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AN INERTIAL CENSOR-SEGAL ALGORITHM FOR SPLIT COMMON FIXED-POINT PROBLEMS

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Abstract. In this paper we study the split common fixed-point problem in Hilbert spaces. To speed up its convergence, we modify the algorithm recently introduced by Censor and Segal. Moreover, the step-size in our algorithm is independent of the norm of the given linear mapping. Under some mild conditions, we establish two weak convergence theorems of the proposed algorithm.
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1. Introduction

In recent years there has been growing interest in the study of the split common fixed-point problem (SCFP) because of its various applications in signal processing and image reconstruction [7, 14, 21]. More specifically, the SCFP consists in finding $\bar{x} \in H_1$ satisfying

$$\bar{x} \in F(U), \text{ s.t. } A\bar{x} \in F(T), \tag{1.1}$$

where $F(U)$ and $F(T)$ stand for the fixed point sets of mappings $U : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear mapping. Here H_1 and H_2 are two Hilbert spaces. Let C and Q be two nonempty closed convex subsets of H_1 and H_2, respectively. It is trivial to see that $C = F(P_C)$ and $Q = F(P_Q)$, where P_C and P_Q denote the metric projections onto C and Q, respectively. Thus, if we let $U = P_C$ and $T = P_Q$ in (1.1), then it is reduced to

$$\bar{x} \in C, \text{ s.t. } A\bar{x} \in Q, \tag{1.2}$$

which is known as the split feasibility problem (SFP); see e.g. [9, 19, 20].

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In [10], Censor and Segal introduced an algorithm for solving the SCFP. More specifically, their algorithm is defined as:

$$x^{k+1} = U(x^k - \tau A^*(I - T)Ax^k), \tag{1.3}$$

where I stands for the identity mapping, A^* is the adjoint mapping of A, and the step-size τ is chosen such that

$$0 < \tau < \frac{2}{\|A\|^2}. \tag{1.4}$$

If U and T are firmly quasi-nonexpansive mappings, then the Censor-Segal algorithm converges weakly to a solution of (1.1). Subsequently, this algorithm was further extended to the case of quasi-nonexpansive mappings [15], demicontractive mappings [16]; see [3, 13, 18] for other variants of method (1.3).

In particular, if we let $U = P_C$ and $T = P_Q$ in (1.3), then it is reduced to

$$x^{k+1} = P_C(x^k - \tau A^*(I - P_Q)Ax^k), \tag{1.5}$$

which is Byrne's CQ algorithm for solving the SFP. The inertial technique recently developed by Alvarez and Attouch [1] is a novel way to speed up the convergence of various algorithms; see also [5, 4]. To improve its performance, Dang *et al.* [12] recently applied the inertial technique to (1.5) and proposed the inertial CQ algorithm:

$$\left[\begin{array}{l} w^k = x^k + \theta_k(x^k - x^{k-1}) \\ x^{k+1} = P_C(x^k - \tau_k A^*(I - P_Q)Ax^k), \end{array}\right. \tag{1.6}$$

where $0 \le \theta_k < \theta < 1$ and

$$0 < \tau < \frac{2}{\|A\|^2}. \tag{1.7}$$

It was shown that the inertial CQ algorithm with (1.7) converges weakly to a solution of the SFP provided that

$$\sum_{k=1}^{\infty} \theta_k \|x^k - x^{k-1}\|^2 < \infty.$$

In this paper, we continue to study the SCFP. To speed up its convergence, we propose an inertial Censor-Segal algorithm that can improve the performance of the original algorithm. Moreover, the step-size in our algorithm is independent of the norm of the given linear mapping. Under some mild conditions, we establish two weak convergence theorems of the proposed algorithm.

2. Preliminaries

In this section, we assume that "$\rightharpoonup$" stands for weak convergence, H is a Hilbert space and C is a nonempty closed convex subset in H.

Definition 2.1 Let T be a mapping from C into H.

(i) T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C;$$

(ii) T is firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Definition 2.2 Let $T : C \to H$ be a mapping with $F(T) \neq \emptyset$.

(i) T is quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \quad \forall x \in C, y \in F(T);$$

(ii) T is firmly quasi-nonexpansive if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \|(I - T)x\|^2, \quad \forall x \in C, y \in F(T).$$

Definition 2.3 Let $T : C \to H$ be a mapping with $F(T) \neq \emptyset$. Then $I - T$ is said to be *demiclosed at 0*, if, for any $\{x^k\}$ in H, there holds the following implication:

$$\left.\begin{array}{c} x^k \rightharpoonup x \\ (I - T)x^k \to 0 \end{array}\right] \Rightarrow x \in F(T).$$

It is well known that if T is a nonexpansive mapping, then $I - T$ is demiclosed at 0. Clearly, such a property is also shared by firmly nonexpansive mappings.

Lemma 2.4 Let $T : C \to H$ be a mapping with $F(T) \neq \emptyset$. Then the following are equivalent.

(i) T is a firmly quasi-nonexpansive mapping;
(ii) $\langle Tx - z, (I - T)x \rangle \geq 0, \ \forall \ z \in F(T), x \in C$;
(iii) $\langle x - z, (I - T)x \rangle \geq \|(I - T)x\|^2, \ \forall \ z \in F(T), x \in C$.

Typical examples of firmly quasi-nonexpansive mappings include subgradient projections and orthogonal projections; see [2].

Lemma 2.5 [2] Assume that $\{x^k\}$ is a sequence in H such that

(i) for each $z \in C$, the limit of sequence $\{\|x^k - z\|\}$ exists;
(ii) any weak cluster point of sequence $\{x^k\}$ belongs to C.

Then the sequence $\{x^k\}$ is weakly convergent to an element z in C.

Lemma 2.6 [1] Let $\{\phi_k\}$ and $\{\delta_k\}$ be two nonnegative real sequences such that

$$\phi_{k+1} - \phi_k \leq \theta_k(\phi_k - \phi_{k-1}) + \delta_k, \sum_{k=0}^{\infty} \delta_k < \infty,$$

where $\{\theta_k\} \subset [0, \theta]$ with $0 < \theta < 1$. Then the sequence $\{\phi_k\}$ is convergent.

Lemma 2.7 [2] Let $s, t \in \mathbb{R}$ and $x, y \in H$. It then follows that

$$\|tx + sy\|^2 = t(t + s)\|x\|^2 + s(t + s)\|y\|^2 - ts\|x - y\|^2.$$

3. The proposed algorithms

In this section, we propose an inertial Censor-Segal algorithm for split common fixed-point problems. We need the following basic assumption of the SCFP:

(i) The solution set S of the SCFP is nonempty;
(ii) $U : H_1 \to H_1$ is firmly quasi-nonexpansive such that $I - U$ is demiclosed at 0;
(iii) $T : H_2 \to H_2$ is firmly quasi-nonexpansive such that $I - T$ is demiclosed at 0.

Algorithm 1 Let x^0, x^1 be arbitrary. Given x^k, x^{k-1}, update the next iteration via

$$\left[\begin{array}{l} w^k = x^k + \theta_k(x^k - x^{k-1}) \\ x^{k+1} = U(w^k - \tau_k A^*(I-T)Aw^k), \end{array}\right. \tag{3.1}$$

where $0 < \rho_k < 2, 0 \le \theta_k < \theta < 1$, and

$$\tau_k = \begin{cases} \rho_k \dfrac{\|(I-T)Aw^k\|^2}{\|A^*(I-T)Aw^k\|^2}, & \|A^*(I-T)Aw^k\| \neq 0, \\ 0, & \|A^*(I-T)Aw^k\| = 0. \end{cases} \tag{3.2}$$

Remark 3.1 In comparison with (1.4), our step-size (3.2) is independent of the norm $\|A\|$, so that the calculation or estimation of $\|A\|$ is avoided.

Lemma 3.2 *Assume that the sequence $\{w^k\}$ is generated by Algorithm 1. Then, $\|A^*(I-T)Aw^k\| = 0$ if and only if $\|(I-T)Aw^k\| = 0$.*

Proof. It is easy to see that $\|A^*(I-T)Aw^k\| = 0$ if $\|(I-T)Aw^k\| = 0$. To see the converse, let $\|A^*(I-T)Aw^k\| = 0$ and $z \in S$. It then follows from Lemma 2 that

$$\begin{aligned} \|(I-T)Aw^k\|^2 &\le \langle (I-T)Aw^k, Aw^k - Az\rangle \\ &= \langle A^*(I-T)Aw^k, w^k - z\rangle \\ &= \|A^*(I-T)Aw^k\|\|w^k - z\|. \end{aligned}$$

This yields $\|(I-T)Aw^k\| = 0$. Hence the proof is complete.

Lemma 3.3 *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm 1. Then, for any $\tau \in [0,1]$ and $z \in S$, it follows that*

$$\|x^{k+1} - z\|^2 \le \|w^k - z\|^2 - (1-\tau)\|w^k - x^{k+1}\|^2 - \rho_k\left(2 - \frac{\rho_k}{\tau}\right)\delta_k, \tag{3.3}$$

where

$$\delta_k = \begin{cases} 0, & \|A^*(I-T)Aw^k\| = 0, \\ \dfrac{\|(I-T)Aw^k\|^4}{\|A^*(I-T)Aw^k\|^2}, & \|A^*(I-T)Aw^k\| \neq 0. \end{cases} \tag{3.4}$$

Proof. Since U is firmly quasi-nonexpansive, we have

$$\begin{aligned} \|x^{k+1} - z\|^2 &= \|U(w^k - \tau_k A^*(I-T)Aw^k) - Uz\|^2 \\ &\le \|(w^k - z) - \tau_k A^*(I-T)Aw^k\|^2 - \|(w^k - x^{k+1}) - \tau_k A^*(I-T)Aw^k\|^2 \\ &= \|w^k - z\|^2 - \|w^k - x^{k+1}\|^2 - 2\tau_k\langle A^*(I-T)Aw^k, w^k - z\rangle \\ &\quad + 2\tau_k\langle A^*(I-T)Aw^k, w^k - x^{k+1}\rangle. \end{aligned}$$

It then follows from Lemma 2 that

$$\begin{aligned} 2\tau_k\langle A^*(I-T)Aw^k, w^k - z\rangle &= 2\tau_k\langle (I-T)Aw^k - (I-T)Az, Aw^k - Az\rangle \\ &\ge 2\tau_k\|(I-T)Aw^k\|^2. \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}2\tau_k|\langle A^*(I-T)Aw^k, w^k - x^{k+1}\rangle| &\le 2\tau_k\|A^*(I-T)Aw^k\|\|w^k - x^{k+1}\|\\ &\le \tau\|w^k - x^{k+1}\|^2 + \frac{\tau_k^2}{\tau}\|A^*(I-T)Aw^k\|^2.\end{aligned}$$

If $\|A^*(I-T)Aw^k\| = 0$, then $\tau_k = 0$, so that

$$\|x^{k+1} - z\|^2 \le \|w^k - z\|^2 - (1-\tau)\|w^k - x^{k+1}\|^2.$$

Otherwise, if $\|A^*(I-T)Aw^k\| \ne 0$, we have

$$\begin{aligned}&\|x^{k+1} - z\|^2\\ &\le \|w^k - z\|^2 - (1-\tau)\|w^k - x^{k+1}\|^2 + \frac{\tau_k^2}{\tau}\|A^*(I-T)Aw^k\|^2 - 2\tau_k\|(I-T)Aw^k\|^2\\ &= \|w^k - z\|^2 - (1-\tau)\|w^k - x^{k+1}\|^2 + \frac{\rho_k^2}{\tau}\frac{\|(I-T)Aw^k\|^4}{\|A^*(I-T)Aw^k\|^2} - 2\rho_k\frac{\|(I-T)Aw^k\|^4}{\|A^*(I-T)Aw^k\|^2}\\ &= \|w^k - z\|^2 - (1-\tau)\|w^k - x^{k+1}\|^2 - \rho_k(2 - \frac{\rho_k}{\tau})\frac{\|(I-T)Aw^k\|^4}{\|A^*(I-T)Aw^k\|^2}.\end{aligned}$$

Hence the desired inequality (3.3) follows.

Lemma 3.4 *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm 1. Assume that $\{x^k\}$ and $\{w^k\}$ are bounded such that*

$$\lim_{k\to\infty}\delta_k = \lim_{k\to\infty}\|x^k - w^k\| = \lim_{k\to\infty}\|x^{k+1} - w^k\| = 0, \tag{3.5}$$

where δ_k is defined as in (3.4)*. Then each weak cluster point of $\{x^k\}$ belongs to S.*

Proof. Let $\bar{x}$ be any weak cluster point of $\{x^k\}$. Thus, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\{x^{k_i}\}$ is weakly convergent to $\bar{x}$. In view of (3.5), $\{w^{k_i}\}$ is also weakly convergent to $\bar{x}$, which implies that $\{Aw^{k_i}\}$ weakly converges to $A\bar{x}$.

We first show $\bar{x} \in F(U)$. To see this, let $y^k = w^k - \tau_k A^*(I-T)Aw^k$. We have

$$\begin{aligned}\|y^k - w^k\| &= \tau_k\|A^*(I-T)Aw^k\|\\ &= \rho_k\sqrt{\delta_k} \le 2\sqrt{\delta_k}.\end{aligned}$$

Hence, by (3.5), $\lim_{k\to\infty}\|y^k - w^k\| = 0$, which yields that $\{y^{k_i}\}$ also weakly converges to $\bar{x}$. Moreover, it follows again from (3.5) that

$$\begin{aligned}\|(I-U)y^k\| &= \|y^k - x^{k+1}\|\\ &\le \|y^k - w^k\| + \|w^k - x^{k+1}\| \to 0.\end{aligned}$$

Since $I - U$ is demiclosed at 0, we have $\bar{x} \in F(U)$.

We next show $A\bar{x} \in F(T)$. Indeed, if $\|A^*(I-T)Aw^k\| \ne 0$, then

$$\|(I-T)Aw^k\|^2 = \frac{\|(I-T)Aw^k\|^4}{\|(I-T)Aw^k\|^2} \le \|A\|^2\frac{\|(I-T)Aw^k\|^4}{\|A^*(I-T)Aw^k\|^2}.$$

Hence, by (3.5), we have

$$\|(I-T)Aw^k\|^2 \le \|A\|^2\delta_k \to 0.$$

Since $Aw^{k_i} \rightharpoonup A\bar{x}$ and $I - T$ is demiclosed at 0, we have $A\bar{x} \in F(T)$.

Altogether, $\bar{x}$ is a solution of the SFP. This ends the proof.

4. Convergence analysis

In this section, we will establish the convergence of the proposed algorithm. We first study the convergence of Algorithm 1 under the condition:

$$\sum_{k=1}^{\infty} \theta_k \|x^k - x^{k-1}\|^2 < \infty. \tag{a1}$$

It is readily seen that condition (a1) indicates

$$\lim_{k\to\infty} \theta_k \|x^k - x^{k-1}\|^2 = 0. \tag{4.1}$$

Theorem 4.1 *Assume that* $\liminf_{k\to\infty} \rho_k(2-\rho_k) > 0$ *and* θ_k *satisfies condition (a1). Then the sequence* $\{x^k\}$ *generated by Algorithm 1 weakly converges to a solution of SCFP* (1.1).

Proof. We first show, for any $z \in S$, the sequence $\{\|x^k - z\|\}$ is convergent. To see this, it suffices to show the following inequality:

$$\phi_{k+1} - \phi_k \le \theta_k(\phi_k - \phi_{k-1}) + 2\theta_k \|x^k - x^{k-1}\|^2 - \rho_k(2-\rho_k)\delta_k, \tag{4.2}$$

where $\phi_k = \|x^k - z\|^2$ and δ_k is defined as in (3.4). Indeed, by applying Lemma 3.3 with $\tau = 1$, we have

$$\|x^{k+1} - z\|^2 \le \|w^k - z\|^2 - \rho_k(2-\rho_k)\delta_k. \tag{4.3}$$

On the other hand, we deduce from Lemma 2 that

$$\begin{aligned}\|w^k - z\|^2 &= \|(1+\theta_k)(x^k - z) - \theta_k(x^{k-1} - z)\|^2 \\ &= (1+\theta_k)\|x^k - z\|^2 - \theta_k\|x^{k-1} - z\|^2 + \theta_k(1+\theta_k)\|x^k - x^{k-1}\|^2 \quad (4.4) \\ &\le (1+\theta_k)\|x^k - z\|^2 - \theta_k\|x^{k-1} - z\|^2 + 2\theta_k\|x^k - x^{k-1}\|^2. \quad (4.5)\end{aligned}$$

Combining (4.3) and (4.5) at once yields (4.2) as desired. Note that by (a1)

$$\sum_{k=1}^{\infty} 2\theta_k \|x^k - x^{k-1}\|^2 < \infty.$$

By Lemma 2, the limit of $\{\phi_k\}$ exists. That is, the sequence $\{\|x^k - z\|\}$ is convergent. We next show that each weak cluster point of $\{x^k\}$ belongs to S. By Lemma 3.4, it suffices to show that condition (3.5) holds true. Since $\{\phi_k\}$ is convergent, $\{x^k\}$ is bounded. By inequality (4.5), the sequence $\{w^k\}$ is also bounded. Thus, we may assume that for all $k \ge 0$, there is $M > 0$ such that

$$4\|x^k - z\| + \|A\|^2\|w^k - z\| \le M.$$

By passing to the limit in (4.2) and our hypothesis on ρ_k, we have

$$\lim_{k\to\infty} \delta_k = 0. \tag{4.6}$$

Moreover, by (4.1) we have

$$\lim_{k\to\infty} \|w^k - x^k\|^2 = \lim_{k\to\infty} \theta_k \|x^k - x^{k-1}\|^2 = 0. \tag{4.7}$$

On the other hand, by Lemma 2, we have

$$\langle w^k - \tau_k A^*(I-T)Aw^k - x^{k+1}, x^{k+1} - z\rangle \geq 0,$$

which at once yields

$$\langle x^{k+1} - w^k, x^{k+1} - z\rangle \leq -\tau_k \langle A^*(I-T)Aw^k, x^{k+1} - z\rangle.$$

From this inequality it then follows that

$$\begin{aligned}
\|x^{k+1} - w^k\|^2 &= (\|w^k - z\|^2 - \|x^{k+1} - z\|^2) + 2\langle x^{k+1} - w^k, x^{k+1} - z\rangle \\
&\leq (\|w^k - z\|^2 - \|x^{k+1} - z\|^2) - 2\tau_k \langle A^*(I-T)Aw^k, x^{k+1} - z\rangle \\
&\leq (\|w^k - z\|^2 - \|x^{k+1} - z\|^2) + 2\tau_k \|A^*(I-T)Aw^k\| \|x^{k+1} - z\| \\
&= (\|w^k - z\|^2 - \|x^{k+1} - z\|^2) + 2\rho_k \sqrt{\delta_k} \|x^{k+1} - z\| \\
&\leq (\|w^k - z\|^2 - \|x^{k+1} - z\|^2) + 4\sqrt{\delta_k} \|x^{k+1} - z\| \\
&\leq (\|w^k - z\|^2 - \|x^{k+1} - z\|^2) + M\sqrt{\delta_k}.
\end{aligned}$$

Thanks to (4.6) and (4.7), $\lim_{k\to\infty} \|x^{k+1} - w^k\| = 0$. By Lemma 3.4, we conclude that each weak cluster point of $\{x^k\}$ belongs to S.

Finally, by Lemma 2, the sequence $\{x^k\}$ converges weakly to a solution of SCFP (1.1).

Remark 4.2 For some $\theta \in (0, 1)$ let us define a sequence $\{\theta_k\}$ as

$$\theta_k = \begin{cases} \min\left(\theta, \dfrac{1}{(k+1)^2\|x^k - x^{k-1}\|^2}\right), & x^k \neq x^{k-1}, \\ \theta, & x^k = x^{k-1}. \end{cases}$$

It is clear that such a $\{\theta_k\}$ fulfills condition (a1).

We next study the convergence of Algorithm 1 under the condition:

$$\theta_k \uparrow \theta \in [0, \sqrt{5} - 2); \tag{a2}$$

namely, the sequence $\{\theta_k\}$ is nondecreasing and converges to $\theta \in [0, \sqrt{5} - 2)$.

Theorem 4.3 *Assume that* $\liminf_{k\to\infty} \rho_k(1 - \rho_k) > 0$ *and* θ_k *satisfies condition (a2). Then the sequence* $\{x^k\}$ *generated by Algorithm 1 weakly converges to a solution of SCFP* (1.1).

Proof. We first show, for each $z \in S$, the following inequality:

$$\phi_k - \phi_{k+1} \geq \frac{1}{2}(1 - 4\theta_{k+1} - \theta_{k+1}^2)\|x^k - x^{k+1}\|^2 + 2\rho_k(1 - \rho_k)\delta_k, \tag{4.8}$$

where δ_k is defined as in (3.4) and

$$\phi_k = \|x^k - z\|^2 - \theta_k\|x^{k-1} - z\|^2 + \frac{\theta_k}{2}(3 + \theta_k)\|x^k - x^{k-1}\|^2. \tag{4.9}$$

Indeed, it follows from the Cauchy-Schwartz inequality that

$$\begin{aligned}\|w^k - x^{k+1}\|^2 &= \|x^k - x^{k+1} + \theta_k(x^k - x^{k-1})\|^2 \\ &= \|x^k - x^{k+1}\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 + 2\theta_k\langle x^k - x^{k+1}, x^k - x^{k-1}\rangle \\ &\ge \|x^k - x^{k+1}\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 - 2\theta_k\|x^k - x^{k+1}\|\|x^k - x^{k-1}\| \\ &\ge \|x^k - x^{k+1}\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 - \theta_k(\|x^k - x^{k+1}\|^2 + \|x^k - x^{k-1}\|^2) \\ &= (1-\theta_k)\|x^k - x^{k+1}\|^2 - \theta_k(1-\theta_k)\|x^k - x^{k-1}\|^2. \qquad (4.10)\end{aligned}$$

Applying Lemma 2 with $\tau = 1/2$, we have

$$\|x^{k+1} - z\|^2 \le \|w^k - z\|^2 - \frac{1}{2}\|w^k - x^{k+1}\|^2 - 2\rho_k(1-\rho_k)\delta_k. \qquad (4.11)$$

Combining (4.10) and (4.11) yields

$$\begin{aligned}\|x^{k+1} - z\|^2 &\le \|w^k - z\|^2 - 2\rho_k(1-\rho_k)\delta_k \\ &\quad - \frac{1}{2}(1-\theta_k)\|x^k - x^{k+1}\|^2 + \frac{\theta_k}{2}(1-\theta_k)\|x^k - x^{k-1}\|^2 \\ &= (1+\theta_k)\|x^k - z\|^2 - \theta_k\|x^{k-1} - z\|^2 + \theta_k(1+\theta_k)\|x^k - x^{k-1}\|^2 \\ &\quad - \frac{1}{2}(1-\theta_k)\|x^k - x^{k+1}\|^2 + \frac{\theta_k}{2}(1-\theta_k)\|x^k - x^{k-1}\|^2 - 2\rho_k(1-\rho_k)\delta_k \\ &= (1+\theta_k)\|x^k - z\|^2 - \theta_k\|x^{k-1} - z\|^2 + \frac{\theta_k}{2}(3+\theta_k)\|x^k - x^{k-1}\|^2 \\ &\quad - \frac{1}{2}(1-\theta_k)\|x^k - x^{k+1}\|^2 - 2\rho_k(1-\rho_k)\delta_k \\ &\le (1+\theta_{k+1})\|x^k - z\|^2 - \theta_k\|x^{k-1} - z\|^2 + \frac{\theta_k}{2}(3+\theta_k)\|x^k - x^{k-1}\|^2 \\ &\quad - \frac{1}{2}(1-\theta_{k+1})\|x^k - x^{k+1}\|^2 - 2\rho_k(1-\rho_k)\delta_k,\end{aligned}$$

where the first equality follows from (4.4) and the last inequality follows from the monotone property of θ_k. By the definition of ϕ_k and our hypothesis on θ_k, we get inequality (4.8) as desired.

We next show that each weak cluster point of $\{x^k\}$ belongs to S. Indeed, by (4.8) the sequence $\{\phi_k\}$ is clearly nonincreasing. On the other hand,

$$\|x^k - z\|^2 \le \theta_k\|x^{k-1} - z\|^2 + \phi_k \le \theta\|x^{k-1} - z\|^2 + \phi_k,$$

which by induction yields

$$\|x^k - z\|^2 \le \|x^0 - z\|^2 + \frac{\phi_1}{1-\theta}.$$

Thus $\{x^k\}$ is bounded, and so is the sequence $\{w^k\}$ by (4.10). It then follows from the definition of ϕ_k that

$$\begin{aligned}\phi_{k+1} &\ge -\theta_{k+1}\|x^k - z\|^2 \ge -\|x^k - z\|^2 \\ &\ge -\|x^0 - z\|^2 - \frac{\phi_1}{1-\theta}.\end{aligned}$$

This implies that $\{\phi_k\}$ is bounded from below, and thus $\{\phi_k\}$ is a convergent sequence. Passing to the limit in (4.8) yields

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty}(1 - 4\theta_{k+1} - \theta_{k+1}^2)\|x^{k+1} - x^k\|^2 = 0.$$

However, from (a2) we have $1 - 4\theta_{k+1} - \theta_{k+1}^2 \geq 1 - 4\theta - \theta^2 > 0$, which yields

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{4.12}$$

This together with (5.2) yields

$$\|x^k - w^k\| = \theta_k\|x^{k-1} - x^k\| \leq \|x^{k-1} - x^k\| \to 0,$$

and

$$\|x^{k+1} - w^k\| \leq \|x^{k+1} - x^k\| + \|x^k - w^k\| \to 0.$$

Thus, condition (3.5) holds true. By Lemma 3.4, the desired assertion follows.

Finally, we show that the sequence $\{x^k\}$ converges weakly to a solution of SCFP (1.1). By Lemma 2, it suffices to show the convergence of $\{\|x^k - z\|\}$. Thanks to (4.9), we have

$$\|x^k - z\|^2 = \frac{1}{1-\theta_k}\left(\phi_k + \theta_k(\|x^{k-1} - z\|^2 - \|x^k - z\|^2) - \frac{\theta_k(3+\theta_k)}{2}\|x^k - x^{k-1}\|^2\right).$$

Since

$$\begin{aligned}
&\theta_k \left|\|x^{k-1} - z\|^2 - \|x^k - z\|^2\right| \\
&= \theta_k \left|\|x^{k-1} - z\| - \|x^k - z\|\right| (\|x^{k-1} - z\| + \|x^k - z\|) \\
&\leq \theta\|x^{k-1} - x^k\|(\|x^{k-1} - z\| + \|x^k - z\|),
\end{aligned}$$

it follows from (4.12) that

$$\lim_{k\to\infty} \theta_k(\|x^{k-1} - z\|^2 - \|x^k - z\|^2) = 0.$$

Consequently, $\{\|x^k - z\|\}$ is convergent, which ends the proof.

5. An application

In statistics and machine learning, the elastic net is a regression analysis method that performs both variable selection and regularization in order to enhance the prediction accuracy and interpretability of the statistical model it produces. It is a regularized regression method that linearly combines the L_1 and L_2 penalties of the LASSO and ridge methods. Here the L_1 penalty is defined as

$$\|x\|_1 = \sum_{i=1}^n |x_i|,$$

and the L_2 penalty is defined as

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}.$$

The elastic net requires to solve the problem:

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & \frac{1}{2}\|Ax-y\|_2^2, \\ \text{s.t.} & (1-\lambda)\|x\|_1+\lambda\|x\|_2^2\le t, \end{array} \tag{5.1}$$

where λ is a positive parameter, $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ and $t > 0$ is a constant. This problem is a specific SCFP. To see this, it suffices to let $Tz = y, \forall z \in \mathbb{R}^m$ and $U = P_C$ with

$$C = \{x \in \mathbb{R}^n \mid (1-\lambda)\|x\|_1 + \lambda\|x\|_2^2 \le t\}.$$

By applying Algorithm 1, we get an algorithm for solving problem (5.1). Let x^0, x^1 be arbitrary. Given x^k, x^{k-1}, update the next iteration via

$$\left[\begin{array}{l} w^k = x^k + \theta_k(x^k - x^{k-1}) \\ x^{k+1} = U(w^k - \tau_k A^*(Aw^k - y)), \end{array}\right. \tag{5.2}$$

where $0 < \rho_k < 2, 0 \le \theta_k < \theta < 1$, and

$$\tau_k = \begin{cases} \rho_k \dfrac{\|Aw^k-y\|^2}{\|A^*(Aw^k-y)\|^2}, & \|A^*(Aw^k-y)\| \ne 0, \\ 0, & \|A^*(Aw^k-y)\| = 0. \end{cases} \tag{5.3}$$

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