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FIXED AND PERIODIC POINT RESULTS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS; A SURVEY

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Abstract. In this paper, we consider the concept of cone b-metric spaces over Banach algebras and obtain some fixed point results for various definitions of contractive mappings. Moreover, we discuss about the property P and the property Q of fixed point problems. Our results are significant, since we omit the assumptions of normality of cones under which can be generalized and unified a number of recently announced results in the existing literature. In particular, we refer to the results of Huang et al. [H. Huang, G. Deng, S. Radenović, Some topological properties and fixed point results in cone metric spaces over Banach algebras, Positivity. (2018), in press].
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1. Introduction

Fixed point theory is an important and useful tool for different branches of mathematical analysis and it has a wide range of applications in applied mathematics and sciences. It may be discussed as an essential subject of nonlinear analysis. Since the first results of Banach in 1922 [6], various authors have been studying fixed points of mappings in metric spaces. Also, it is well-known that ordered normed spaces and cones have many applications in applied mathematics and related topics. Hence, fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century (see [9, 35]). In 2007, Huang and Zhang [18] reintroduced such spaces under the name of cone metric spaces by substituting an ordered normed space for the real numbers and obtained some fixed point results (also, see [1, 20] and references contained therein).

On the other hand, the concept of metric type or b-metric space was studied by Bakhtin [5] (also, see [8, 23]). Then, analogously with definition of a b-metric space and cone metric space, Hussain and Shah [19] and Ćvetković et al. [7] defined cone b-metric spaces (or same cone metric type spaces) and proved several fixed and common fixed point theorems. Then some authors obtained several well-known results in fixed point theory in the framework of non-normal cone b-metric spaces such as Rahimi et al. [27], Kadelburg et al. [22] and others. In the sequel, some authors presented any cone metric space or cone b-metric space is just equivalent to a metric space or b-metric space, respectively, if the metric or b-metric function is defined by a nonlinear scalarization function ξ_e or by a Minkowski functional q_e (for more details, see [10, 11]).

In 2013, Liu and Xu [25] introduced the concept of cone metric space over Banach algebra and obtained some fixed point theorems in such spaces. The results of Liu and Xu are significant, since can be proved that cone metric spaces over Banach algebras are not equivalent to metric spaces. Hence, some interesting results about fixed point theory in cone metric spaces over Banach algebra and cone b-metric spaces over Banach algebra with its applications were proved in [13, 17] and references contained therein.

Following the idea of Huang et al. [12], the purpose of this paper is to prove fixed point and common fixed point theorems for various definitions of contractive conditions in cone b-metric spaces over Banach algebra with a solid cone P. Also, we compare and arrange all of obtained results about fixed point theory from 1922 until now in the framework of various cone metric spaces over Banach algebras. Furthermore, we prove some theorems about periodic points in related to fixed point problems. Our paper generalize, extend and unify some well-known results in the literature such as: Abbas and Rhoades [2], Ahmed and Salunke [3], Altun et al. [4], Huang et al. [12, 13, 17], Huang and Zhang [18], Liu and Xu [25], Kumam et al. [24], Rahimi et al. [26], Reich [28] Rhoades [29] and others.

2. Preliminaries

Consistent with the content of Huang et al. [13], the following definitions and results will be needed in the sequel.

A Banach algebra $\mathcal{A}$ is a Banach space over field $\mathbf{K} \in \{\mathbb{R}, \mathbb{C}\}$ together with an associative and distributive multiplication such that $\lambda(xy) = (\lambda x)y = x(\lambda y)$ and $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$ and $\alpha \in \mathbf{K}$, where $\|\cdot\|$ is norm on $\mathcal{A}$. Let $\mathcal{A}$ be a Banach algebra with unit e and zero element θ. A nonempty closed convex subset P of Banach algebra $\mathcal{A}$ is named a cone if $\{\theta, e\} \subset P$, $P^2 = PP \subset P, P \cap (-P) = \{\theta\}$ and $\alpha P + \beta P \subset P$ for all $\alpha, \beta \geq 0$. For a given cone P we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Moreover, we write $x \ll y \Longleftrightarrow y - x \in \text{int}P$ (where $\text{int}P$ is the interior of P). If $\text{int}P \neq \emptyset$, the cone P is called solid. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y \Longrightarrow \|x\| \leq k\|y\|$. The least positive number satisfying the above condition is called the normal constant of P.

Definition 2.1. [13] Let X be a nonempty set, $K \geq 1$ be a constant and $\mathcal{A}$ be a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies the following conditions:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x,z) \preceq K[d(x,y)+d(y,z)]$ for all $x,y,z \in X$.
Then d is called a cone b-metric on X and (X,d) is called a cone b-metric space over Banach algebra.

Obviously, for $K=1$, cone b-metric space over Banach algebra $\mathcal{A}$ is a cone metric space over Banach algebra $\mathcal{A}$. Also, for definition such as convergent and Cauchy sequences, c-sequence, completeness, continuity and examples in cone b-metric spaces over Banach algebra $\mathcal{A}$, we refer to [13, 17]. Also, we consider the following example.

Example 2.2. Let $X=[0,1]$, $\mathcal{A}=C^1_{\mathbf{R}}[0,1]$ with the norm $\|f\|=\|f\|_\infty+\|f'\|_\infty$ and multiplication in $\mathcal{A}$ be just pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with a unit $e(t)=1$ for all $t\in[0,1]$. Take a solid cone $P=\{f\in\mathcal{A}|f(t)\geq 0,\ \forall t\in[0,1]\}$ and define a $d: X\times X\to P\subseteq\mathcal{A}$ by $d(x,y)=|x-y|^2 2^t$, where $2^t\in P\subset\mathcal{A}$. Then (X,d) is a cone b-metric space over Banach algebra $\mathcal{A}$ with the non-normal solid cone P and $K=2$.

Lemma 2.3. [7, 27] *Let (X,d) be a cone b-metric space over Banach algebra $\mathcal{A}$ and $u,v,w\in\mathcal{A}$. Then the following properties are often used, particularly in dealing with cone b-metric spaces in which the cone need not be normal.*

(p_1) *If $u\preceq v$ and $v\ll w$, then $u\ll w$.*
(p_2) *If $\theta\preceq u\ll c$ for each $c\in intP$, then $u=\theta$.*
(p_3) *If $u\preceq\lambda u$ where $u\in P$ and $0\leq\lambda<1$, then $u=\theta$.*
(p_4) *Let $x_n\to\theta$ in $\mathcal{A}$ and $\theta\ll c$. Then there exists a positive integer n_0 such that $x_n\ll c$ for each $n>n_0$.*

Lemma 2.4. [30] *Let $\mathcal{A}$ be a Banach algebra with a unit e. Then the spectral radius $\rho(u)$ of $u\in\mathcal{A}$ holds*

$$\rho(u)=\lim_{n\to\infty}\|u^n\|^{\frac{1}{n}}=\inf\|u^n\|^{\frac{1}{n}}.$$

If $\rho(u)<|C|$ and C is a complex constant, then $Ce-u$ is invertible in $\mathcal{A}$. Moreover,

$$(Ce-u)^{-1}=\sum_{i=0}^{\infty}\frac{u^i}{C^{i+1}}.$$

Lemma 2.5. [30] *Let $\mathcal{A}$ be a Banach algebra with a unit e and $u,v\in\mathcal{A}$. If u commutes with v,*

$$\rho(u+v)\leq\rho(u)+\rho(v),\quad \rho(uv)\leq\rho(u)\rho(v).$$

Lemma 2.6. [13] *Let $\{u_n\}$ be a sequence in $\mathcal{A}$ with $\{u_n\}\to\theta$ as $n\to\infty$. Then $\{u_n\}$ is a c-sequence.*

Lemma 2.7. [13] *Let P be a solid cone in Banach algebra $\mathcal{A}$ and $\{u_n\}$ be a c-sequence in P. If $\beta\in P$ is an arbitrarily given vector, then $\{\beta u_n\}$ is a c-sequence.*

Lemma 2.8. [13] *Let $\mathcal{A}$ be a Banach algebra with a unit e. Let $\alpha\in\mathcal{A}$ and $\rho(\alpha)<1$ Then $\{\alpha^n\}$ is a c-sequence.*

Lemma 2.9. [13] *Let $\mathcal{A}$ be a Banach algebra with a unit e and $u\in\mathcal{A}$. If $\rho(u)<|C|$ and C is a complex constant, then*

$$\rho((Ce-u)^{-1})\leq\frac{1}{|C|-\rho(u)}.$$

Lemma 2.10. [13] *Let $\mathcal{A}$ be a Banach algebra with a unit e and P be a solid cone in $\mathcal{A}$. Let $u,\alpha,\beta\in P$ hold $\alpha\preceq\beta$ and $u\preceq\alpha u$. If $\rho(\beta)<1$, then $u=\theta$.*

3. Main results

The following is the main result of this paper. We prove a common fixed point theorem in a complete cone b-metric space over Banach algebra $\mathcal{A}$. All of the results from the past until now can be obtained from this theorem.

Theorem 3.1. *Let (X,d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with constant $K \geq 1$ and P be a solid cone in $\mathcal{A}$. Suppose the mappings f and g are two self-maps of X satisfying*

$$d(fx,gy) \preceq ad(x,y) + b[d(x,fx)+d(y,gy)] + c[d(x,gy)+d(y,fx)] \tag{3.1}$$

for all $x,y \in X$, where $a,b,c \in P$ are some vectors commute together such that

$$K\rho(a) + (K+1)\rho(b) + (K^2+K)\rho(c) < 1. \tag{3.2}$$

Then f and g have a unique common fixed point in X. Also, any fixed point of f is a fixed point of g and conversely.

Proof. Let x_0 be an arbitrary point of X. Define $\{x_n\}$ by

$$x_1 = fx_0\,, x_2 = gx_1\,, \cdots, x_{2n+1} = fx_{2n}\,, x_{2n+2} = gx_{2n+1}$$

for $n = 0,1,2,\cdots$. Now, we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(fx_{2n},gx_{2n+1}) \\ &\preceq ad(x_{2n},x_{2n+1}) + b[d(x_{2n},fx_{2n}) + d(x_{2n+1},gx_{2n+1})] \\ &\quad + c[d(x_{2n},gx_{2n+1}) + d(x_{2n+1},fx_{2n})] \\ &= ad(x_{2n},x_{2n+1}) + b[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\ &\quad + c[d(x_{2n},x_{2n+2}) + d(x_{2n+1},x_{2n+1})] \\ &\preceq (a+b)d(x_{2n},x_{2n+1}) + bd(x_{2n+1},x_{2n+2}) \\ &\quad + Kc[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\ &= (a+b+Kc)d(x_{2n},x_{2n+1}) + (b+Kc)d(x_{2n+1},x_{2n+2}), \end{aligned}$$

which implies that

$$(e-(b+Kc))d(x_{2n+1},x_{2n+2}) \preceq (a+b+Kc)d(x_{2n},x_{2n+1}). \tag{3.3}$$

Similarly, we have

$$d(x_{2n+3},x_{2n+2}) \preceq (a+b+Kc)d(x_{2n+2},x_{2n+1}) + (b+Kc)d(x_{2n+3},x_{2n+2}),$$

which implies that

$$(e-(b+Kc))d(x_{2n+3},x_{2n+2}) \preceq (a+b+Kc)d(x_{2n+2},x_{2n+1}). \tag{3.4}$$

On the other hand, from Lemma 2.5, we get

$$\rho(b+Kc) \leq K\rho(a) + (K+1)\rho(b) + (K^2+K)\rho(c).$$

Using (3.2), it follows from Lemma 2.4 that $(e-b-cK)$ is invertible. Now, by (3.3) and (3.4), we have

$$d(x_n,x_{n+1}) \preceq \lambda d(x_{n-1},x_n) \preceq \lambda^2 d(x_{n-2},x_{n-1}) \cdots \preceq \lambda^n d(x_0,x_1)$$

for all $n \in \mathbb{N}$, where

$$\lambda = \frac{(a+b+Kc)}{e-(b+Kc)} = (a+b+Kc)(e-(b+Kc))^{-1}.$$

Take advantage of Lemma 2.5 and Lemma 2.9, it follows from (3.2) that

$$\rho(\lambda) = \rho\Big((a+b+cK)(e-b-cK)^{-1}\Big) \leq \frac{\rho(a)+\rho(b)+K\rho(c)}{\rho(e)-\rho(b)-K\rho(c)} < \frac{1}{K},$$

which implies that $\rho(K\lambda) = K\rho(\lambda) < 1$. Thus, by Lemma 2.4, $e - K\lambda$ is invertible and $(e-K\lambda)^{-1} = \sum_{i=0}^{\infty}(K\lambda)^i$.

Now, let $m, n \in \mathbb{N}$ with $m > n$. Then we get

$$\begin{aligned}
d(x_n, x_m) &\preceq K[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&\preceq Kd(x_n, x_{n+1}) + K^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&\preceq Kd(x_n, x_{n+1}) + K^2 d(x_{n+1}, x_{n+2}) + K^3[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_m)] \\
&\;\;\vdots \\
&\preceq Kd(x_n, x_{n+1}) + K^2 d(x_{n+1}, x_{n+2}) + \cdots + K^{m-n} d(x_{m-1}, x_m) \\
&\preceq K\lambda^n d(x_0, x_1) + K^2\lambda^{n+1} d(x_0, x_1) + \cdots + K^{m-n}\lambda^{m-1} d(x_0, x_1) \\
&\preceq K\lambda^n(e + K\lambda + K^2\lambda^2 + \cdots + K^{m-n-1}\lambda^{m-n-1}) d(x_0, x_1) \\
&\preceq K\lambda^n(e-K\lambda)^{-1} d(x_0, x_1).
\end{aligned}$$

On the other hand, since $\rho(\lambda) < \frac{1}{K}$ and $K \geq 1$, then $\rho(\lambda) < 1$. By Lemma 2.8, we conclude that $\{\lambda^n\}$ is a c-sequence. Now, using Lemma 2.7 and Lemma 2.3 (p_2), then $\{x_n\}$ is a Cauchy sequence. Since cone b-metric space X is complete, so there exists z in X such that $x_n \to z$ as $n \to \infty$. We prove that $gz = fz = z$. By (3.1), we get

$$\begin{aligned}
d(z, gz) &\preceq K[d(z, x_{2n+1}) + d(x_{2n+1}, gz)] \\
&= Kd(z, x_{2n+1}) + Kd(fx_{2n}, gz) \\
&\preceq Kd(z, x_{2n+1}) + K\big(ad(x_{2n}, z) + b[d(x_{2n}, fx_{2n}) + d(z, gz)] \\
&\quad + c[d(x_{2n}, gz) + d(z, fx_{2n})]\big) \\
&\preceq Kd(z, x_{2n+1}) + Kad(x_{2n}, z) + Kb[d(x_{2n}, x_{2n+1}) + d(z, gz)] \\
&\quad + Kc[K(d(x_{2n}, z) + d(z, gz)) + d(z, fx_{2n})] \\
&= K(1+c)d(z, x_{2n+1}) + K(a+cK)d(x_{2n}, z) + Kbd(x_{2n}, x_{2n+1}) \\
&\quad + K(b+cK)d(z, gz),
\end{aligned}$$

which yields that

$$\begin{aligned}
(e - K(b+cK))d(z, gz) \preceq\; & K(1+c)d(z, x_{2n+1}) + K(a+cK)d(x_{2n}, z) \\
& + bKd(x_{2n}, x_{2n+1}).
\end{aligned}$$

Since

$$\rho(K(b+cK)) \leq K\rho(a) + (K+1)\rho(b) + (K^2+K)\rho(c),$$

it follows from Lemma 2.4 and (3.2) that $e-K(b+cK)$ is invertible. Hence, we have

$$\begin{aligned} d(z,gz) \preceq (e-K(b+cK))^{-1}\Big(&K(1+c)d(z,x_{2n+1})\\ &+K(a+cK)d(x_{2n},z)+bKd(x_{2n},x_{2n+1})\Big). \end{aligned}$$

Because $\{x_n\}$ is a Cauchy sequence and $x_n \to z$ as $n \to \infty$, it means $\{d(z,x_{2n+1})\}$, $\{d(x_{2n},z)\}$ and $\{d(x_{2n},x_{2n+1}\}$ are c-sequences. Hence, by Lemma 2.3 (p_2), we get $d(z,gz)=\theta$. Thus, $gz=z$. Now, we have

$$\begin{aligned} d(fz,z) &= d(fz,gz)\\ &\preceq ad(z,z)+b[d(z,fz)+d(z,gz)]+c[d(z,gz)+d(z,fz)]\\ &=(b+c)d(fz,z). \end{aligned}$$

On the other hand, we have

$$\rho\big(K(b+c)\big) \le K\rho(a)+(K+1)\rho(b)+(K^2+K)\rho(c)<1$$

and $b+c\preceq K(b+c)$. From Lemma 2.10, we get $d(fz,z)=\theta$; that is, $fz=z$. Therefore, $fz=gz=z$ and z is a common fixed point of f and g. On the other hand, if z_1 is another common fixed point of f and g, then $fz_1=gz_1=z_1$ and

$$\begin{aligned} d(z,z_1) &= d(fz,gz_1)\\ &\preceq ad(z,z_1)+b[d(z,fz)+d(z_1,gz_1)]+c[d(z,gz_1)+d(z_1,fz)]\\ &=(a+2c)d(z,z_1). \end{aligned}$$

Note that

$$\rho\big(K(a+2c)\big) \le K\rho(a)+(K+1)\rho(b)+(K^2+K)\rho(c)<1$$

and $a+2c\preceq K(a+2c)$. From Lemma 2.10, we get $d(z,z_1)=\theta$; that is, $z=z_1$. □

In Theorem 3.1, set $K=1$. Then we obtain following result in the framework of cone metric spaces over Banach algebras.

Theorem 3.2. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and P be a solid cone in $\mathcal{A}$. Suppose the mappings f and g are two self-maps of X satisfying*

$$d(fx,gy)\preceq ad(x,y)+b[d(x,fx)+d(y,gy)]+c[d(x,gy)+d(y,fx)] \tag{3.5}$$

for all $x,y\in X$, where $a,b,c\in P$ are some vectors commute together such that

$$\rho(a)+2\rho(b)+2\rho(c)<1. \tag{3.6}$$

Then f and g have a unique common fixed point in X. Also, any fixed point of f is a fixed point of g and conversely.

The following result is a consequence of Theorem 3.1.

Corollary 3.3. *Let (X,d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with constant $K\ge 1$ and P be a solid cone in $\mathcal{A}$. Suppose a self-map f of X satisfies*

$$d(f^px,f^qy)\preceq ad(x,y)+b[d(x,f^px)+d(y,f^qy)]+c[d(x,f^qy)+d(y,f^px)] \tag{3.7}$$

for all $x, y \in X$, where p and q are fixed positive integers and $a, b, c \in P$ are some vectors commute together such that

$$K\rho(a) + (K+1)\rho(b) + (K^2+K)\rho(c) < 1. \tag{3.8}$$

Then f has a unique fixed point in X.

Proof. Set $f \equiv f^p$ and $g \equiv f^q$ in inequality (3.1) and use the Theorem 3.1. □

Theorem 3.4. *Let (X, d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with constant $K \geq 1$ and P be a solid cone in $\mathcal{A}$. Suppose a self-map f of X satisfies*

$$d(fx, fy) \preceq ad(x, y) + b[d(x, fx) + d(y, fy)] + c[d(x, fy) + d(y, fx)] \tag{3.9}$$

for all $x, y \in X$, where $a, b, c \in P$ are some vectors commute together such that

$$K\rho(a) + (K+1)\rho(b) + (K^2+K)\rho(c) < 1. \tag{3.10}$$

Then f has a unique fixed point in X.

Proof. In Corollary 3.3, set $p = q = 1$. □

Remark 3.5. In Theorem 3.1 and Theorem 3.4, let $a, b, c \in \mathbb{R}^+$. Then we can obtain same theorems in b-metric and cone b-metric spaces introduced by Kumam et al. [24]. Also, set $a, b, c \in \mathbb{R}^+$ in Theorem 3.2. Then we have well-known theorems in metric and cone metric spaces proved by Rus [31, 32]. Note that the used contractions in Theorems 3.1, 3.2 and 3.4 defined by Rus [31].

The following corollary are some consequences of Theorem 3.4.

Corollary 3.6. *Let (X, d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with constant $K \geq 1$ and P be a solid cone in $\mathcal{A}$. Suppose a self-map f of X satisfies in one of the following contractive conditions.*

(i) Banach contraction [6, 32]*: for all $x, y \in X$, where $a \in P$ with $\rho(a) < \frac{1}{K}$,*

$$d(fx, fy) \preceq ad(x, y); \tag{3.11}$$

(ii) Kannan contraction [29, 32]*: for all $x, y \in X$, where $b \in P$ with $\rho(b) < \frac{1}{K+1}$,*

$$d(fx, fy) \preceq b[d(x, fx) + d(y, fy)]; \tag{3.12}$$

(iii) Chatterjea contraction [29, 32]*: for all $x, y \in X$, where $c \in P$ with $\rho(c) < \frac{1}{K^2+K}$,*

$$d(fx, fy) \preceq c[d(x, fy) + d(y, fx)]; \tag{3.13}$$

(iv) Rus contraction [31, 32]*: for all $x, y \in X$, where $a, b \in P$ commute together and $K\rho(a) + (K+1)\rho(b) < 1$,*

$$d(fx, fy) \preceq ad(x, y) + b[d(x, fx) + d(y, fy)]; \tag{3.14}$$

(v) Rus contraction [31, 32]*: for all $x, y \in X$, where $a, c \in P$ commute together and $K\rho(a) + (K^2+K)\rho(c) < 1$,*

$$d(fx, fy) \preceq ad(x, y) + c[d(x, fy) + d(y, fx)]; \tag{3.15}$$

(vi) Hardy-Rogers contraction [29]*: for all $x, y \in X$, where $\alpha_i \in P$ for $i = 1, 2, \cdots, 5$ commute together and $2K\rho(\alpha_1) + (K+1)\rho(\alpha_2+\alpha_3) + (K^2+K)\rho(\alpha_4+\alpha_5) < 2$,*

$$d(fx, fy) \preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx). \tag{3.16}$$

Then f has a unique fixed point in X.

Example 3.7. Let X, $\mathcal{A}$, $\|\cdot\|$, P, d and K as in Example 2.2. Then (X,d) is a complete cone b-metric space over Banach Algebra $\mathcal{A}$. Also, let a mapping $f : X \to X$ be defined by $fx = \frac{\sqrt{3}}{4}x$ for all $x \in X$. Put $a = \frac{3}{16} + \frac{1}{4}t$. Then, we have

$$\begin{aligned} d(fx, fy)(t) &= \Big|\frac{\sqrt{3}}{4}x - \frac{\sqrt{3}}{4}y\Big|^2 2^t \\ &= \frac{3}{16}|x-y|^2 2^t \\ &\preceq ad(x,y)(t) \end{aligned}$$

for all $x, y, t \in X$, where $a \in P$ is a vector with $\rho(a) \leq \frac{7}{16} < \frac{1}{2}$. Thus, (3.11) is satisfied with $\rho(a) < \frac{1}{K}$. Hence, the conditions of Corollary 3.6 (i) are satisfied and so f has a unique fixed point $x = 0$.

Remark 3.8. In 2017, Huang et al. [15] applied $\rho(a) < 1$ instead of $\rho(a) < \frac{1}{K}$ and proved the same result of Corollary 3.6 (i). Also, in 2018, Ahmed and Salunke [3] proved this section of Corollary 3.6 by applying $\rho(a) < 1$. As a new work, it seems a researcher can be easily obtained all of the results of this paper with changing contraction conditions by considering the same techniques in [3, 15]. Further, Theorem 2.1 of Huang and Radenović [14] (or same Zamfirescu-type contraction [29]) comes from the Corollary 3.6 (i)-(iii). Also, Corollary 3.6 (i) is same Theorem 2.1 of Huang and Xu [16]. Moreover, if we consider $a, b, c \in \mathbb{R}^+$, then we can obtain old well-known theorems in b-metric and cone b-metric spaces.

In Corollary 3.6, set $K = 1$. Then we have the following corollary.

Corollary 3.9. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and P be a solid cone in $\mathcal{A}$. Suppose a self-map f of X satisfies in one of the following contractive conditions.*

(i) *Banach contraction* [6, 32]*: for all $x, y \in X$, where $a \in P$ with $\rho(a) < 1$,*

$$d(fx, fy) \preceq ad(x,y); \tag{3.17}$$

(ii) *Kannan contraction* [29, 32]*: for all $x, y \in X$, where $b \in P$ with $\rho(b) < \frac{1}{2}$,*

$$d(fx, fy) \preceq b[d(x, fx) + d(y, fy)]; \tag{3.18}$$

(iii) *Chatterjea contraction* [29, 32]*: for all $x, y \in X$, where $c \in P$ with $\rho(c) < \frac{1}{2}$,*

$$d(fx, fy) \preceq c[d(x, fy) + d(y, fx)]; \tag{3.19}$$

(iv) *Rus contraction* [31, 32]*: for all $x, y \in X$, where $a, b \in P$ commute together and $\rho(a) + 2\rho(b) < 1$,*

$$d(fx, fy) \preceq ad(x,y) + b[d(x, fx) + d(y, fy)]; \tag{3.20}$$

(v) *Rus contraction* [31, 32]*: for all $x, y \in X$, where $a, c \in P$ commute together and $\rho(a) + 2\rho(c) < 1$,*

$$d(fx, fy) \preceq ad(x,y) + c[d(x, fy) + d(y, fx)]; \tag{3.21}$$

(vi) *Rus contraction* [31]*: for all $x, y \in X$, where $a, b, c \in P$ commute together and $\rho(a) + 2\rho(b) + 2\rho(c) < 1$,*

$$d(fx, fy) \preceq ad(x,y) + b[d(x, fx) + d(y, fy)] + c[d(x, fy) + d(y, fx)]; \tag{3.22}$$

(*vii*) *Hardy-Rogers contraction* [29]*: for all* $x, y \in X$*, where* $\alpha_i \in P$ *for* $i = 1, 2, \cdots, 5$ *commute together and* $\rho(\alpha_1) + \rho(\alpha_2 + \alpha_3) + \rho(\alpha_4 + \alpha_5) < 1$,

$$d(fx, fy) \preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx). \quad (3.23)$$

Then f *has a unique fixed point in* X.

Remark 3.10. In 2013, Liu and Xu [25] obtained Corollary 3.9 (i) in a complete cone metric space over Banach algebra $\mathcal{A}$ with considering the normality of cone P. In 2014, Xu and Radenović [33] omitted the normality of cones by using c-sequences. Moreover, very recently, Huang et al. proved Corollary 3.9 (i) in Proposition 2.6, Theorem 2.20 and Corollary 2.21 of [12]. Further, consider Corollary 3.9 (i)-(iii) simultaneously. Then we can obtain same fixed point result in a complete cone metric space over Banach algebra $\mathcal{A}$ on Zamfirescu-type contraction [29, 32]. Also, in 2016, Yan et al. [34] considered ordered cone metric spaces over Banach algebras and obtained the results of Corollary 3.9 in such spaces. As the continuation of this article, the reader can be proved the results of this work in the framework of ordered cone b-metric spaces over Banach algebras as a new paper.

Remark 3.11. If we consider $a, b, c \in \mathbb{R}^+$, then we can obtain old well-known theorems in metric and cone metric spaces. In 2010, Altun et al. [4] proved ordered cone metric space version of Corollary 3.9 (vi) by considering the normality of cone P. Also, Abbas and Rhoades [2] obtained Corollary 3.9 in cone metric spaces with a normal cone P. Moreover, the mentioned corollary is hold in cone metric spaces with a non-normal solid cone P.

4. Periodic point results

Clearly, if f is a mapping which has a fixed point y, then y is also a fixed point of f^n for each $n \in \mathbb{N}$. However the converse need not be true [2]. Let $Fix(f)$ be the set of fixed points of f. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbb{N}$, then f is said to have property P [21]. Moreover, recall that two mappings $f, g : X \to X$ are said to have property Q if $Fix(f) \cap Fix(g) = Fix(f^n) \cap Fix(g^n)$ for each $n \in \mathbb{N}$. For more details on periodic point results, we refer to Rus' works [31, 32].

Theorem 4.1. *Let* (X, d) *be a complete cone b-metric space over Banach algebra* $\mathcal{A}$ *with constant* $K \geq 1$ *and* P *be a solid cone in* $\mathcal{A}$*. Suppose the mappings* f *and* g *are two self-maps of* X *satisfying* (3.1) *and* (3.2) *of Theorem* 3.1*. Then* f *and* g *have property Q.*

Proof. By Theorem 3.1, f and g have a unique common fixed point in X.
Let $z \in Fix(f^n) \cap Fix(g^n)$. From (3.1), we have

$$\begin{aligned} d(z, gz) &= d(f(f^{n-1}z), g(g^n z)) \\ &\preceq ad(f^{n-1}z, g^n z) + b[d(f^{n-1}z, f^n z) + d(g^n z, g^{n+1}z)] \\ &\quad + c[d(f^{n-1}z, g^{n+1}z) + d(g^n z, f^n z)] \\ &= ad(f^{n-1}z, z) + b[d(f^{n-1}z, z) + d(z, gz)] + cd(f^{n-1}z, gz), \end{aligned}$$

which implies that $d(z,gz) \preceq \lambda d(f^{n-1}z,z)$, where $\lambda = (a+b+cK)(e-b-cK)^{-1}$ (by a similar argument of Theorem 3.1). Thus, we get

$$d(z,gz) = d(f^n z, g^{n+1}z) \preceq \lambda d(f^{n-1}z,z) \preceq \cdots \preceq \lambda^n d(fz,z),$$

where $\rho(\lambda) < \frac{1}{K}$ by (3.2). Since $K \geq 1$, then $\rho(\lambda) < 1$. Hence, $\lambda^n d(fz,z)$ is a c-sequence. Thus, by Lemma 2.3 (p_2), we get $d(z,gz) = \theta$; that is, $gz = z$. Also, Theorem 3.1 implies that $fz = z$ and $z \in Fix(f) \cap Fix(g)$. This completes the proof. □

Corollary 4.2. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and P be a solid cone in $\mathcal{A}$. Suppose the mappings f and g are two self-maps of X satisfying* (3.5) *and* (3.6) *of Theorem* 3.2*. Then f and g have property Q.*

Theorem 4.3. *Let (X,d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with constant $K \geq 1$ and P be a solid cone in $\mathcal{A}$. Suppose the mapping f is a self-map of X satisfying* (3.9) *and* (3.10) *of Theorem* 3.4*. Then f has property P.*

Proof. Using Theorem 3.4, f has a unique fixed point in X. Let $z \in Fix(f^n)$. Then, from (3.9), we have

$$\begin{aligned} d(z,fz) &= d(f(f^{n-1}z), f(f^n z)) \\ &\preceq ad(f^{n-1}z, f^n z) + b[d(f^{n-1}z, f^n z) + d(f^n z, f^{n+1}z)] \\ &\quad + c[d(f^{n-1}z, f^{n+1}z) + d(f^n z, f^n z)] \\ &\preceq ad(f^{n-1}z,z) + b[d(f^{n-1}z,z) + d(z,fz)] \\ &\quad + cK[d(f^{n-1}z,z) + d(z,fz)], \end{aligned}$$

which implies that $d(z,fz) \preceq \lambda d(f^{n-1}z,z)$, where $\lambda = (a+b+cK)(e-b-cK)^{-1}$ (by a similar argument of Theorem 3.1). Thus, we have

$$d(z,fz) = d(f^n z, f^{n+1}z) \preceq \lambda d(f^{n-1}z,z) \preceq \cdots \preceq \lambda^n d(fz,z),$$

where $\rho(\lambda) < \frac{1}{K}$ by (3.10). Since $K \geq 1$, then $\rho(\lambda) < 1$. Hence, $\lambda^n d(fz,z)$ is a c-sequence. Thus, by Lemma 2.3 (p_2), we get $d(z,fz) = \theta$; that is, $fz = z$. Hence, $z \in Fix(f)$ and the proof is complete. □

Corollary 4.4. *Let (X,d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with constant $K \geq 1$ and P be a solid cone in $\mathcal{A}$. Suppose a self-map f satisfies any one of the inequalities of Corollary* 3.6*. Then f has property P.*

Corollary 4.5. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and P be a solid cone in $\mathcal{A}$. Suppose a self-map f satisfies any one of the inequalities Corollary* 3.9*. Then f has property P.*

Remark 4.6. Now, by applying the results of this section, all theorems and corollaries of [2, 12, 14, 21, 24, 31, 32] in related to properties P and Q can be obtained.

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