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A NEW ALGORITHM FOR SOLVING A SYSTEM OF GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FINDING COMMON FIXED POINTS OF A FINITE FAMILY OF BREGMAN NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce a new general iterative algorithm for finding a common element of the set of common fixed points of a finite family of Bregman nonexpansive mappings and the set of solutions of systems of generalized mixed equilibrium problems.
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1. Introduction

Let E be a real reflexive Banach space and C a nonempty, closed and convex subset of E and E^* be the dual space of E. Let Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $\Psi : C \to E^*$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real valued function. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(\Theta)$, that is

$$GMEP(\Theta) = \{x \in C : \ \Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C\}.$$

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Let $\Lambda = \{\Theta_j, \varphi_j, \Psi_j\}_{j\in I}$ be a finite family of bifunction from $C \times C$ into $\mathbb{R}$, real-valued function from C into $\mathbb{R}$ and monotone mapping from C to E^*, respectively.
The system of generalized mixed equilibrium problems is to determine common generalized mixed equilibrium points for Λ. i.e., the set

$$GMEP(\Lambda) = \{x \in C : \ \Theta_j(x, y) + \langle \Psi_j x, y - x\rangle + \varphi_j(y) \geq \varphi_j(x), \quad \forall y \in C, \quad \forall j \in I\}.$$

We can write $GMEP(\Lambda) = \cap_{j\in I} GMEP(\Theta_j)$.

In particular, if $\Psi \equiv 0$, the problem (1.1) is reduced to the *mixed equilibrium problem* [12] for finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $MEP(\Theta, \varphi)$.

If $\varphi \equiv 0$, the problem (1.1) is reduced to the *generalized equilibrium problem* [38] for finding $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solution (1.3) is denoted by $GEP(\Theta, \Psi)$.

If $\Theta \equiv 0$, the problem (1.1) is reduced to the *mixed variational inequality of Browder type* [6] for finding inC such that

$$\langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.4}$$

The set of solution of (1.4) is denoted by $MVI(C, \varphi, \Psi)$.

If $\Psi \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced to the *equilibrium problem* [4] for finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $EP(\Theta)$. This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization and economics. Some methods have been proposed to solve the equilibrium problem, (see [14, 15, 16, 18, 20, 34, 37, 39]).

The above formulation (1.5) was shown in [4] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games.

Equilibrium problems which were introduced by Blum and Oettli [4] and Noor and Oettli [2] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization.

In [36], Suantai, et al., used the following Halpern's iterative scheme for Bregman strongly nonexpansive self mapping T on E; for $x_1 \in E$ let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the above sequence converges strongly to a fixed point of T.

The authors of [22] introduced the following algorithm:

$$\begin{aligned} x_1 &= x \in C \qquad \text{chosen arbitrarily,} \\ z_n &= Res_H^f(x_n), \\ y_n &= \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T_n(z_n))) \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n(y_n))), \end{aligned} \tag{1.6}$$

where H is an equilibrium bifunction and T_n is a Bregman strongly nonexpansive mapping for any $n \in \mathbb{N}$. They proved the sequence (1.6) converges strongly to the point $proj_{F(T)\cap EP(H)}x$.

The author of [17] presented the following iterative scheme:

$$\begin{aligned} x_1 &= x \in C \qquad \text{chosen arbitrarily,} \\ y_n &= Res^f_{\Theta_M,\varphi_M,\Psi_M} \circ \ldots \circ Res^f_{\Theta_2,\varphi_2,\Psi_2} \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n), \\ x_{n+1} &= proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))). \end{aligned} \tag{1.7}$$

It was proved that the sequence $\{x_n\}$ defined in (1.7) converges strongly to the point

$$proj_{(\cap_{i=1}^N F(T_i))\cap\left(\cap_{j=1}^M GMEP(\Theta_j)\right)}x.$$

In this paper, motivated by the above algorithms, we present the following iterative scheme:

$$\begin{aligned} x_1 &= x \in C \qquad \text{chosen arbitrarily,} \\ z_n &= Res^f_{\Theta_M,\varphi_M,\Psi_M} \circ \ldots \circ Res^f_{\Theta_2,\varphi_2,\Psi_2} \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n), \\ y_n &= proj_C^f \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T(z_n))) \\ x_{n+1} &= proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned} \tag{1.8}$$

where $T = T_N \circ T_{N-1} \circ \ldots \circ T_1$ such that each T_i is Bregman strongly nonexpansive mapping for $i = 1, 2, \ldots, N$, $\varphi_j : C \to \mathbb{R}$ are real-valued functions, $\Psi_j : C \to E^*$ are continuous monotone mappings, $\Theta_j : C \times C \to \mathbb{R}$ are equilibrium bifunctions, for $j \in \{1, 2, \ldots M\}$. We will prove that the sequence $\{x_n\}$ defined in (1.8) converges strongly to the point

$$proj_{(\cap_{i=1}^N F(T_i))\cap\left(\cap_{j=1}^M GMEP(\Theta_j)\right)}x.$$

Also, we give some examples and numerical results to support our theorem.

2. Preliminaries

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. We denote by $\text{dom}f$, the domain of f, that is the set $\{x \in E : f(x) < +\infty\}$. Let $x \in \text{int}(\text{dom}f)$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y), \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

For any $x \in \text{int}(\text{dom}f)$, the right-hand derivative of f at x in the derivation $y \in E$ is defined by

$$f^{'}(x,y) := \lim_{t \searrow 0} \frac{f(x+ty)-f(x)}{t}.$$

The function f is called Gâteaux differentiable at x if $\lim_{t \searrow 0} \frac{f(x+ty)-f(x)}{t}$ exists for all $y \in E$. In this case, $f^{'}(x,y)$ coincides with $\nabla f(x)$, the value of the gradient (∇f) of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom}f)$ and f is called Fréchet differentiable at x if this limit is attain uniformly for all y which satisfies $\|y\| = 1$. The function f is uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for any $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int}(\text{dom}f)$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $\text{int}(\text{dom}f)$ (see [5]).

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function

$$D_f : \text{dom}f \times \text{int}(\text{dom}f) \to [0, +\infty)$$

defined as follows:

$$D_f(x,y) := f(x) - f(y) - \langle \nabla f(y), x-y \rangle \tag{2.1}$$

is called the Bregman distance with respect to f, [13].

The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [3]. It is well known that in reflexive spaces, f is Legendre function if and only if it satisfies the following conditions:

(L_1) The interior of the domain of f, $\text{int}(\text{dom}f)$, is nonempty, f is Gâteaux differentiable on $\text{int}(\text{dom}f)$ and $\text{dom}f = \text{int}(\text{dom}f)$;

(L_2) The interior of the domain of f^*, $\text{int}(\text{dom}f^*)$, is nonempty, f^* is Gâteaux differentiable on $\text{int}(\text{dom}f^*)$ and $\text{dom}f^* = \text{int}(\text{dom}f^*)$.

Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see [5]). This , with (L_1) and (L_2), imply the following equalities:

$$\nabla f = (\nabla f^*)^{-1}, \quad \text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$$

and

$$\text{ran}\nabla f^* = \text{dom}(\nabla f) = \text{int}(\text{dom}f),$$

where $\text{ran}\nabla f$ denotes the range of ∇f.

When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$, [26]. By Bauschke et al., [3] the conditions (L_1) and (L_2) also yields that the function f and f^* are strictly convex on the interior of their respective domains.

If E is a smooth and strictly convex Banach space, then an important and interesting Legendre function is $f(x) := \frac{1}{p}\|x\|^p (1 < p < \infty)$. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p (1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces. From now on we assume that the convex function $f : E \to (-\infty, \infty]$ is Legendre. In connection with Legendre functions, see also the recent paper [27].

Definition 2.1. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in \text{int}(\text{dom} f)$ onto the nonempty, closed and convex subset $C \subset \text{dom} f$ is the necessary unique vector $proj_C^f(x) \in C$ satisfying

$$D_f(proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 2.2. If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f(x) = 2Jx$ for all $x \in E$, where J is the normalized duality mapping from E in to 2^{E^*}, and hence $D_f(x, y)$ reduced to

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,$$

for all $x, y \in E$, which is the Lyapunov function introduced by Alber [1] and Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ which is defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x).$$

If $E = H$, a Hilbert space, J is the identity mapping and hence Bregman projection $P_C^f(x)$ reduced to the metric projection of H onto C, $P_C(x)$.

Definition 2.3. [10, 8] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:

(1) *totally convex* at $x \in \text{int}(\text{dom} f)$ if its modulus of total convexity at x, that is, the function $\nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom} f, \|y - x\| = t\},$$

is positive whenever $t > 0$;

(2) totally convex if it is totally convex at every point $x \in \text{int}(\text{dom} f)$;

(3) totally convex on bounded sets if $\nu_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \text{dom} f\}.$$

The set $lev_{\leq}^f(r) = \{x \in E : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a sublevel of f.

Definition 2.4. [8, 32] The function $f : E \to (-\infty, +\infty]$ is called;

(1) *cofinite* if $\text{dom} f^* = E^*$;

(2) *coercive* [19] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty;$$

(3) *strongly coercive* if $\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$;

(4) *sequentially consistent* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n \to \infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \to \infty} \|y_n - x_n\| = 0.$$

Lemma 2.5. [11] *The function f is totally convex on bounded subsets if and only if it is sequentially consistent.*

Lemma 2.6. [32, Proposition 2.3] *If $f : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then f is cofinite.*

Lemma 2.7. [11] *Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points.Then the following statements hold:*

(1) f is sequentially consistent if and only if it is totally convex on bounded sets;
(2) If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets;
(3) If f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain and Fréchet derivative ∇f is uniformly continuous on bounded sets.

Lemma 2.8. [30, Proposition 2.1] *Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.9. [32, Lemma 3.1] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Let $T : C \to C$ be a nonlinear mapping. The fixed point set of T is denoted by $F(T)$, that is $F(T) = \{x \in C : Tx = x\}$. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$. A point $p \in C$ is called an asymptotic fixed point of T (see [29]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T.

A mapping $T : C \to \text{int}(\text{dom}f)$ with $F(T) \neq \emptyset$ is called:

(1) quasi-Bregman nonexpansive [32] with respect to f if

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

(2) Bregman relatively nonexpansive [9, 32] with respect to f if,

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in F(T), \quad \text{and} \quad \widehat{F}(T) = F(T).$$

(3) Bregman strongly nonexpansive (see [7, 32]) with respect to f and $\widehat{F}(T)$ if,

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in \widehat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{z\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0.$$

(4) Bregman firmly nonexpansive (for short BFNE [31]) with respect to f if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

equivalently,

$$D_f(Tx,Ty)+D_f(Ty,Tx)+D_f(Tx,x)+D_f(Ty,y)\le D_f(Tx,y)+D_f(Ty,x). \quad (2.2)$$

The existence and approximation of Bregman firmly nonexpansive mappings was studied in [29]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E, then $F(T)=\widehat{F}(T)$ and $F(T)$ is closed and convex. It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T)=\widehat{F}(T)$.

Lemma 2.10. [11] *Let C be a nonempty, closed and convex subset of E. Let $f:E\to\mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x\in E$, then*

1) $z=proj_C^f(x)$ if and only if

$$\langle\nabla f(x)-\nabla f(z),y-z\rangle\le 0,\quad \forall y\in C.$$

2) $D_f(y,proj_C^f(x))+D_f(proj_C^f(x),x)\le D_f(y,x),\quad \forall x\in E,y\in C.$

Let $f:E\to\mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [1] and [13], we make use of the function $V_f:E\times E^*\to[0,\infty)$ associated with f, which is defined by

$$V_f(x,x^*)=f(x)-\langle x^*,x\rangle+f^*(x^*),\quad \forall x\in E,x^*\in E^*.$$

Then V_f is nonexpansive and $V_f(x,x^*)=D_f(x,\nabla f^*(x^*))$ for all $x\in E$ and $x^*\in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x,x^*)+\langle y^*,\nabla f^*(x^*)-x\rangle\le V_f(x,x^*+y^*) \quad (2.3)$$

for all $x\in E$ and $x^*,y^*\in E^*$ [21]. In addition, if $f:E\to(-\infty,+\infty]$ is a proper lower semicontinuous function, then $f^*:E^*\to(-\infty,+\infty]$ is a proper weak* lower semicontinuous and convex function (see [23]). Hence, V_f is convex in the second variable. Thus, for all $z\in E$,

$$D_f\left(z,\nabla f^*\left(\sum_{i=1}^N t_i\nabla f(x_i)\right)\right)\le\sum_{i=1}^N t_iD_f(z,x_i),$$

where $\{x_i\}_{i=1}^N\subset E$ and $\{t_i\}_{i=1}^N\subset(0,1)$ with $\sum_{i=1}^N t_i=1$.

Lemma 2.11. [23] *Let $f:E\to(-\infty,+\infty]$ be a bounded, uniformly Fréchet differentiable and totally convex function on bounded subsets of E. Assume that ∇f^* is bounded on bounded subsets of $domf^*=E^*$ and let C be a nonempty subset of $int(domf)$. If $\{T_i:i=1,2,\dots,N\}$ be N Bregman strongly nonexpansive mappings from C into itself satisfying $\cap_{i=1}^N\widehat{F}(T_i)\neq\emptyset$. Let $T=T_N\circ T_{N-1}\circ\ldots\circ T_1$, then T is Bregman strongly nonexpansive mapping and $\widehat{F}(T)=\cap_{i=1}^N\widehat{F}(T_i)$.*

Lemma 2.12. [33] *Let C be a nonempty, closed and convex subset of $int(domf)$ and $T:C\to C$ be a quasi-Bregman nonexpansive mappings with respect to f. Then $F(T)$ is closed and convex.*

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A_1) $\Theta(x, x) = 0$ for all $x \in C$;
(A_2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;
(A_3) for each $y \in C, x \mapsto \Theta(x, y)$ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \searrow 0} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(A_4) for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous (see [25]).

Definition 2.13. Let C be a nonempty, closed and convex subsets of a real reflexive Banach space and let φ be a lower semicontinuous and convex functional from C to $\mathbb{R}$ and $\Psi : C \to E^*$ be a continuous monotone mapping. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctional satisfying (A_1)-(A_4). The *mixed resolvent* of Θ is the operator $Res^f_{\Theta,\varphi,\Psi} : E \to 2^C$

$$\begin{aligned} Res^f_{\Theta,\varphi,\Psi}(x) = \{z \in C : &\Theta(z, y) + \varphi(y) + \langle \Psi x, y - z\rangle + \langle \nabla f(z) - \nabla f(x), y - z\rangle \\ &\geq \varphi(z), \quad \forall y \in C\}. \end{aligned} \tag{2.4}$$

Lemma 2.14. [17] *Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. Assume that $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional, $\Psi : C \to E^*$ be a continuous monotone mapping and the bifunctional $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4), then $dom(Res^f_{\Theta,\varphi,\Psi}) = E$.*

Lemma 2.15. [17] *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of E. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4), then*

(1) $Res^f_{\Theta,\varphi,\Psi}$ *is single-valued;*
(2) $Res^f_{\Theta,\varphi,\Psi}$ *is a BFNE operator;*
(3) $F\left(Res^f_{\Theta,\varphi,\Psi}\right) = GMEP(\Theta)$;
(4) $GMEP(\Theta)$ *is closed and convex;*
(5) $D_f\left(p, Res^f_{\Theta,\varphi,\Psi}(x)\right) + D_f\left(Res^f_{\Theta,\varphi,\Psi}(x), x\right) \leq D_f(p, x)$,
$\forall p \in F\left(Res^f_{\Theta,\varphi,\Psi}\right), x \in E$.

Lemma 2.16. [28, 40] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \beta_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\beta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(2) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main result

Theorem 3.1. *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of $int(domf)$. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T_i : C \to C$, for $i = 1, 2, \dots, N$, be a finite family of Bregman strongly nonexpansive mappings with respect to f such that $F(T_i) = \widehat{F}(T_i)$ and each T_i is uniformly continuous. Let $\Theta_j : C \times C \to \mathbb{R}$ satisfying conditions (A_1)-(A_4), $\varphi_j : C \to \mathbb{R}$ are real-valued convex functions, $\Psi_j : C \to E^*$ are continuous monotone mappings for $j \in \{1, 2, \dots, M\}$. Assume that $\big(\cap_{i=1}^{N} F(T_i)\big) \cap \big(\cap_{j=1}^{M} GMEP(\Theta_j)\big)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C \qquad \text{chosen arbitrarily,} \\ z_n &= Res^f_{\Theta_M,\varphi_M,\Psi_M} \circ \dots \circ Res^f_{\Theta_2,\varphi_2,\Psi_2} \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n), \\ y_n &= proj^f_C \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T(z_n))) \\ x_{n+1} &= proj^f_C \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned} \tag{3.1}$$

where $T = T_N \circ T_{N-1} \circ \dots \circ T_1$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to $proj^f_{\left(\cap_{i=1}^{N} F(T_i)\right)\cap\left(\cap_{j=1}^{M} GMEP(\Theta_j)\right)} x$.

Proof. We note from Lemma 2.12 that $F(T_i)$, for each $i \in \{1, 2, \dots, N\}$ is closed and convex and hence $\cap_{i=1}^{N} F(T_i)$ is closed and convex.
Let $p = proj_{\left(\cap_{i=1}^{N} F(T_i)\right)\cap\left(\cap_{j=1}^{M} GMEP(\Theta_j)\right)} x \in \big(\cap_{i=1}^{N} F(T_i)\big) \cap \big(\cap_{j=1}^{M} GMEP(\Theta_j)\big)$. Then $p \in (\cap_{i=1}^{N} F(T_i))$ and $p \in \cap_{j=1}^{M} GMEP(\Theta_j)$. Now, by using (3.1) and Lemma 2.15, we have

$$\begin{aligned} D_f(p, z_n) &= D_f(p, Res^f_{\Theta_M,\varphi_M,\Psi_M} \circ \dots \circ Res^f_{\Theta_2,\varphi_2,\Psi_2} \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n)) \\ &\leq D_f(p, Res^f_{\Theta_{M-1},\varphi_{M-1},\Psi_{M-1}} \circ \dots \circ Res^f_{\Theta_2,\varphi_2,\Psi_2} \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n)) \\ &\vdots \\ &\leq D_f(p, Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n)) \\ &\leq D_f(p, x_n). \end{aligned}$$

Also,

$$\begin{aligned} D_f(p, y_n) &= D_f(p, proj^f_C \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T(z_n)))) \\ &\leq D_f(p, \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T(z_n)))) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T(z_n)) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, z_n) \\ &\leq \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, x_n) \\ &\leq D_f(p, x_n). \end{aligned}$$

So,

$$\begin{aligned} D_f(p,x_{n+1}) &= D_f(p, proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\leq D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\leq \alpha_n D_f(p,x_n) + (1-\alpha_n) D_f(p, T(y_n)) \\ &\leq \alpha_n D_f(p,x_n) + (1-\alpha_n) D_f(p, y_n) \\ &\leq \alpha_n D_f(p,x_n) + (1-\alpha_n) D_f(p, x_n) \\ &\leq D_f(p,x_n). \end{aligned}$$

Hence $\{D_f(p,x_n)\}$ and $D_f(p,Ty_n)$ are bounded. Moreover, by Lemma 2.9 we get that the sequences $\{x_n\}$ and $\{T(y_n)\}$ are bounded.
From the fact that $\alpha_n \to 0$ as $n\to\infty$, Lemma 2.10 we get that

$$\begin{aligned} D_f(T(y_n),x_{n+1}) &\leq D_f(T(y_n), proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\leq D_f(T(y_n), \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\leq \alpha_n D_f(T(y_n),x_n) + (1-\alpha_n) D_f(T(y_n), T(y_n)) \\ &= \alpha_n D_f(T(y_n),x_n). \end{aligned}$$

Therefore, by Lemma 2.5, we have

$$\|x_{n+1} - T(y_n)\| \to 0, \quad as \quad n\to\infty. \tag{3.2}$$

From Lemma 2.10 and (3.1), we have

$$\begin{aligned} \lim_{n\to\infty} D_f(x_n,z_n) &= \lim_{n\to\infty} D_f(x_n, Res^f_{\Theta_M,\varphi_M,\Psi_M} \circ \ldots \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n)) \\ &\leq \lim_{n\to\infty} D_f(x_n, Res^f_{\Theta_{M-1},\varphi_{M-1},\Psi_{M-1}} \circ \ldots \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n)) \\ &\vdots \\ &\leq \lim_{n\to\infty} D_f(x_n, Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n)) \\ &\leq \lim_{n\to\infty} [D_f(p, Res^f_{\Theta_1,\varphi_1,\Psi_1} x_n) - D_f(p,x_n)] \\ &\leq \lim_{n\to\infty} [D_f(p,x_n) - D_f(p,x_n)] \\ &= 0. \end{aligned}$$

By Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.3}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 2.8, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. So,

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_n)\|_* = 0. \tag{3.4}$$

Since f is uniformly Fréchet differentiable, it is also uniformly continuous, we get

$$\lim_{n\to\infty} \|f(x_n) - f(z_n)\| = 0. \tag{3.5}$$

By Bregman distance we have

$$\begin{aligned}
&D_f(p,x_n)-D_f(p,z_n)\\
&=f(p)-f(x_n)-\langle\nabla f(x_n),p-x_n\rangle-f(p)+f(z_n)+\langle\nabla f(z_n),p-z_n\rangle\\
&=f(z_n)-f(x_n)+\langle\nabla f(z_n),p-z_n\rangle-\langle\nabla f(x_n),p-x_n\rangle\\
&=f(z_n)-f(x_n)+\langle\nabla f(z_n),x_n-z_n\rangle-\langle\nabla f(z_n)-\nabla f(x_n),p-x_n\rangle,
\end{aligned}$$

for each $p\in\cap_{i=1}^{N}F(T_i)$. By (3.3)-(3.5), we obtain

$$\lim_{n\to\infty}(D_f(p,x_n)-D_f(p,z_n))=0. \tag{3.6}$$

By the above equation, we have

$$\begin{aligned}
D_f(z_n,y_n) &= D_f(p,y_n)-D_f(p,z_n)\\
&= D_f(p,proj_C^f\nabla f^*(\alpha_n\nabla f(x_n)+(1-\alpha_n)\nabla f(T(z_n))-D_f(p,z_n))\\
&\le D_f(p,\nabla f^*(\alpha_n\nabla f(x_n)+(1-\alpha_n)\nabla f(T(z_n))-D_f(p,z_n))\\
&\le \alpha_nD_f(p,x_n)+(1-\alpha_n)D_f(p,T(z_n)-D_f(p,z_n)\\
&\le \alpha_nD_f(p,x_n)+(1-\alpha_n)D_f(p,z_n)-D_f(p,z_n)\\
&= \alpha_n(D_f(p,x_n)-D_f(p,z_n))\\
&= 0.
\end{aligned}$$

By (3.6), we have

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{3.7}$$

Note that

$$\|x_n-y_n\|\le\|x_n-z_n\|+\|z_n-y_n\|.$$

By applying (3.3) and (3.7), we can write

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.8}$$

Now, we claim that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.9}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 2.8, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. So,

$$\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(y_n)\|_*=0. \tag{3.10}$$

Since f is uniformly Fréchet differentiable, it is also uniformly continuous, we get

$$\lim_{n\to\infty}\|f(x_n)-f(y_n)\|=0. \tag{3.11}$$

Then, we have

$$\begin{aligned}
&D_f(p,x_n)-D_f(p,y_n)\\
&=f(p)-f(x_n)-\langle\nabla f(x_n),p-x_n\rangle-f(p)+f(y_n)+\langle\nabla f(y_n),p-y_n\rangle\\
&=f(y_n)-f(x_n)+\langle\nabla f(y_n),p-y_n\rangle-\langle\nabla f(x_n),p-x_n\rangle\\
&=f(y_n)-f(x_n)+\langle\nabla f(y_n),x_n-y_n\rangle+\langle\nabla f(y_n)-\nabla f(x_n),p-x_n\rangle,
\end{aligned}$$

for each $p \in (\cap_{i=1}^{N} F(T_i))$. By (3.8)-(3.11), we obtain

$$\lim_{n\to\infty}(D_f(p, x_n) - D_f(p, y_n)) = 0. \tag{3.12}$$

By the above equation, we have

$$\begin{aligned}
D_f(y_n, x_{n+1}) &= D_f(p, x_{n+1}) - D_f(p, y_n)\\
&= D_f(p, proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(x_n)) - D_f(p, y_n))\\
&\leq D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(x_n)) - D_f(p, y_n))\\
&\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, T(y_n) - D_f(p, y_n)\\
&\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, y_n) - D_f(p, y_n)\\
&= \alpha_n (D_f(p, x_n) - D_f(p, y_n))\\
&= 0.
\end{aligned}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0.$$

From the above equation and (3.2), we can write

$$\begin{aligned}
\|y_n - T(y_n)\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - T(y_n)\|\\
&= 0
\end{aligned} \tag{3.13}$$

when $n \to \infty$. By applying the triangle inequality, we get

$$\|x_n - T(x_n)\| \leq \|x_n - y_n\| + \|y_n - T(y_n)\| + \|T(y_n) - T(x_n)\|.$$

By (3.8), (3.13) and since each T_i for $i \in \{1, 2, \ldots N\}$ is uniformly continuous, we have

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$. From (3.9) we have $\|x_{n_k} - T(x_{n_k})\| \to 0$ as $k \to \infty$ and hence $q \in (\cap_{i=1}^{N} F(T_i))$.
From (3.3) we can write

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0.$$

Here, we prove that $q \in \cap_{j=1}^{M} GMEP(\Theta_j)$. For this reason, consider that

$$z_n = Res^f_{\Theta_M,\varphi_M,\Psi_M} \circ \ldots \circ Res^f_{\Theta_2,\varphi_2,\Psi_2} \circ Res^f_{\Theta_1,\varphi_1,\Psi_1}(x_n),$$

so we have

$$\Theta_j(z_n, z) + \langle \Psi_j x_n, z - z_n\rangle + \varphi_j(z) + \langle Jz_n - Jx_n, z - z_n\rangle \geq \varphi_j(z_n),$$

for all $j \in \{1, 2, \ldots, M\}$ and $z \in C$.
From (A_2), we have

$$\Theta_j(z, z_n) \leq -\Theta_j(z_n, z) \leq \langle \Psi_j x_n, z - z_n\rangle + \varphi_j(z) - \varphi_j(z_n) + \langle Jz_n - Jx_n, z - z_n\rangle,$$

for all $j \in \{1, 2, \ldots, M\}$ and $z \in C$.
Hence,

$$\Theta_j(z, z_{n_i}) \leq \langle \Psi_j x_{n_i}, z - z_{n_i}\rangle + \varphi_j(z) - \varphi_j(z_{n_i}) + \langle Jz_{n_i} - Jx_{n_i}, z - z_{n_i}\rangle,$$

for all $j \in \{1, 2, \dots, M\}$ and $z \in C$.
Since $z_{n_i} \rightharpoonup q$, from continuity of Ψ and weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we also have

$$\Theta_j(z, q) + \langle \Psi_j q, q - z\rangle + \varphi_j(q) - \varphi_j(z) \leq 0,$$

for all $j \in \{1, 2, \dots, M\}$ and $z \in C$.
For t with $0 \leq t \leq 1$ and $z \in C$, let $z_t = tz + (1-t)q$. Since $z \in C$ and $q \in C$ we have $z_t \in C$ and hence $\Theta_j(z_t, q) + \langle \Psi_j q, q - z_t\rangle + \varphi_j(q) - \varphi_j(z_t) \leq 0$. So, we have

$$\begin{aligned} 0 &= \Theta_j(z_t, z_t) + \langle \Psi_j q, z_t - z_t\rangle + \varphi_j(z_t) - \varphi_j(z_t) \\ &\leq t\Theta_j(z_t, z) + (1-t)\Theta_j(z_t, q) + t\langle \Psi_j q, z - z_t\rangle + (1-t)\langle \Psi_j q, q - z_t\rangle \\ &\quad + t\varphi_j(z) + (1-t)\varphi_j(q) - \varphi_j(z_t) \\ &\leq t[\Theta_j(z_t, z) + \langle \Psi_j q, z - z_t\rangle + \varphi_j(z) - \varphi_j(z_t)]. \end{aligned}$$

Therefore, $\Theta_j(z_t, z) + \langle \Psi_j q, z - z_t\rangle + \varphi_j(z) - \varphi_j(z_t) \geq 0$. Then, we have

$$\Theta_j(q, z) + \langle \Psi_j q, z - q\rangle + \varphi_j(z) - \varphi_j(q) \geq 0,$$

for all $j \in \{1, 2, \dots, M\}$ and $z \in C$.
Hence we have $q \in \cap_{j=1}^{M} GMEP(\Theta_j)$. We showed that

$$q \in (\cap_{i=1}^{N} F(T_i)) \cap \left(\cap_{j=1}^{M} GMEP(\Theta_j)\right).$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup q \in C$ and

$$\limsup_{n\to\infty} \langle \nabla f(x_n) - \nabla f(p), x_n - p\rangle = \langle \nabla f(x_n) - \nabla f(p), q - p\rangle.$$

On the other hand, since $\|x_{n_k} - Tx_{n_k}\| \to 0$ as $k \to \infty$, we have $q \in (\cap_{i=1}^{N} F(T_i))$. It follows from the definition of the Bregman projection that

$$\limsup_{n\to\infty} \langle \nabla f(x_n) - \nabla f(p), x_n - p\rangle = \langle \nabla f(x_n) - \nabla f(p), q - p\rangle \leq 0. \tag{3.14}$$

From (2.3), we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\leq D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &= V_f(p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))) \\ &\leq V_f(p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)) - \alpha_n(\nabla f(x_n) - \nabla f(p))) \\ &\quad + \langle \alpha_n(\nabla f(x_n) - \nabla f(p)), x_{n+1} - p\rangle \\ &= V_f(p, \alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(T(y_n)) \\ &\quad + \alpha_n \langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle \\ &\leq \alpha_n V_f(p, \nabla f(p)) + (1-\alpha_n)V_f(p, \nabla f(T(y_n))) \\ &\quad + \alpha_n \langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle \\ &= (1-\alpha_n)D_f(p, T(y_n)) + \alpha_n \langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle \\ &\leq (1-\alpha_n)D_f(p, x_n) + \alpha_n \langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle. \end{aligned}$$

By Lemma 2.16 and (3.14), we can conclude that

$$\lim_{n\to\infty} D_f(p, x_n) = 0.$$

Therefore, by Lemma 2.5, $x_n \to p$. This completes the proof. □

4. Examples and numerical results

Example 4.1. Consider the following problem: Find an element

$$x^* \in S = (\cap_{i=1}^N F(T_i)) \cap (\cap_{j=1}^M GMEP(\Theta_j)),$$

where

$$T_i(x) = \frac{1}{i}\sin ix \text{ for all } i = 1, 2, ..., 100,$$
$$\Theta_j(x, y) = j(y^2 - x^2),$$
$$\varphi_j(x) = x^2$$

and

$$\Psi_j(x) = jx \text{ for all } j = 1, 2, ..., 50 \text{ and for all } x, y \in \mathbb{R}.$$

We can see that Θ_j satisfies the conditions (A1)-(A4), φ_j is a continuous convex function, Ψ_j is a continuous monotone mapping for all $j = 1, 2, ..., 50$ and T_i is a Bregman strongly nonexpansive mapping with

$$F(T_i) = \hat{F}(T_i) = \{0\}.$$

It is easy to see that $S = \{0\}$.

Now, with $f(x) = \frac{1}{2}x^2$, from the definition of $Res^f_{\Theta_j,\varphi_j,\Psi_j}$, for each $x \in \mathbb{R}$, we have

$$Res^f_{\Theta_j,\varphi_j,\Psi_j}(x) = \{z \in \mathbb{R} :\ j(y^2 - z^2) + y^2 + jx(y - z) + (z - x)(y - z) \geq z^2,\ \forall y \in \mathbb{R}\}.$$

Hence, we obtain that

$$Res^f_{\Theta_j,\varphi_j,\Psi_j}(x) = \frac{j-1}{2j+3}x,$$

for all $j = 1, 2, ..., 50$.

Now, apply iterative method (3.1) with $x_1 = 5$, $\beta_n = 1/2$ and $\alpha_n = 1/n$ for all $n \geq 1$, we obtain the following table of numerical results:

TOL	$\|x_n - x^*\|$	n	x^n	$\|x_n - x^*\|$	n	x^n
	$\beta_n = 1/2$ and $\alpha_n = 1/n$			$\beta_n = 1/2$ and $\alpha_n = 1/n^{0.5}$		
10^{-6}	9.70×10^{-7}	20	9.70×10^{-7}	6.83×10^{-7}	28	6.83×10^{-7}
10^{-7}	7.22×10^{-8}	24	7.22×10^{-8}	8.31×10^{-8}	32	8.31×10^{-8}
10^{-8}	5.23×10^{-9}	28	5.23×10^{-9}	9.74×10^{-9}	36	9.74×10^{-9}
10^{-9}	7.22×10^{-10}	31	7.22×10^{-10}	6.40×10^{-10}	41	6.40×10^{-10}
10^{-10}	9.87×10^{-11}	34	9.87×10^{-11}	7.03×10^{-11}	45	7.03×10^{-11}

Table 1. Table of numerical results

The strong convergence of the iterative method (3.1) for the Example 4.1 is also described in Fig. 1.

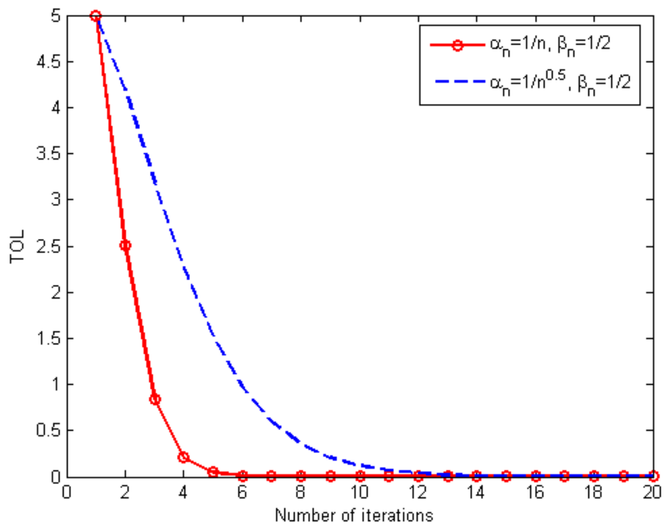

FIGURE 1

Example 4.2. Consider the following problem: Find an element

$$x^* \in S = (\cap_{i=1}^N C_i) \cap (\cap_{j=1}^M GMEP(\Theta_j)),$$

where $\Theta_j(x, y) = j(\|y\|^2 - \|x\|^2)$, $\varphi_j(x) = \|x\|^2$ and $\Psi_j(x) = jx$ for all $j = 1, 2, \ldots, 50$ and for all $x, y \in \mathbb{R}$ and

$$C_i = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \left(x_1 - \frac{1}{i}\right)^2 + x_2^2 + \left(x_3 + \frac{1}{i}\right)^2 \leq 4 \right\},$$

for all $i = 1, 2, ..., 100$.

We can see that Θ_j satisfies the conditions (A1)-(A4), φ_j is a continuous convex function, Ψ_j is a continuous monotone mapping for all $j = 1, 2, \ldots, 50$. Let $T_i = P_{C_i}$, then we have T_i is a Bregman strongly nonexpansive mapping with $F(T_i) = \hat{F}(T_i) = C_i$, for all $j = 1, 2, ..., 100$.

It is easy to see that $S = \{(0, 0, 0)\}$.

Now, with $f(x) = \frac{1}{2}\|x\|^2$, from the definition of $Res^f_{\Theta_j, \varphi_j, \Psi_j}$, for each $x \in \mathbb{R}^3$, we have

$$Res^f_{\Theta_j, \varphi_j, \Psi_j}(x) = \{z \in \mathbb{R}^3 : \ j(\|y\|^2 - \|z\|^2) + \|y\|^2 + j\langle x, y - z\rangle + \langle z - x, y - z\rangle \geq \|z\|^2, \ \forall y \in \mathbb{R}^3\}.$$

Hence, we obtain that

$$Res^f_{\Theta_j, \varphi_j, \Psi_j}(x) = \frac{j - 1}{2j + 3}x,$$

for all $x \in \mathbb{R}^3$ and for all $j = 1, 2, ..., 50$. Now, apply iterative method (3.1) with $x_1 = (1, 2, 3)$, $\beta_n = 1/2$ and $\alpha_n = 1/n^{1/2}$ for all $n \geq 1$, we obtain the following table of numerical results:

TOL	$\|x_n - x^*\|$	n	x^n
$\beta_n = 1/2$ and $\alpha_n = 1/n$			
10^{-6}	7.08×10^{-7}	27	$(2.08 \times 10^{-7}, 4.17 \times 10^{-7}, 6.25 \times 10^{-7})$
10^{-7}	5.57×10^{-8}	31	$(1.49 \times 10^{-8}, 2.98 \times 10^{-8}, 4.47 \times 10^{-8})$
10^{-8}	7.62×10^{-9}	34	$(2.03 \times 10^{-9}, 4.07 \times 10^{-9}, 6.11 \times 10^{-9})$
10^{-9}	5.30×10^{-10}	38	$(1.41 \times 10^{-10}, 2.83 \times 10^{-10}, 4.25 \times 10^{-10})$
10^{-10}	7.14×10^{-11}	41	$(1.90 \times 10^{-11}, 3.81 \times 10^{-11}, 5.72 \times 10^{-11})$
$\beta_n = 1/2$ and $\alpha_n = 1/n^{0.5}$			
10^{-6}	7.92×10^{-7}	35	$(2.11 \times 10^{-7}, 4.23 \times 10^{-7}, 6.35 \times 10^{-7})$
10^{-7}	9.07×10^{-8}	39	$(2.42 \times 10^{-8}, 4.85 \times 10^{-8}, 7.27 \times 10^{-8})$
10^{-8}	5.81×10^{-9}	44	$(1.55 \times 10^{-9}, 3.10 \times 10^{-9}, 4.66 \times 10^{-9})$
10^{-9}	6.28×10^{-10}	48	$(1.67 \times 10^{-10}, 3.35 \times 10^{-10}, 5.03 \times 10^{-10})$
10^{-10}	6.64×10^{-11}	52	$(1.77 \times 10^{-11}, 3.55 \times 10^{-11}, 5.33 \times 10^{-11})$

TABLE 2. Table of numerical results

The strong convergence of the iterative method (3.1) for the Example 4.2 is also described in Fig. 2.

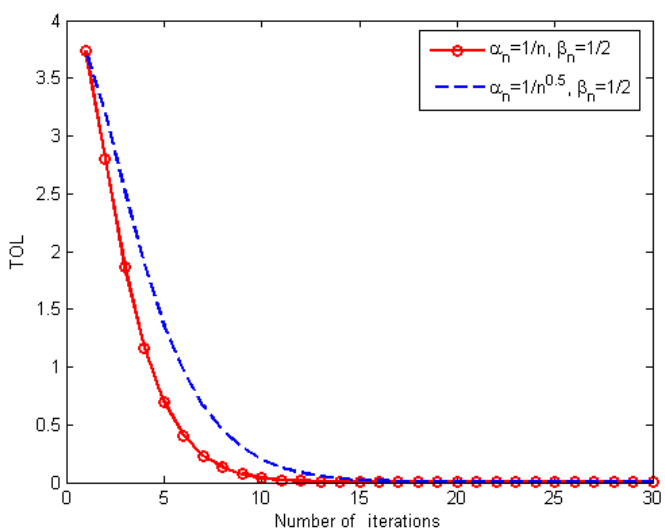

FIGURE 2

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