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A NOVEL ITERATIVE ALGORITHM WITH CONVERGENCE ANALYSIS FOR SPLIT COMMON FIXED POINTS AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. We propose a new algorithm which can be considered as a combination between the subgradient extragradient method and viscosity methods for solving split common fixed points problem and variational inequality problem. We find a point which belongs to the set of common fixed points of a finite family of demimetric mappings and the common solutions to a system of variational inequalities problem for a family of monotone and Lipschitz continuous operators in a Hilbert space such that its image under a linear transformation belongs to the set of common fixed points of a finite family of demimetric mappings in uniformly convex and smooth Banach space in the image space. The strong convergence of the sequences generated by the algorithm is proved. We also give some numerical results which show that our proposed algorithms are efficient and implementable from the numerical point of view.
Key Words and Phrases: Variational inequality, subgradient extragradient method, split common fixed point problems, demimetric mapping.
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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ be an operator. The classical variational inequality is formulated as the following problem:

$$\text{Finding a point} \quad x^\star \in C \quad \text{such that} \quad \langle Fx^\star, y - x^\star \rangle \geq 0, \qquad \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(C, F)$. Many problems in science and engineering can be recast as variational inequalities (see, for example, [17, 1, 32]). Several iterative methods have been developed for solving variational inequality and related optimization problems, see the books [2, 18, 15].

For solving the variational inequality in Euclidean space, Korpelevich [19] introduced the extragradient method where two metric projections onto feasible sets must be found at each iterative step. In 2011, Censor et al. [9] have replaced the second projection onto any closed convex set in the extragradient method by one onto a half-space and proposed the subgradient extragradient method for variational inequalities in Hilbert spaces, see also [10]. Recently, some authors proposed subgradient extragradient-like algorithms for solving variational inequality problems, see, for example, ([31, 20, 28, 29, 30]).

We recall the following definitions concerning an operator $F : \mathcal{H} \to \mathcal{H}$.

The operator F is called:

• L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

• Monotone if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}.$$

• Quasi-nonexpansive if

$$\|F(x) - p\| \leq \|x - p\|, \qquad \forall x \in \mathcal{H}, \quad \forall p \in Fix(F),$$

where $Fix(F) = \{x \in \mathcal{H} : F(x) = x\}$.

In [20], Kraikaew and Saejung presented an algorithm based on the subgradient extragradient method and the Halpern method for the problem of finding a common element of the solution set of a variational inequality and the fixed-point set of a quasi-nonexpansive mapping in real Hilbert spaces. In particular, they proved the following strong convergence theorem.

Theorem 1.1. *Let $S : \mathcal{H} \to \mathcal{H}$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $F : \mathcal{H} \to \mathcal{H}$ a monotone and L-Lipschitz mapping on C. Let τ be a positive real number such that $\tau L < 1$. Suppose that $Fix(S) \cap VI(C, F) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in \mathcal{H}, \\ y_n = P_C(x_n - \tau F(x_n)), \\ T_n = \{x \in \mathcal{H} : \langle x_n - \tau F(x_n) - y_n, x - y_n\rangle \leq 0\}, \\ z_n = \alpha_n x_0 + (1 - \alpha_n)P_{T_n}(x_n - \tau F(y_n)), \\ x_{n+1} = \beta_n z_n + (1 - \beta_n)Sz_n \qquad \forall n \geq 0, \end{cases} \tag{1.1}$$

where $\{\beta_n\} \subset [a, b] \subset]0, 1[$ for some $a, b \in]0, 1[$ and $\{\alpha_n\}$ is a sequence in $]0, 1[$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to $P_{VI(C,F)\bigcap Fix(S)}x_0$.

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) was recently introduced by Censor and Elfving [8] and is formulated as

$$to\, finding \quad x^* \in C \quad \text{such that} \quad \mathcal{A}x^* \in Q, \tag{1.2}$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Such models were successfully developed for instance in radiation therapy treatment planning, sensor networks, resolution enhancement and so on [3, 6, 7]. Initiated by SFP, several split type problems have been investigated and studied, for example, the split common fixed point problem (SCFP)[12], the split variational inequality problem (SVIP)[11], and the split null point problem (SCNP)[4]. Algorithms for solving the SFP receive great attention, (see [21-26] and references therein).

Recently, Takahashi [26, 27] introduced a new nonlinear mapping as follows: Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : E \to E$ with $Fix(U) \neq \emptyset$ is called $\eta-$demimetric if, for any $x \in E$ and $p \in Fix(U)$,

$$\langle x - p, J(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2,$$

where J is the duality mapping on E.

The demimetric mappings covers strict pseudo-contractions and generalized hybrid mappings in Hilbert spaces, and the metric projections and the metric resolvents in Banach spaces (see [26]). Such a class of operators is fundamental because it includes many types of nonlinear operators arising in applied mathematics and optimization.

Now in this paper, we propose a new algorithm which can be considered a combination between the subgradient extragradient method and viscosity methods for solving split common fixed points problem and variational inequality problem. We find a point which belongs to the set of common fixed points of a finite family of demimetric mappings and the common solutions to a system of variational inequalities problem for a family of monotone and Lipschitz continuous operators in a Hilbert space such that its image under a linear transformation belongs to the set of common fixed points of a finite family of demimetric mappings in uniformly convex and smooth Banach space in the image space. The strong convergence of the sequences generated by the algorithm is proved. We also give some numerical results which show that our proposed algorithms are efficient and implementable from the numerical point of view. Our results improve and generalize the results of Censor et al. [9, 11], Kraikaew and Saejung [20] and many others.

2. Preliminaries

Let E be a real Banach space with norm $\|.\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by:

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\},$$

for every ε with $0 \leq \varepsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. The duality mapping J from E into 2^{E^*} is defined by:

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

for every $x \in E$. Let $U = \{x \in E, \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [25]. Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. We know that for any $x \in E$, there exists a unique element $z \in C$, such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C. We will use the following Lemmas.

Lemma 2.1. ([20]) *Let $F : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz mapping on C and λ be a positive number and suppose that $VI(C, F)$ is nonempty. Let $x \in \mathcal{H}$. Define*

$$\begin{cases} U(x) = P_C(x - \lambda F(x)), \\ T^x = \{w \in H : \langle x - \lambda F(x) - U(x), w - U(x) \rangle \le 0\}, \\ V(x) = P_{T^x}(x - \lambda F(U(x))), \end{cases}$$

Then for all $x^* \in VI(C, F)$, we have

$$\|V(x) - x^*\|^2 \le \|x - x^*\|^2 - (1 - \lambda L)\|x - U(x)\|^2 - (1 - \lambda L)\|V(x) - U(x)\|^2.$$

Lemma 2.2. ([16]) *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n \delta_n, \quad n \ge 0, \\ s_{n+1} \le s_n - \eta_n + \mu_n, \quad n \ge 0, \end{cases}$$

where (λ_n) is a sequence in $(0, 1)$, (η_n) is a sequence of nonnegative real numbers and (δ_n) and (μ_n) are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(ii) $\lim_{n \to \infty} \mu_n = 0$
(iii) $\lim_{k \to \infty} \eta_{n_k} = 0$, *implies* $\limsup_{k \to \infty} \delta_{n_k} \le 0$ *for any subsequence* $(n_k) \subset (n)$.

Then $\lim_{n \to \infty} s_n = 0$.

Definition 2.3. Let $\mathcal{H}$ be a real Hilbert space. A mapping $U : \mathcal{H} \to \mathcal{H}$ is said to be β-strict pseudo-contractive if there exists a constant $\beta \in [0, 1)$ such that

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + \beta\|(x - Ux) - (y - Uy)\|^2, \qquad \forall x, y \in \mathcal{H}.$$

The following example show that the class of demimetric mappings is more general than the class of strict pseudo-contraction mappings and quasi-nonexpansive mappings.

Example 2.4. Let $\mathcal{H}$ be the real line. Define T on $\mathbb{R}$ by

$$T(x) = \begin{cases} \frac{4}{5} x \sin \frac{1}{2x}, & 0 < x \leq 2 \\ -2x, & o.w \end{cases}$$

Clearly, 0 is the only fixed point of T. We show that T is $\frac{1}{3}-$ demimetric mapping. For each $x \in \mathbb{R} - (0, 2]$ we have

$$3x^2 = \langle x - p, x - Tx \rangle = \frac{1-\eta}{2} \|x - Tx\|^2 = 3x^2.$$

For each $x \in (0, 2]$ since

$$1 \geq \frac{1}{3}\left(1 - \frac{4}{5} \sin \frac{1}{2x}\right)$$

we have

$$x^2 \left(1 - \frac{4}{5} \sin \frac{1}{2x}\right) = \langle x - p, x - Tx \rangle \geq \frac{1-\eta}{2} \|x - Tx\|^2 = \frac{1}{3} x^2 \left(1 - \frac{4}{5} \sin \frac{1}{2x}\right)^2.$$

Thus T is $\frac{1}{3}-$ demimetric mapping. Let $x = \frac{1}{\pi}$ and $y = \frac{1}{3\pi}$, for each $\beta < 1$ we have

$$\frac{256}{225\pi^2} = \|Tx - Ty\|^2 > \|x - y\|^2 + \beta \|(x - Tx) - (y - Ty)\|^2 = \frac{4}{9\pi^2} + \beta \frac{36}{225\pi^2}.$$

Therefore T is not strict pseudo-contractive mapping. Putting $p = 0$ and $x = 3$ we see that T is not quasi-nonexpansive.

Definition 2.5. Let $T : C \to C$ be a mapping, then $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, imply $x = Tx$.

Lemma 2.6. [22] *Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $T : C \to C$ be a β-strict pseudo-contractive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.7. [27]. *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $Fix(U)$ is closed and convex.*

3. Algorithm and convergence analysis

In this section, we present our algorithm for solving split common fixed points problem and variational inequality problem.

Theorem 3.1. *Let H be a Hilbert space and E be a uniformly convex and smooth Banach spaces. Let*

$$\{\mu^{(i)}\}_{i=1}^m, \{\kappa^{(i)}\}_{i=1}^m \subset (-\infty, 1).$$

Let for $i = 1, 2, ..., m$, $T^{(i)} : H \to H$ be a finite family of $\mu^{(i)}-$demimetric mappings and $S^{(i)} : E \to E$ be a finite family of $\kappa^{(i)}-$demimetric mappings.
Assume that $S^{(i)} - I$ and $T^{(i)} - I$ are demiclosed at 0. Let for each $i = 1, 2, .., m$,

$C^{(i)}$ be a nonempty closed convex subset of H and let $F^{(i)}: H \to H$ be a monotone and $L^{(i)}$- Lipschitz continuous operator on $C^{(i)}$. Let $A: H \to E$ be a bounded linear operator such that $A \neq 0$. Suppose that

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{m} (VI(C^{(i)}, F^{(i)}) \cap Fix(T^{(i)})) : \ Ax^* \in \bigcap_{i=1}^{m} Fix(S^{(i)}) \right\} \neq \emptyset.$$

Assume that f be a contraction of H into itself with constant $k \in (0,1)$. Let $\{x_n\}$ be a sequence defined by:

$$\begin{cases} x_1 \in H \ \ \text{is chosen arbitrarily}, \\ z_n = x_n - \sum_{i=1}^{m} \alpha_n^{(i)} \theta^{(i)} A^* J_E(Ax_n - S^{(i)} Ax_n) \\ u_n^{(i)} = P_{C^{(i)}}(z_n - \lambda^{(i)} F^{(i)}(z_n)), \\ U_n^{(i)} = \{x \in H : \langle z_n - \lambda^{(i)} F^{(i)}(z_n) - u_n^{(i)}, x - u_n^{(i)} \rangle \leq 0\}, \\ v_n^{(i)} = P_{U_n^{(i)}}(z_n - \lambda^{(i)} F^{(i)}(u_n^{(i)})), \\ w_n^{(i)} = v_n^{(i)} + \dfrac{1-\mu^{(i)}}{3}(T^{(i)} v_n^{(i)} - v_n^{(i)}), \quad i \in \{1, 2, ..., m\}, \\ x_{n+1} = \beta_n^{(0)} f(x_n) + \sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)}, \qquad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\lambda^{(i)}\}$ and $\{\theta^{(i)}\}$ satisfy the following conditions:

(i) $\{\alpha_n^{(i)}\}_{i=1}^m, \{\beta_n^{(i)}\}_{i=0}^m \subset (0,1)$ *and* $\sum_{i=1}^{m} \alpha_n^{(i)} = \sum_{i=0}^{m} \beta_n^{(i)} = 1$,

(ii) $\liminf_n \alpha_n^{(i)} > 0$, $\liminf_n \beta_n^{(i)} > 0$ *for each* $i \in \{1, 2, ..., m\}$,

(iii) $\lim_{n\to\infty} \beta_n^{(0)} = 0$ *and* $\sum_{n=0}^{\infty} \beta_n^{(0)} = \infty$,

(iv) $\lambda^{(i)} L^{(i)} < 1$ *and* $0 < \theta^{(i)} \leq \dfrac{1-\kappa^{(i)}}{\|A\|^2}$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^\star \in \Omega$ which solves the variational inequality:

$$\langle x^\star - f(x^\star), x - x^\star \rangle \geq 0, \qquad \forall x \in \Omega. \tag{3.2}$$

Proof. First we show that $\{x_n\}$ is bounded. Note that $P_\Omega(f)$ is a contraction of H into itself. By the Banach contraction principle there exists a unique element $x^\star \in H$ such that $x^\star = P_\Omega(f)x^\star$. Since for each $i \in \{1, 2, ..., m\}$, $S^{(i)} : E \to E$ is $\kappa^{(i)}-$demimetric mapping, from the convexity of $\|.\|^2$ we have

$$\begin{aligned}
\|z_n - x^\star\|^2 &= \|x_n - \sum_{i=1}^{m} \alpha_n^{(i)}\, \theta^{(i)}\, A^* J_E(Ax_n - S^{(i)} Ax_n) - x^\star\|^2 \\
&\le \sum_{i=1}^{m} \alpha_n^{(i)} \|x_n - x^\star - \theta^{(i)} A^* J_E(Ax_n - S^{(i)} Ax_n)\|^2 \\
&= \sum_{i=1}^{m} \alpha_n^{(i)} (\|x_n - x^\star\|^2 - 2\langle x_n - x^\star, \theta^{(i)} A^* J_E(Ax_n - S^{(i)} Ax_n)\rangle \\
&\quad + \|\theta^{(i)} A^* J_E(Ax_n - S^{(i)} Ax_n)\|^2) \\
&\le \sum_{i=1}^{m} \alpha_n^{(i)} (\|x_n - x^\star\|^2 - 2\theta^{(i)} \langle Ax_n - Ax^\star, J_E(Ax_n - S^{(i)} Ax_n)\rangle \\
&\quad + (\theta^{(i)})^2 \|A\|^2 \|J_E(Ax_n - S^{(i)} Ax_n)\|^2) \\
&\le \sum_{i=1}^{m} \alpha_n^{(i)} (\|x_n - x^\star\|^2 - (1 - \kappa^{(i)})\theta^{(i)} \|Ax_n - S^{(i)} Ax_n\|^2 \\
&\quad + (\theta^{(i)})^2 \|A\|^2 \|Ax_n - S^{(i)} Ax_n\|^2) \\
&= \|x_n - x^\star\|^2 + \sum_{i=1}^{m} \alpha_n^{(i)}\, \theta^{(i)} (\theta^{(i)} \|A\|^2 - (1 - \kappa^{(i)})) \|Ax_n - S^{(i)} Ax_n\|^2.
\end{aligned} \tag{3.3}$$

Since $0 < \theta^{(i)} \le \dfrac{1 - \kappa^{(i)}}{\|A\|^2}$ for all $i \in \{1, 2, ..., m\}$, we have that

$$\|z_n - x^\star\| \le \|x_n - x^\star\|.$$

Utilizing Lemma (2.1) for each $i \in \{1, 2, ..., m\}$ we have

$$\|v_n^{(i)} - x^\star\|^2 \le \|z_n - x^\star\|^2 - (1 - \lambda^{(i)} L^{(i)}) \|z_n - u_n^{(i)}\|^2 - (1 - \lambda^{(i)} L^{(i)}) \|v_n^{(i)} - u_n^{(i)}\|^2. \tag{3.4}$$

Since for each $i \in \{1, 2, ..., m\}$, $T^{(i)}$ is $\mu^{(i)}$ - demicontractive, we arrive at

$$\begin{aligned}
\|w_n^{(i)} - x^\star\|^2 &= \|v_n^{(i)} + \frac{1 - \mu^{(i)}}{3} (T^{(i)} v_n^{(i)} - v_n^{(i)}) - x^\star\|^2 \\
&= \|v_n^{(i)} - x^\star\|^2 + 2\langle v_n^{(i)} - x^\star, \frac{1 - \mu^{(i)}}{3} (T^{(i)} v_n^{(i)} - v_n^{(i)})\rangle \\
&\quad + \|\frac{1 - \mu^{(i)}}{3} (T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 \\
&\le \|v_n^{(i)} - x^\star\|^2 - 2 \left(\frac{1 - \mu^{(i)}}{3}\right) \left(\frac{1 - \mu^{(i)}}{2}\right) \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 \\
&\quad + \left(\frac{1 - \mu^{(i)}}{3}\right)^2 \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 \\
&= \|v_n^{(i)} - x^\star\|^2 - \frac{2}{9} (1 - \mu^{(i)})^2 \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2.
\end{aligned} \tag{3.5}$$

Thus we get that

$$\|w_n^{(i)} - x^\star\| \le \|v_n^{(i)} - x^\star\| \le \|z_n - x^\star\| \le \|x_n - x^\star\|. \tag{3.6}$$

Now, since f is a contraction, from the algorithm (3.1) and the inequality (3.6) we obtain

$$\begin{aligned}
\|x_{n+1} - x^\star\| &= \|\beta_n^{(0)} f(x_n) + \sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)} - x^\star\| \\
&\le \beta_n^{(0)} \|f(x_n) - x^\star\| + \sum_{i=1}^{m} \beta_n^{(i)} \|w_n^{(i)} - x^\star\| \\
&\le \beta_n^{(0)} \|f(x_n) - x^\star\| + \sum_{i=1}^{m} \beta_n^{(i)} \|x_n - x^\star\| \\
&\le \beta_n^{(0)} \|f(x_n) - x^\star\| + (1 - \beta_n^{(0)}) \|x_n - x^\star\| \\
&\le \beta_n^{(0)} \|f(x_n) - f(x^\star)\| + \beta_n^{(0)} \|f(x^\star) - x^\star\| + (1 - \beta_n^{(0)}) \|x_n - x^\star\| \\
&\le \beta_n^{(0)} k \|x_n - x^\star\| + \beta_n^{(0)} \|f(x^\star) - x^\star\| + (1 - \beta_n^{(0)}) \|x_n - x^\star\| \\
&\le (1 - (1-k))\beta_n^{(0)} \|x_n - x^\star\| + (1-k) \frac{\beta_n^{(0)}}{1-k} \|f(x^\star) - x^\star\| \\
&\le \max\{\|x_n - x^\star\|, \frac{1}{1-k} \|f(x^\star) - x^\star\|\} \\
&\le \ldots \\
&\le \max\{\|x_0 - x^\star\|, \frac{1}{1-k} \|f(x^\star) - x^\star\|\},
\end{aligned}$$

which implies that $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{w_n^{(i)}\}$ are bounded. Since

$$x_{n+1} = \beta_n^{(0)} f(x_n) + \sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)},$$

applying inequality

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$$

in a Hilbert space, we arrive at

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &\le \| \sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)} - (1 - \beta_n^{(0)}) x^\star\|^2 + 2\beta_n^{(0)} \langle f(x_n) - x^\star, x_{n+1} - x^\star\rangle \\
&= \left\| (1 - \beta_n^{(0)}) \left(\sum_{i=1}^{m} \frac{\beta_n^{(i)}}{1 - \beta_n^{(0)}} w_n^{(i)} - x^\star \right) \right\|^2 + 2\beta_n^{(0)} \langle f(x_n) - x^\star, x_{n+1} - x^\star\rangle \\
&\le (1 - \beta_n^{(0)})^2 \sum_{i=1}^{m} \frac{\beta_n^{(i)}}{1 - \beta_n^{(0)}} \|w_n^{(i)} - x^\star\|^2 + 2\beta_n^{(0)} \langle f(x_n) - x^\star, x_{n+1} - x^\star\rangle
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\beta_n^{(0)})\sum_{i=1}^{m}\beta_n^{(i)}\|x_n-x^\star\|^2+2\beta_n^{(0)}\langle f(x_n)-f(x^\star),x_{n+1}-x^\star\rangle\\
&+2\beta_n^{(0)}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle\\
&=(1-\beta_n^{(0)})^2\|x_n-x^\star\|^2+2\beta_n^{(0)}k\|x_n-x^\star\|\|x_{n+1}-x^\star\|\\
&+2\beta_n^{(0)}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle\\
&\le (1-\beta_n^{(0)})^2\|x_n-x^\star\|^2+\beta_n^{(0)}k(\|x_n-x^\star\|^2+\|x_{n+1}-x^\star\|^2)\\
&+2\beta_n^{(0)}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1}-x^\star\|^2 &\le \frac{(1-\beta_n^{(0)})^2+\beta_n^{(0)}k}{1-\beta_n^{(0)}k}\|x_n-x^\star\|^2\\
&+\frac{2\beta_n^{(0)}}{1-\beta_n^{(0)}k}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle\\
&=\frac{1-2\beta_n^{(0)}+\beta_n^{(0)}k}{1-\beta_n^{(0)}k}\|x_n-x^\star\|^2+\frac{(\beta_n^{(0)})^2}{1-\beta_n^{(0)}k}\|x_n-x^\star\|^2\\
&+\frac{2\beta_n^{(0)}}{1-\beta_n^{(0)}k}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle\\
&\le\left(1-\frac{2(1-k)\beta_n^{(0)}}{1-\beta_n^{(0)}k}\right)\|x_n-x^\star\|^2\\
&+\frac{2(1-k)\beta_n^{(0)}}{1-\beta_n^{(0)}k}\left\{\frac{\beta_n^{(0)}M}{2(1-k)}+\frac{1}{1-k}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle\right\}\\
&\le (1-\eta_n)\|x_n-x^\star\|^2+\eta_n\delta_n, \qquad (3.7)
\end{aligned}
$$

where

$$
\delta_n=\frac{\beta_n^{(0)}M}{2(1-k)}+\frac{1}{1-k}\langle f(x^\star)-x^\star,x_{n+1}-x^\star\rangle,
$$

$$
M=\sup\{\|x_n-x^\star\|^2 : n\ge 0\} \text{ and } \eta_n=\frac{2(1-k)\beta_n^{(0)}}{1-\beta_n^{(0)}k}.
$$

We observe that

$$
\eta_n\to 0,\ \sum_{n=1}^{\infty}\eta_n=\infty.
$$

By using inequalities (3.3), (3.4) and (3.5), and the convexity of $\|.\|^2$ we get

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\beta_n^{(0)} f(x_n) + \sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)} - x^\star\|^2 \\
&\leq \beta_n^{(0)} \|f(x_n) - x^\star\|^2 + \sum_{i=1}^{m} \beta_n^{(i)} \|w_n^{(i)} - x^\star\|^2 \\
&\leq \beta_n^{(0)} \|f(x_n) - x^\star\|^2 + \sum_{i=1}^{m} \beta_n^{(i)} \|v_n^{(i)} - x^\star\|^2 \\
&- \sum_{i=1}^{m} \beta_n^{(i)} \frac{2}{9}(1-\mu^{(i)})^2 \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 \\
&\leq \beta_n^{(0)} \|f(x_n) - x^\star\|^2 + (1-\beta_n^{(0)}) \|z_n - x^\star\|^2 \\
&- \sum_{i=1}^{m} \beta_n^{(i)} \frac{2}{9}(1-\mu^{(i)})^2 \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 \\
&- \sum_{i=1}^{m} \beta_n^{(i)} (1-\lambda^{(i)} L^{(i)}) \|z_n - u_n^{(i)}\|^2 - \sum_{i=1}^{m} \beta_n^{(i)} (1-\lambda^{(i)} L^{(i)}) \|v_n^{(i)} - u_n^{(i)}\|^2 \\
&\leq \beta_n^{(0)} \|f(x_n) - x^\star\|^2 + (1-\beta_n^{(0)}) \|x_n - x^\star\|^2 \\
&- (1-\beta_n^{(0)}) \sum_{i=1}^{m} \alpha_n^{(i)} \theta^{(i)} ((1-\kappa^{(i)}) \\
&- \theta^{(i)} \|A\|^2) \|Ax_n - S^{(i)} Ax_n\|^2 - \sum_{i=1}^{m} \beta_n^{(i)} \frac{2}{9}(1-\mu^{(i)})^2 \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 \\
&- \sum_{i=1}^{m} \beta_n^{(i)} (1-\lambda^{(i)} L^{(i)}) \|z_n - u_n^{(i)}\|^2 \\
&- \sum_{i=1}^{m} \beta_n^{(i)} (1-\lambda^{(i)} L^{(i)}) \|v_n^{(i)} - u_n^{(i)}\|^2. \qquad (3.8)
\end{aligned}$$

Now by setting

$$\begin{aligned}
\xi_n &= (1-\beta_n^{(0)}) \sum_{i=1}^{m} \alpha_n^{(i)} \theta^{(i)} ((1-\kappa^{(i)}) - \theta^{(i)} \|A\|^2) \|Ax_n - S^{(i)} Ax_n\|^2 \\
&+ \sum_{i=1}^{m} \beta_n^{(i)} \frac{2}{9}(1-\mu^{(i)})^2 \|(T^{(i)} v_n^{(i)} - v_n^{(i)})\|^2 + \sum_{i=1}^{m} \beta_n^{(i)} (1-\lambda^{(i)} L^{(i)}) \|z_n - u_n^{(i)}\|^2 \\
&+ \sum_{i=1}^{m} \beta_n^{(i)} (1-\lambda^{(i)} L^{(i)}) \|v_n^{(i)} - u_n^{(i)}\|^2. \qquad (3.9)
\end{aligned}$$

and

$$\zeta_n = \beta_n^{(0)} \|f(x_n) - x^\star\|^2, \quad s_n = \|x_n - x^\star\|^2, \qquad (3.10)$$

the inequality (3.8) can be rewritten in the following form:

$$s_{n+1} \leq s_n - \xi_n + \zeta_n. \tag{3.11}$$

To use Lemma 2.2 (considering inequalities (3.7) and (3.11)), it suffices to verify that, for all subsequences $\{n_k\} \subset \{n\}$, $\lim\limits_{k\to\infty} \xi_{n_k} = 0$ implies

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0.$$

We assume that $\lim\limits_{k\to\infty} \xi_{n_k} = 0$. By our assumption that

$$0 < \theta^{(i)} \leq \frac{1-\kappa_i}{\|A\|^2} \text{ and } \liminf_{n\to\infty} \alpha_n^{(i)} > 0,$$

we get that

$$\lim_{k\to\infty} \|Ax_{n_k} - S^{(i)}Ax_{n_k}\| = 0. \tag{3.12}$$

By similar argument, we obtain

$$\lim_{k\to\infty} \|T^{(i)}v_{n_k}^{(i)} - v_{n_k}^{(i)}\| = \lim_{k\to\infty} \|v_{n_k}^{(i)} - u_{n_k}^{(i)}\| = \lim_{k\to\infty} \|u_{n_k}^{(i)} - x_{n_k}\| = 0. \tag{3.13}$$

Thus

$$\lim_{k\to\infty} \|v_{n_k}^{(i)} - x_{n_k}\| = 0.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \widehat{x}$. Since $\lim\limits_{k\to\infty} \|v_{n_k}^{(i)} - x_{n_k}\| = 0$, we have $v_{n_k}^{(i)} \rightharpoonup \widehat{x}$. From the demiclosedness of $I - T^{(i)}$ we have $\widehat{x} \in Fix(T^{(i)})$. From (3.12) and the demiclosedness of $I - S^{(i)}$ we have $A\widehat{x} \in Fix(S^{(i)})$. Now we show that $\widehat{x} \in VI(C^{(i)}, F^{(i)})$, for each $1 \leq i \leq m$. From $u_n^{(i)} = P_{C^{(i)}}(z_n - \lambda^{(i)}F^{(i)}(z_n))$, by the variational characterization of the metric projection onto $C^{(i)}$, we have

$$\langle x - u_n^{(i)}, z_n - \lambda^{(i)}F^{(i)}(z_n) - u_n^{(i)}\rangle \leq 0, \quad \forall x \in C^{(i)}. \tag{3.14}$$

Since, $F^{(i)}$ is monotone, for each $x \in C^{(i)}$ we get

$$\langle \lambda^{(i)}F^{(i)}(x), z_n - x\rangle \leq \langle \lambda^{(i)}F^{(i)}(z_n), z_n - x\rangle. \tag{3.15}$$

Utilizing the inequalities (3.14) and (3.15) we have

$$\begin{aligned}
\langle \lambda^{(i)}F^{(i)}(x), z_n - x\rangle &\leq \langle \lambda^{(i)}F^{(i)}(z_n), z_n - x\rangle \\
&= \langle \lambda^{(i)}F^{(i)}(z_n), z_n - u_n^{(i)}\rangle + \langle \lambda^{(i)}F^{(i)}(z_n), u_n^{(i)} - x\rangle \\
&= \langle \lambda^{(i)}F^{(i)}(z_n), z_n - u_n^{(i)}\rangle + \langle z_n - u_n^{(i)}, u_n^{(i)} - x\rangle \\
&+ \langle \lambda^{(i)}F^{(i)}(z_n) - z_n + u_n^{(i)}, u_n^{(i)} - x\rangle \\
&\leq \lambda^{(i)}\langle F^{(i)}(z_n), z_n - u_n^{(i)}\rangle + \langle z_n - u_n^{(i)}, u_n^{(i)} - x\rangle \\
&\leq \lambda^{(i)}\|F^{(i)}(z_n)\|\|z_n - u_n^{(i)}\| + \|z_n - u_n^{(i)}\|\|u_n^{(i)} - x\|.
\end{aligned} \tag{3.16}$$

Hence

$$\langle F^{(i)}x, z_n - x\rangle \leq \|F^{(i)}(z_n)\|\|z_n - u_n^{(i)}\| + \frac{1}{\lambda^{(i)}}\|z_n - u_n^{(i)}\|\|u_n^{(i)} - x\|.$$

Since $\{F^{(i)}(z_n)\}$ is bounded, $z_{n_l} - u_{n_l}^{(i)} \to 0$ and $z_{n_l} \rightharpoonup \widehat{x}$, we have

$$\langle F^{(i)}(x), \widehat{x} - x\rangle = \lim_{l\to\infty}\langle F^{(i)}(x), z_{n_l} - x\rangle \leq 0, \qquad \forall x \in C^{(i)}.$$

This implies that $\widehat{x} \in VI(C^{(i)}, F^{(i)})$. Thus $\widehat{x} \in \Omega$. Now we show that

$$\limsup_{k\to\infty}\delta_{n_k} = \limsup_{k\to\infty}\langle f(x^\star) - x^\star, x_{n_k} - x^\star\rangle \leq 0. \tag{3.17}$$

To show this inequality, we choose a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\langle f(x^\star) - x^\star, x_{n_{k_j}} - x^\star\rangle = \limsup_{k\to\infty}\langle f(x^\star) - x^\star, x_{n_k} - x^\star\rangle.$$

Since $\{x_{n_{k_j}}\}$ converges weakly to $\widehat{x}$, it follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(x^\star) - x^\star, x_{n_k} - x^\star\rangle &= \lim_{j\to\infty}\langle f(x^\star) - x^\star, x_{n_{k_j}} - x^\star\rangle\\ &= \langle f(x^\star) - x^\star, \widehat{x} - x^\star\rangle \leq 0.\end{aligned} \tag{3.18}$$

Hence, all conditions of Lemma 2.2 are satisfied. Therefore, we immediately deduce that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\|x_n - x^\star\| = 0,$$

that is $\{x_n\}$ converges strongly to $x^\star = P_\Omega(f(x^\star))$, which completes the proof. □

Theorem 3.2. *Let H and E be two Hilbert spaces and let $A : H \to E$ be a bounded linear operator. Let for each $i = 1, 2, .., m$, $C^{(i)}$ be a nonempty closed convex subset of H and let $F^{(i)} : H \to H$ be a monotone and $L^{(i)}$- Lipschitz continuous operator on $C^{(i)}$. Let $\{\mu^{(i)}\}_{i=1}^m, \{\kappa^{(i)}\}_{i=1}^m \subset (-\infty, 1)$. Let for $i = 1, 2, ..., m$, $T^{(i)} : H \to H$ be a finite family of $\mu^{(i)}-$strict pseudo-contractive mappings and $S^{(i)} : E \to E$ be a finite family of $\kappa^{(i)}-$strict pseudo-contractive mappings. Suppose that*

$$\Omega = \left\{x^* \in \bigcap_{i=1}^m (VI(C^{(i)}, F^{(i)}) \cap Fix(T^{(i)})) : \quad Ax^* \in \bigcap_{i=1}^m Fix(S^{(i)})\right\} \neq \emptyset.$$

Assume that f be a contraction of H into itself with constant $k \in (0,1)$. Let $\{x_n\}$ be a sequence defined by:

$$\begin{cases} x_1 \in H \ \ is \ \ chosen \ \ arbitrarily, \\ z_n = x_n - \sum_{i=1}^{m} \alpha_n^{(i)} \, \theta^{(i)} \, A^*(Ax_n - S^{(i)}Ax_n) \\ u_n^{(i)} = P_{C^{(i)}}(z_n - \lambda^{(i)} F^{(i)}(z_n)), \\ U_n^{(i)} = \{x \in H : \langle z_n - \lambda^{(i)} F^{(i)}(z_n) - u_n^{(i)}, x - u_n^{(i)} \rangle \leq 0\}, \\ v_n^{(i)} = P_{U_n^{(i)}}(z_n - \lambda^{(i)} F^{(i)}(u_n^{(i)})), \\ w_n^{(i)} = v_n^{(i)} + \dfrac{1-\mu^{(i)}}{3}(T^{(i)} v_n^{(i)} - v_n^{(i)}), \quad i \in \{1, 2, ..., m\}, \\ x_{n+1} = \beta_n^{(0)} \, f(x_n) + \sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)}, \qquad \forall n \geq 0, \end{cases} \tag{3.19}$$

where $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\lambda^{(i)}\}$ and $\{\theta^{(i)}\}$ satisfy the following conditions:

(i) $\{\alpha_n^{(i)}\}_{i=1}^m, \{\beta_n^{(i)}\}_{i=0}^m \subset (0,1)$ *and* $\sum_{i=1}^{m} \alpha_n^{(i)} = \sum_{i=0}^{m} \beta_n^{(i)} = 1$,

(ii) $\liminf_n \alpha_n^{(i)} > 0$, $\liminf_n \beta_n^{(i)} > 0$ *for each* $i \in \{1, 2, ..., m\}$,

(iii) $\lim_{n \to \infty} \beta_n^{(0)} = 0$ *and* $\sum_{n=0}^{\infty} \beta_n^{(0)} = \infty$,

(iv) $\lambda^{(i)} \, L^{(i)} < 1$ *and* $0 < \theta^{(i)} \leq \dfrac{1-\kappa^{(i)}}{\|A\|^2}$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^\star \in \Omega$ which solves the variational inequality;

$$\langle x^\star - f(x^\star), x - x^\star \rangle \geq 0, \qquad \forall x \in \Omega. \tag{3.20}$$

Proof. We note that every α-strict pseudo-contractive mappings is α-demimetric (see [7]). Also, from Lemma (2.7) we know that for each strict pseudo-contractive mapping U, U is demiclosed (see [23]). Thus we obtain the desired result by Theorem 3.2. $\square$

Theorem 3.3. *Let H be a Hilbert space. Let for each $i = 1, 2, .., m$, $C^{(i)}$ be a nonempty closed convex subset of H and let $F^{(i)} : H \to H$ be a monotone and $L^{(i)}$-Lipschitz continuous operator on $C^{(i)}$. Let $\{\mu^{(i)}\}_{i=1}^m \subset (-\infty, 1)$ and $\{T^{(i)}\}_{i=1}^m : H \to H$ be a finite family of $\mu^{(i)}-$demimetric mappings such that $T^{(i)} - I$ are demiclosed at 0. Suppose that*

$$\Omega = \bigcap_{i=1}^{m} (VI(C^{(i)}, F^{(i)}) \cap Fix(T^{(i)})) \neq \emptyset.$$

Assume that f be a contraction of H into itself with constant $k \in (0,1)$. Let $\{x_n\}$ be a sequence defined by:

$$\begin{cases} x_1 \in H \ \ is \ \ chosen \ \ arbitrarily, \\ u_n^{(i)} = P_{C^{(i)}}(x_n - \lambda^{(i)} F^{(i)}(x_n)), \\ U_n^{(i)} = \{x \in H : \langle x_n - \lambda^{(i)} F^{(i)}(x_n) - u_n^{(i)}, x - u_n^{(i)} \rangle \leq 0\}, \\ v_n^{(i)} = P_{U_n^{(i)}}(x_n - \lambda^{(i)} F^{(i)}(u_n^{(i)})), \\ w_n^{(i)} = v_n^{(i)} + \dfrac{1-\mu^{(i)}}{3}(T^{(i)} v_n^{(i)} - v_n^{(i)}), \quad i \in \{1,2,...,m\}, \\ x_{n+1} = \beta_n^{(0)} f(x_n) + \displaystyle\sum_{i=1}^{m} \beta_n^{(i)} w_n^{(i)}, \qquad \forall n \geq 0, \end{cases} \tag{3.21}$$

where $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}$ and $\{\lambda^{(i)}\}$ satisfy the following conditions:

(i) $\{\beta_n^{(i)}\}_{i=0}^m \subset (0,1)$ *and* $\displaystyle\sum_{i=0}^{m} \beta_n^{(i)} = 1$,

(ii) $\liminf_n \beta_n^{(i)} > 0$ *for each* $i \in \{1,2,...,m\}$,

(iii) $\lim_{n\to\infty} \beta_n^{(0)} = 0$ *and* $\displaystyle\sum_{n=0}^{\infty} \beta_n^{(0)} = \infty$,

(iv) $\lambda^{(i)} L^{(i)} < 1$ *for each* $i \in \{1,2,...,m\}$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^\star \in \Omega$ which solves the variational inequality;

$$\langle x^\star - f(x^\star), x - x^\star \rangle \geq 0, \qquad \forall x \in \Omega. \tag{3.22}$$

4. Numerical experiments

In this section, we provide some concrete example including numerical results of the problem considered in Section 3 of this paper. All codes were written in MATLAB R2016a and run on DELL i5 Dual-Core 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let $E = H = L^2([\alpha,\beta])$ in (3.19) with $m = 2$. Let

$$\beta_n^{(i)} = \frac{n}{2(n+1)}, \ i = 1,2 \ \beta_n^{(0)} = \frac{1}{n+1}, \ i = 1,2.$$

We take

$$\alpha_n^{(1)} = \frac{n}{2(n+1)} \text{ and } \alpha_n^{(2)} = \frac{n+2}{2(n+1)}.$$

Furthermore, let us take

$$C^{(1)} := \{x \in L^2([\alpha,\beta]) : \langle a, x \rangle \leq b\},$$

where $0 \neq a \in L^2([\alpha,\beta])$ and $b \in \mathbb{R}$, then (see, for example, [5])

$$P_{C^{(1)}}(x) = \begin{cases} \dfrac{b - \langle a, x\rangle}{||a||_{L^2}^2} a + x, & \langle a, x\rangle > b \\ x, & \langle a, x\rangle \leq b. \end{cases}$$

Let

$$C^{(2)} = \{x \in L^2([\alpha, \beta]) : ||x - d||_{L^2} \leq r\},$$

be a closed ball centered at $d \in L^2([\alpha, \beta])$ with radius $r > 0$, then

$$P_{C^{(2)}}(x) = \begin{cases} d + r\dfrac{x-d}{||x-d||}, & x \notin C^{(2)} \\ x, & x \in C^{(2)}. \end{cases}$$

Suppose that $A : L^2([0,1]) \to L^2([0,1])$ is defined by

$$(Ax)(t) = \int_0^t x(s)ds, \ \forall x \in L^2([0,1]).$$

It can be easily shown that A is a bounded linear operator with $\|A\| = \dfrac{2}{\pi}$ and the adjoint A^* of A is defined by

$$(A^*x)(t) = \int_t^1 x(s)ds, \ \forall x \in L^2([0,1]).$$

Now, suppose

$$C^{(1)} := \left\{x \in L^2([0,1]) : \int_0^1 (t^2+1)x(t)dt \leq 1\right\}$$

and

$$C^{(2)} = \left\{x \in L^2([0,1]) : \int_0^1 |x(t) - \sin t|^2 dt \leq 16\right\}.$$

Assume that $S^{(i)} := I$, $i = 1,2$, where I is the identity mapping, $T^{(i)} := P_{C^i}$, $i = 1,2$ and $f(x) := \dfrac{x}{2}$, $x \in L^2([0,1])$. Then $S^{(i)}$, $i = 1,2$ is demimetric with $\kappa^{(i)} = 0$, $i = 1,2$ and $T^{(i)}$, $i = 1,2$ is demimetric with $\mu^{(i)} = 0$, $i = 1,2$. Define

$$F^{(i)}x(t) := \max\{0, x(t) : t \in [0,1]\}, \ i = 1,2.$$

Then $F^{(i)}$ is monotone with $L^{(i)} = 1$, $i = 1,2$.
We perform the numerical computations using different choices of initial points and $\lambda^{(i)}$. To terminate the algorithm, we use the stopping criteria

$$\frac{||x_{n+1} - x_n||_2}{||x_n||_2} < \varepsilon$$

with a tolerance $\varepsilon = 0.01$.
Case 1. We choose $x_1 = 3(t^2 - t)e^{2t} + 2e^{3t}$ with $\lambda^{(i)} = 0.001$, $\lambda^{(i)} = 0.5$, and $\lambda^{(i)} = 0.99$, $i = 1,2$.
Case 2. We choose $x_1 = (t^3 - t)\cos(3t) + 2e^t$ with $\lambda^{(i)} = 0.001$, $\lambda^{(i)} = 0.5$, and $\lambda^{(i)} = 0.99$, $i = 1,2$.

TABLE 1. Numerical results obtained using our proposed algorithm with different cases

	Case 1		Case 2	
$\lambda^{(i)},\ i=1,2$	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
0.001	17	3.2793×10^{-3}	8	1.7663×10^{-3}
0.1	72	1.0094×10^{-2}	62	5.2771×10^{-3}
0.5	26	5.2922×10^{-3}	25	2.4403×10^{-3}
0.99	24	3.5444×10^{-3}	23	2.3217×10^{-3}

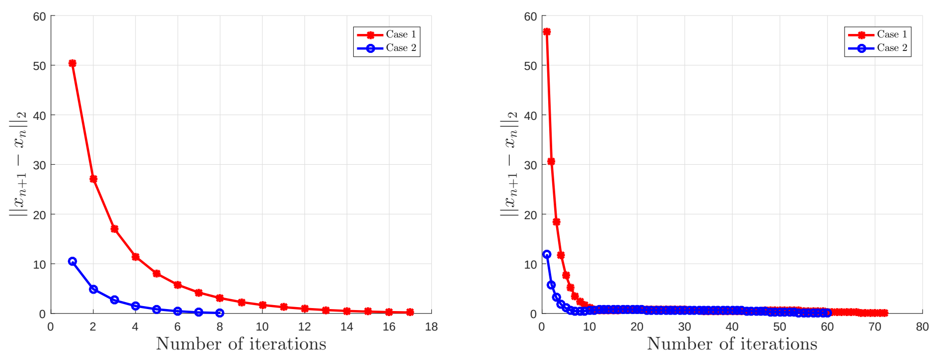

FIGURE 1. Proposed Algorithm with: left $\lambda = 0.001$ and right $\lambda = 0.1$

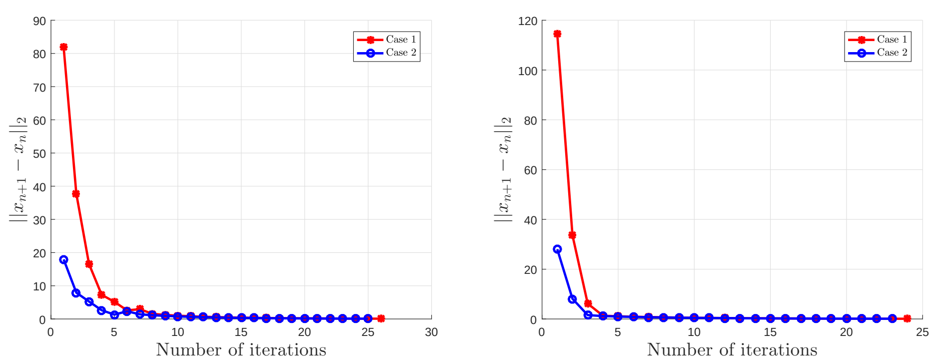

FIGURE 2. Proposed Algorithm with: left $\lambda = 0.5$ and right $\lambda = 0.99$

Remark 4.2.

(1) From the above Example 4.1, the numerical results show that our proposed algorithm is efficient and easy to implement in both cases.
(2) We also observe that except for small λ ($\lambda = 0.001$), the choice of initial point (with different values of λ) has no significant effect on the number of iterations but less CPU time is required for Case 2 compared to Case 1.
(3) The choice of λ has great effect on the number of iterations and CPU time with different cases. However, $\lambda = 0.001$ gives better results in terms of CPU time and number of iterations required to reach the stopping criterion.

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