(i). $y_i = T_r^{f_i}(x)$ *iff* $\langle y_i + rA_i(y_i) - x, y - y_i \rangle \geq 0, \ \ \forall y \in C,$

(ii). $u_j = T_r^{g_j}(u)$ *iff* $\langle u_j + rB_j(u_j) - u, v - u_j \rangle \geq 0, \ \ \forall v \in D.$

Proof. (i). For each $r > 0$, we have

$$\begin{aligned} y_j = T_r^{f_i}(x) &\Leftrightarrow f_i(y_i, y) + \tfrac{1}{r}\langle y - y_i, y_i - x \rangle \geq 0, \ \forall y \in C \\ &\Leftrightarrow \langle A_i(y_i), y - y_i \rangle + \tfrac{1}{r}\langle y - y_i, y_i - x \rangle \geq 0, \ \forall y \in C \\ &\Leftrightarrow \langle y_i + rA_i(y_i) - x, y - y_i \rangle \geq 0, \ \ \forall y \in C. \end{aligned}$$

(ii). The proof follows similar step of (i) above. □

If each A_i is monotone on C and each B_j is monotone on D for all $i \in I$, $j \in J$, then each f_i and each g_j satisfies Condition A on C and D, respectively. Hence, using Theorem 3.1 and 3.2, we get the following results for approximate solution of (4.1).

Theorem 4.1 *Let each A_i is monotone on C and each B_j is monotone on D for all $i \in I$, $j \in J$. Let $\{y^k\}$, $\{t^k\}$, $\{u^k\}$ and $\{x^k\}$ be a sequences generated by iterative algorithm*

$$\begin{cases} x^0 \in C, \\ \langle y_i^k + r_k A_i(y_i^k) - x^k, y - y_i^k \rangle \geq 0, \ \ \forall y \in C, \\ y^k = \sum\limits_{i=1}^{N} \xi_i^k y_i^k, \\ t_{i'}^k = \delta_k y^k + (1 - \delta_k) U_{i'}(y^k), \\ t^k = \arg\max\{\|v - x^k\| : v \in \{t_{i'}^k : i' \in I'\}\}, \\ \langle u_j^k + r_k B_j(u_j^k) - At^k, v - u_j^k \rangle \geq 0, \ \ \forall v \in D, \\ u^k = \arg\max\{\|v - At^k\| : v \in \{u_j^k : j \in J\}\}, \\ x^{k+1} = P_C\Big(t^k + \mu_k A^*\Big(\sum\limits_{j'=1}^{M'} \theta_{j'}^k V_{j'}(u^k) - At^k\Big)\Big), \end{cases}$$

where $0 < \sigma_1 \leq \delta_k \leq \sigma_2 < 1$, $0 < \xi \leq \xi_i^k \leq 1$ $(i \in I)$, $r_k \geq r > 0$, $0 < \gamma_1 \leq \mu_k \leq \gamma_2 < \frac{1}{\sigma^2}$ for some $\sigma \geq \|A\|$, $0 < \theta \leq \theta_{j'}^k \leq 1$ $(j' \in J')$ with $\sum_{j'=1}^{M'} \theta_{j'}^k = 1$ and $\sum_{i=1}^{N} \xi_i^k = 1$. Then, $\{y^k\}$, $\{t^k\}$ and $\{x^k\}$ converge weakly to $p \in \Phi$ and $\{u^k\}$ converges weakly to $Ap \in [\bigcap_{j'=1}^{M'} FixV_{j'}] \bigcap [\bigcap_{j=1}^{M} SVI(B_j, D)]$.

Theorem 4.2 *Let each A_i is monotone on C and each B_j is monotone on D for all $i \in I$, $j \in J$. Let $\{y^k\}$, $\{t^k\}$, $\{u^k\}$ and $\{x^k\}$ be a sequences generated by iterative algorithm*

$$\begin{cases} x^0 \in C = C_0, \\ \langle y_i^k + r_k A_i(y_i^k) - x^k, y - y_i^k \rangle \geq 0, \ \ \forall y \in C, \\ y^k = \sum\limits_{i=1}^{N} \xi_i^k y_i^k, \\ t_{i'}^k = \delta_k y^k + (1-\delta_k) U_{i'}(y^k), \\ t^k = \arg\max\{\|v - x^k\| : v \in \{t_{i'}^k : i' \in I'\}\}, \\ \langle u_j^k + r_k B_j(u_j^k) - At^k, v - u_j^k \rangle \geq 0, \ \ \forall v \in D, \\ u^k = \arg\max\{\|v - At^k\| : v \in \{u_j^k : j \in J\}\}, \\ s^k = P_C\Big(t^k + \mu_k A^*\Big(\sum\limits_{j'=1}^{M'} \theta_{j'}^k V_{j'}(u^k) - At^k\Big)\Big), \\ C_{k+1} = \{w \in C_k : \|s^k - w\| \leq \|t^k - w\| \leq \|z^k - w\| \leq \|x^k - w\|\}, \\ x^{k+1} = P_{C_{k+1}}(x^0), \end{cases}$$

where $0 < \sigma_1 \leq \delta_k \leq \sigma_2 < 1$, $0 < \xi \leq \xi_i^k \leq 1$, $(i \in I)$, $r_k \geq r > 0$, $0 < \gamma_1 \leq \mu_k \leq \gamma_2 < \frac{1}{\sigma^2}$ for some $\sigma \geq \|A\|$, $0 < \theta \leq \theta_{j'}^k \leq 1$, $(j' \in J')$ with $\sum_{j'=1}^{M'} \theta_{j'}^k = 1$ and $\sum_{i=1}^{N} \xi_i^k = 1$. Then, $\{y^k\}$, $\{t^k\}$ and $\{x^k\}$ converge strongly to $p \in \Phi$ and $\{u^k\}$ converges strongly to $Ap \in [\bigcap_{j'=1}^{M'} FixV_{j'}] \bigcap [\bigcap_{j=1}^{M} SVI(B_j, D)]$.

Setting $U_{i'} = Id_C$ for all $i' \in I'$ and $V_{j'} = Id_D$ for all $j' \in J'$ in problem (4.1) yields split system of variational inequality problem. In [2, Algorithm 6.4], the iterative algorithm can also solve problem (4.1) for $U_{i'} = Id_C$ for all $i' \in I'$ and $V_{j'} = Id_D$ for all $j' \in J'$. However, in order to obtain the convergence, the method requires the restrictive condition that the operators A_i, B_j are α-inverse strongly monotone.

4.2. **Numerical Experiment.** Let $H_1 = \mathbb{R}^n$ and $H_2 = \mathbb{R}$. Consider the problem (1.1) for bifunctions $f_i : C \times C \to \mathbb{R}$ defined by

$$f_i(x, y) = G_i(y) - G_i(x), \ \ i \in I = \{1, 2, \ldots, N\},$$

and $g_1, g_2 : D \times D \to \mathbb{R}$ are given by

$$g_1(u, v) = u(v - u), \quad g_2(u, v) = (2u - u^2)(v - u)$$

where $G_i(y) = \frac{1}{2} y^T Q_i y + q_i^T y$ with $q_i \in \mathbb{R}^n$ and Q_i being a symmetric positive definite $n \times n$ matrix of order n. Lets take the feasible sets

$$C = \{x \in \mathbb{R}^n : -5 \leq x_i \leq 5, \ \forall i = 1, 2, \ldots, n\} \text{ and } D = [-1, +\infty).$$

Let $U_{i'} : C \to C$ given by $U_{i'}(x) = \frac{1}{i'} x, \quad i' \in I' = \{1, 2, \ldots, N'\}$ and $V_{j'} : D \to D$ defined by $V_{j'}(u) = \frac{1}{j'} u, \quad j' \in J' = \{1, 2, \ldots, M'\}$. For a vector ϑ in $\mathbb{R}^n$ take a mapping $A : \mathbb{R}^n \to \mathbb{R}$ given by $A(x) = \langle \vartheta, x \rangle$. Thus, $A^* u = u.\vartheta$ and $\|A\| = \|\vartheta\|$.
It is easy to show that,

- each $U_{i'}$ and $V_{j'}$ is nonexpansive mapping,
- each bifunction f_i satisfy Condition A on C,
- g_1 and g_2 satisfy Condition A on D,

- $\{0\} = \Omega_1$, $\{0\} = \Omega_2$ and $A(0) = 0$. Thus, $\Omega = \{0\}$.

Note that in this example, the resolvent $T^{f_i}_{r_k}$ of the bifunction f_i coincides with the proximal mapping of the function G_i with the constant $r_k > 0$, that is, we need to solve the following optimization program

$$T^{f_i}_{r_k}(x^k) = \arg\min\big\{G_i(y) + \frac{1}{r_k}\|x^k - y\|^2 : y \in C\big\}, \quad i \in I$$

or the following convex quadratic problem

$$T^{f_i}_{r_k}(x^k) = \arg\min\big\{\frac{1}{2}y^T\hat{H}_i y + b_i^T y : y \in C\big\}, \quad i \in I$$

where $\hat{H}_i = 2(Q_i + \frac{1}{r_k}I_d)$ and $b_i = q_i - \frac{2}{r_k}x^k$ where I_d is $n \times n$ identity matrix. For each $i \in I$, the convex quadratic problem $\arg\min\big\{\frac{1}{2}y^T\hat{H}_i y + b_i^T y : y \in C\big\}$ can be effectively solved, for instance, by MATLAB Optimization Toolbox.

Experiment: We use the following data for our numerical experiment.

- $N = 3$, $n = 3$, $N' = 4$, $M' = 5$, $\vartheta = (1, 1, 1)^T$.
- q_1, q_2 and q_3 are zero vectors in $\mathbb{R}^3$ and

$$Q_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

We set $r_k = 1$, $\xi_i^k = \frac{i}{6}$, $\theta_{j'}^k = \frac{j'}{15}$ for $i \in I = \{1, 2, 3\}$, $j' \in J' = \{1, 2, 3, 4, 5\}$ and we will test our methods for different starting points x^0 and different parameters δ_k and μ_k. The results in Table 1 and Table 2 report iteration (k) and CPU time in second for the sequence $\{x^k\}$ generated by the Algorithms for different starting points and parameters. The algorithm has been coded in Matlab R2017a running on MacBook 1.1 GHz Intel Core m3 8 GB 1867 MHz LPDDR3.

TABLE 1. Performance of Algorithm 3.1 and 3.2 for different x^0, δ_k and μ_k

	$\delta_k = 0.5$, $\mu_k = 10^{-2}$			
	Algorithm 3.1		Algorithm 3.2	
x^0	Iter(k)	CPU time(s)	Iter(k)	CPU time(s)
$(-1, 2, 3)^T$	4	0.107	7	0.259
$(4, 1, -4)^T$	6	0.131	12	0.317
$(2, -2, 3)^T$	10	0.225	21	0.548

The study of the numerical experiments here is preliminary and it is obvious that SSFPSCEP depend on the structure of the constrained sets C and D, the bifunctions f_i and g_j, and the mappings $U_{i'}$ and $V_{j'}$. However, the results in Table 1 and Table 2 shows that the number of iterative step and time for execution of the algorithms depend on the starting point x^0 and parameter sequences.

TABLE 2. Performance of Algorithm 3.1 and 3.2 for different x^0, δ_k and μ_k

	$\delta_k = \frac{98k+108}{100(k+1)}$, $\mu_k = 10^{-3}$			
	Algorithm 3.1		Algorithm 3.2	
x^0	Iter(k)	CPU time(s)	Iter(k)	CPU time(s)
$(-1, 2, 3)^T$	5	0.130	7	0.276
$(4, 1, -4)^T$	11	0.280	18	0.485
$(2, -2, 3)^T$	17	0.331	25	0.674

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