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ON AN EQUATION CHARACTERIZING MULTI-CUBIC MAPPINGS AND ITS STABILITY AND HYPERSTABILITY

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Abstract. In this paper, we introduce n-variables mappings which are cubic in each variable. We show that such mappings satisfy a functional equation. The main purpose is to extend the applications of a fixed point method to establish the Hyers-Ulam stability for the multi-cubic mappings. As a consequence, we prove that a multi-cubic functional equation can be hyperstable.
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1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [18] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [12]. Later on, various generalizations and extension of Hyers' result were ascertained by Aoki [1], Th. M. Rassias [17], J. M. Rassias [16] and Găvruţa [11] in different versions. Since then, the stability problems have been extensively investigated for a variety of functional equations and spaces.

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [9] that a mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x)+A(y)$) in each variable. Some facts on such mappings can be found in [15] and many other sources. In addition, f is said to be *multi-quadratic* if it is quadratic (satisfies quadratic functional equation $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$) in each variable [10]. In [19], Zhao et al. proved that the mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if the following relation holds

$$\sum_{t\in\{-1,1\}^n} f(x_1 + tx_2) = 2^n \sum_{j_1,j_2,\cdots,j_n\in\{1,2\}} f(x_{1j_1}, x_{2j_2}, \cdots, x_{nj_n}) \tag{1.1}$$

where $x_j = (x_{1j}, x_{2j}, \cdots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. In [9] and [10], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [19]).

One of the functional equations in the field of stability of functional equations is the cubic functional equation

$$C(x+2y)-3C(x+y)+3C(x)-C(x-y)=6C(y) \tag{1.2}$$

which is introduced by J. M. Rassias in [16] for the first time. It is easy to see that the mapping $f(x)=ax^3$ satisfies (1.2). Thus, every solution of the cubic functional equation (1.2) is said to be a cubic mapping. Rassias established the Ulam-Hyers stability problem for these cubic mappings. The following alternative cubic functional equation

$$\mathfrak{C}(2x+y)+\mathfrak{C}(2x-y)=2\mathfrak{C}(x+y)+2\mathfrak{C}(x-y)+12\mathfrak{C}(x) \tag{1.3}$$

has been introduced by Jun and Kim in [14]. They found out the general solution and proved the Hyers-Ulam stability for functional equation (1.3); for other forms of the (generalized) cubic functional equations and their stabilities on the various Banach spaces refer to [3], [4], [5], [13].

In this paper, we define multi-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-cubic mappings to obtain a single equation. We also prove the generalized Hyers-Ulam stability for multi-cubic functional equations by applying the fixed point method which was introduced and used for the first time by Brzdęk et al., in [6]; for more applications of this approach for the satbility of multi-Cauchy-Jensen mappings in Banach spaces and 2-Banach spaces see [2] and [7], respectively.

2. Characterization of multi-cubic mappings

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0,\infty), n \in \mathbb{N}$. For any $l \in \mathbb{N}_0, m \in \mathbb{N}$, $t=(t_1,\cdots,t_m) \in \{-1,1\}^m$ and $x=(x_1,\cdots,x_m) \in V^m$ we write $lx := (lx_1,\cdots,lx_m)$ and $tx := (t_1x_1,\cdots,t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

From now on, let V and W be vector spaces over the rationals, $n \in \mathbb{N}$ and

$$x_i^n=(x_{i1},x_{i2},\cdots,x_{in}) \in V^n,$$

where $i \in \{1,2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. Let $x_1,x_2 \in V^n$ and $T \in \mathbb{N}_0$ with $0 \le T \le n$. Put

$$\mathcal{M}^n=\{(N_1,N_2,\cdots,N_n)|\ N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\},$$

where $j \in \{1,\cdots,n\}$. Consider

$$\mathcal{M}_T^n := \{\mathfrak{N}_n=(N_1,N_2,\cdots,N_n) \in \mathcal{M}^n|\ \mathrm{Card}\{N_j:\ N_j=x_{1j}\}=T\}.$$

For $r \in \mathbb{R}$, we put $r\mathcal{M}_T^n=\{r\mathfrak{N}_n : \mathfrak{N}_n \in \mathcal{M}_T^n\}$. We say the mapping $f: V^n \longrightarrow W$ is *n-multi-cubic* or *multi-cubic* if f is cubic in each variable (see equation (1.3)). For such mappings, we use the following notations:

$$f\left(\mathcal{M}_T^n\right) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f(\mathfrak{N}_n), \tag{2.1}$$

$$f\left(\mathcal{M}_T^n, z\right) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f(\mathfrak{N}_n, z) \qquad (z \in V).$$

Remark 2.1. It is easily verified that if the mapping h satisfies equation (1.3), then

$$h(2x) = 8h(x). \tag{2.2}$$

But the converse is not true. Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in $\mathcal{A}$ (not necessarily unit). Define the mapping $h : \mathcal{A} \longrightarrow \mathcal{A}$ by $h(a) = \|a\|^3 a_0$ for any $a \in \mathcal{A}$. Clearly, for each $x \in \mathcal{A}$, $h(2x) = 8h(x)$ while relation (1.3) does not hold for h even if we put $x = 0$ and $0 \neq y$. Therefore, condition (2.2) does not imply that h is a cubic mapping.

Proposition 2.2. *If the mapping $f : V^n \longrightarrow W$ is multi-cubic, then f satisfies the equation*

$$\sum_{q \in \{-1,1\}^n} f(2x_1 + qx_2) = \sum_{k=0}^{n} 2^{n-k} 12^k f(\mathcal{M}_k^n), \tag{2.3}$$

where $f\left(\mathcal{M}_k^n\right)$ is defined in (2.1).

Proof. We prove f satisfies equation (2.3) by induction on n. For $n = 1$, it is trivial that f satisfies equation (1.3). If (2.3) is valid for some positive integer $n > 1$, then,

$$\begin{aligned}
\sum_{q \in \{-1,1\}^{n+1}} f(2x_1^{n+1} + qx_2^{n+1}) &= 2 \sum_{q \in \{-1,1\}^n} f(2x_1^n + qx_2^n, x_{1n+1} + x_{2n+1}) \\
&\quad + 2 \sum_{q \in \{-1,1\}^n} f(2x_1^n + qx_2^n, x_{1n+1} - x_{2n+1}) \\
&\quad + 12 \sum_{q \in \{-1,1\}^n} f(2x_1^n + qx_2^n, x_{1n+1}) \\
&= 2 \sum_{k=0}^{n} \sum_{q \in \{-1,1\}} 2^{n-k} 12^k f(\mathcal{M}_k^n, x_{1n+1} + qx_{2n+1}) \\
&\quad + 12 \sum_{k=0}^{n} 2^{n-k} 12^k f(\mathcal{M}_k^n, x_{1n+1}) \\
&= \sum_{k=0}^{n+1} 2^{n+1-k} 12^k f(\mathcal{M}_k^{n+1}).
\end{aligned}$$

This means that (2.3) holds for $n + 1$. $\square$

In the sequel, $\begin{pmatrix} n \\ k \end{pmatrix}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $n!/(k!(n-k)!)$.

We say the mapping $f : V^n \longrightarrow W$ satisfies (has) *the r-power condition* in the jth variable if

$$f(z_1, \cdots, z_{j-1}, 2z_j, z_{j+1}, \cdots, z_n) = 2^r f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n),$$

for all $(z_1, \cdots, z_n) \in V^n$. It follows from Remark 2.1 that the 3-power condition does not imply f is cubic in the jth variable. Using this condition, we show that if f satisfies equation (2.3), then it is multi-cubic as follows:

Proposition 2.3. *If the mapping $f : V^n \longrightarrow W$ satisfies equation* (2.3) *and* 3*-power condition in each variable, then it is multi-cubic.*

Proof. Fix $j \in \{1, \cdots, n\}$. Putting $x_{2k} = 0$ for all $k \in \{1, \cdots, n\} \backslash \{j\}$ in the left side of (2.3) and using the assumption, we get

$$\begin{aligned}
&2^{n-1} \times 2^{3(n-1)}[f(x_{11}, \cdots, x_{1j-1}, 2x_{1j} + x_{2j}, x_{1j+1}, \cdots, x_{1n})\\
&+ f(x_{11}, \cdots, x_{1j-1}, 2x_{1j} - x_{2j}, x_{1j+1}, \cdots, x_{1n})]\\
&= 2^{n-1}[f(2x_{11}, \cdots, 2x_{1j-1}, 2x_{1j} + x_{2j}, 2x_{1j+1}, \cdots, 2x_{1n})\\
&+ f(2x_{11}, \cdots, 2x_{1j-1}, 2x_{1j} - x_{2j}, 2x_{1j+1}, \cdots, 2x_{1n})].
\end{aligned} \tag{2.4}$$

Set

$$\begin{aligned}
f^*(x_{1j}, x_{2j}) :&= f(x_{11}, \cdots, x_{1j-1}, x_{1j} + x_{2j}, x_{1j+1}, \cdots, x_{1n})\\
&\quad + f(x_{11}, \cdots, x_{1j-1}, x_{1j} - x_{2j}, x_{1j+1}, \cdots, x_{1n}).
\end{aligned}$$

By the mentioned replacements in (2.3), it follows from (2.4) that

$$\begin{aligned}
&2^{n-1} \times 2^{3(n-1)}[f(x_{11}, \cdots, x_{1j-1}, 2x_{1j} + x_{2j}, x_{1j+1}, \cdots, x_{1n})\\
&+ f(x_{11}, \cdots, x_{1j-1}, 2x_{1j} - x_{2j}, x_{1j+1}, \cdots, x_{1n})]\\
&= 2^{n-1} \times 2^n f^*(x_{1j}, x_{2j})\\
&+ \sum_{k=1}^{n-1} \left[\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} 2^{2(n-k)} \times 12^k \right] f(x_{11}, \cdots, x_{1n})\\
&+ \sum_{k=1}^{n-1} \left[\begin{pmatrix} n-1 \\ k \end{pmatrix} 2^{2(n-k)-1} \times 12^k \right] f^*(x_{1j}, x_{2j})\\
&+ 12^n f(x_{11}, \cdots, x_{1n})\\
&= \left[2^{2n-1} + \sum_{k=1}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} 2^{2(n-k)-1} \times 12^k \right] f^*(x_{1j}, x_{2j})\\
&+ \left[12^n + \sum_{k=1}^{n-1} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} 2^{2(n-k)} \times 12^k \right] f(x_{11}, \cdots, x_{1n}).
\end{aligned} \tag{2.5}$$

On the other hand, we have

$$\begin{aligned}
2^{2n-1} + \sum_{k=1}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} 2^{2(n-k)-1} \times 12^k &= 2^{2n-1} \left(1 + \sum_{k=1}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} 3^k \right)\\
&= 2^{2n-1} (1+3)^{n-1} = 2^{4n-3}.
\end{aligned} \tag{2.6}$$

In addition,

$$\begin{aligned}
12^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{2(n-k)} \times 12^k &= 12^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{2(n-k)} \times 2^{2k} \times 3^k \\
&= 12^n + 3 \times 2^{2n} \sum_{k=0}^{n-2} \binom{n-1}{k-1} 3^k \\
&= 12^n + 3 \times 2^{2n} \left(\sum_{k=0}^{n-1} \left[\binom{n-1}{k-1} 3^k \right] - 3^{n-1} \right) \\
&= 12^n + 3 \times 2^{2n} \left((1+3)^{n-1} - 3^{n-1} \right) \\
&= 12^n + 3 \times 2^{2n} \left(2^{2(n-1)} - 3^{n-1} \right) \\
&= 12 \times 2^{4(n-1)}. \qquad (2.7)
\end{aligned}$$

The relations (2.5), (2.6) and (2.7) imply that

$$\begin{aligned}
& f\left(x_{11}, \cdots, x_{1j-1}, 2x_{1j} + x_{2j}, x_{1j+1}, \cdots, x_{1n}\right) \\
&+ f\left(x_{11}, \cdots, x_{1j-1}, 2x_{1j} - x_{2j}, x_{1j+1}, \cdots, x_{1n}\right) \\
&= 2f^*(x_{1j}, x_{2j}) + 12 f\left(x_{11}, \cdots, x_{1n}\right).
\end{aligned}$$

This means that f is cubic in the jth variable. Since j is arbitrary, we obtain the desired result. □

3. Stability Results for (2.3)

In this section, we prove the generalized Hyers-Ulam stability of equation (2.3) by a fixed point result (Theorem 3.1) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. We introduce the upcoming three hypotheses:

(A1) Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \cdots, g_j : \mathcal{S} \longrightarrow \mathcal{S}$ and $L_1, \cdots, L_j : \mathcal{S} \longrightarrow \mathbb{R}_+$,

(A2) $\mathcal{T} : Y^{\mathcal{S}} \longrightarrow Y^{\mathcal{S}}$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{j} L_i(x) \left\|\lambda(g_i(x)) - \mu(g_i(x))\right\|, \quad \lambda, \mu \in Y^{\mathcal{S}}, x \in \mathcal{S},$$

(A3) $\Lambda : \mathbb{R}_+^{\mathcal{S}} \longrightarrow \mathbb{R}_+^{\mathcal{S}}$ is an operator defined through

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x) \delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^{\mathcal{S}}, x \in \mathcal{S}.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [6, Theorem 1]. This result plays a key tool to obtain our objective in this paper.

Theorem 3.1. *Let hypotheses* (A1)-(A3) *hold and the function* $\theta : \mathcal{S} \longrightarrow \mathbb{R}_+$ *and the mapping* $\phi : \mathcal{S} \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\phi(x) - \phi(x)\| \leq \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty \qquad (x \in \mathcal{S}).$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *such that*

$$\|\phi(x) - \psi(x)\| \leq \theta^*(x) \qquad (x \in \mathcal{S}).$$

Moreover, $\psi(x) = \lim_{l\to\infty} \mathcal{T}^l \phi(x)$ *for all* $x \in \mathcal{S}$*.*

Here and subsequently, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathfrak{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathfrak{D}f(x_1, x_2) := \sum_{q\in\{-1,1\}^n} f(2x_1 + qx_2) - \sum_{k=0}^{n} 2^{n-k} 12^k f\left(\mathcal{M}_k^n\right),$$

where $f\left(\mathcal{M}_k^n\right)$ is defined in (2.1). With this notation, we have the next stability result for functional equation (2.3).

Theorem 3.2. *Let* $\beta \in \{-1, 1\}$*,* V *be a linear space and* W *be a Banach space. Suppose that* $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying*

$$\lim_{l\to\infty} \left(\frac{1}{2^{3n\beta}}\right)^l \phi(2^{\beta l} x_1, 2^{\beta l} x_2) = 0 \tag{3.1}$$

for all $x_1, x_2 \in V^n$ *and*

$$\Phi(x) = \frac{1}{2^{3n\frac{\beta+1}{2}+n}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{3n\beta}}\right)^l \phi\left(2^{\beta l + \frac{\beta-1}{2}} x, 0\right) < \infty \tag{3.2}$$

for all $x \in V^n$*. Assume also* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\mathfrak{D}f(x_1, x_2)\|_Y \leqslant \phi(x_1, x_2) \tag{3.3}$$

for all $x_1, x_2 \in V^n$*. Then, there exists a unique solution* $\mathcal{C} : V^n \longrightarrow W$ *of (2.3) such that*

$$\|f(x) - \mathcal{C}(x)\| \leq \Phi(x) \tag{3.4}$$

for all $x \in V^n$*.*

Proof. Putting $x = x_1$ and $x_2 = 0$ in (3.3), we have

$$\left\|2^n f(2x) - \left(\sum_{k=0}^{n} \binom{n}{k} 2^{2(n-k)} \times 12^k\right) f(x)\right\| \leq \phi(x, 0) \tag{3.5}$$

for all $x \in V^n$. By an easy computation, we have

$$\sum_{k=0}^{n} \binom{n}{k} 2^{2(n-k)} \times 12^k = 2^{2n} \sum_{k=0}^{n} \binom{n}{k} 3^k = 2^{2n}(1+3)^n = 2^{4n}. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\left\|f(2x) - 2^{3n} f(x)\right\| \leq \frac{1}{2^n}\phi(x, 0) \tag{3.7}$$

for all $x \in V^n$. Set

$$\xi(x) := \frac{1}{2^{3n\frac{\beta+1}{2}+n}}\phi\left(2^{\frac{\beta-1}{2}}x, 0\right), \text{ and } \mathcal{T}\xi(x) := \frac{1}{2^{3n\beta}}\xi(2^\beta x) \qquad (\xi \in W^{V^n}).$$

Then, relation (3.7) can be modified as

$$\|f(x) - \mathcal{T}f(x)\| \leq \xi(x) \qquad (x \in V^n). \tag{3.8}$$

Define $\Lambda\eta(x) := \frac{1}{2^{3n\beta}}\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. We now see that Λ has the form described in (A3) with $\mathcal{S} = V^n$, $g_1(x) = 2^\beta x$ and $L_1(x) = \frac{1}{2^{3n\beta}}$ for all $x \in V^n$. Furthermore, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\|\frac{1}{2^{3n\beta}}\left[\lambda(2^\beta x) - \mu(2^\beta x)\right]\right\| \leq L_1(x)\left\|\lambda(g_1(x)) - \mu(g_1(x))\right\|.$$

The above relation shows that the hypotheis (A2) holds. By induction on l, one can check that for any $l \in \mathbb{N}_0$ and $x \in V^n$,

$$\Lambda^l\xi(x) := \left(\frac{1}{2^{3n\beta}}\right)^l \xi(2^{\beta l}x) = \frac{1}{2^{3n\frac{\beta+1}{2}+n}}\left(\frac{1}{2^{3n\beta}}\right)^l \phi\left(2^{\beta l+\frac{\beta-1}{2}}x, 0\right) \tag{3.9}$$

for all $x \in V^n$. The relations (3.2) and (3.9) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\mathcal{C}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x) = \frac{1}{2^{3n\beta}}\mathcal{C}(2^\beta x) \qquad (x \in V^n),$$

and (3.4) holds. We shall to show that

$$\|\mathfrak{D}(\mathcal{T}^l f)(x_1, x_2)\| \leq \left(\frac{1}{2^{3n\beta}}\right)^l \phi(2^{\beta l}x_1, 2^{\beta l}x_2) \tag{3.10}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. It is easily seen that (3.10) is valid for $l = 0$ by (3.3). Assume that (3.10) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\|\mathfrak{D}(\mathcal{T}^{l+1} f)(x_1, x_2)\| \\
&= \left\|\sum_{q\in\{-1,1\}^n}(\mathcal{T}^{l+1}f)(2x_1 + qx_2) - \sum_{k=0}^{n} 2^{n-k}12^k(\mathcal{T}^{l+1}f)(\mathcal{M}_k^n)\right\| \\
&= \frac{1}{2^{3n\beta}}\left\|\sum_{q\in\{-1,1\}^n}(\mathcal{T}^{l}f)(2^\beta(2x_1 + qx_2)) - \sum_{k=0}^{n} 2^{n-k}12^k(\mathcal{T}^{l}f)(2^\beta\mathcal{M}_k^n)\right\| \\
&= \frac{1}{2^{3n\beta}}\left\|\mathfrak{D}(\mathcal{T}^l f)(2^\beta x_1, 2^\beta x_2)\right\| \leq \left(\frac{1}{2^{3n\beta}}\right)^{l+1}\phi(2^{\beta(l+1)}x_1, 2^{\beta(l+1)}x_2)
\end{aligned} \tag{3.11}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.10) and applying (3.1), we arrive at $\mathfrak{D}\mathcal{C}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathcal{C}$ satisfies (2.3). Finally, assume that $\mathcal{C}' : V^n \longrightarrow W$ is another multi-cubic mapping satisfying equation

(2.3) and inequality (3.4), and fix $x \in V^n$, $j \in \mathbb{N}$. Then

$$\begin{aligned}
&\|\mathcal{C}(x) - \mathcal{C}'(x)\| \\
&= \left\| \frac{1}{2^{3n\beta j}}\mathcal{C}(2^{\beta j}x) - \frac{1}{2^{3n\beta j}}\mathcal{C}'(2^{\beta j}x) \right\| \\
&\leq \frac{1}{2^{3n\beta j}}(\|\mathcal{C}(2^{\beta j}x) - f(2^{\beta j}x)\| + \|\mathcal{C}'(2^{\beta j}x) - f(2^{\beta j}x)\|) \\
&\leq \frac{1}{2^{3n\beta j}} 2\Phi(2^{\beta j}x) \\
&\leq 2\frac{1}{2^{3n\frac{\beta+1}{2}+n}} \sum_{l=j}^{\infty} \left(\frac{1}{2^{3n\beta}}\right)^l \phi\left(2^{\beta l + \frac{\beta-1}{2}}x, 0\right).
\end{aligned}$$

Consequently, letting $j \to \infty$ and using the fact that series (3.2) is convergent for all $x \in V^n$, we obtain $\mathcal{C}(x) = \mathcal{C}'(x)$ for all $x \in V^n$, which finishes the proof. $\square$

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathcal{F}_1\varphi(a_1, \cdots, a_n) = \mathcal{F}_2\varphi(a_1, \cdots, a_n) \tag{3.12}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathcal{F}_1\varphi_0(a_1, \cdots, a_n), \mathcal{F}_2\varphi_0(a_1, \cdots, a_n)) \leq \psi(a_1, \cdots, a_n), \qquad a_1, \cdots, a_n \in A,$$

fulfils (3.12); this definition is introduced in [8]. In other words, a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$.

Under some conditions the functional equation (2.3) can be hyperstable as follows.

Corollary 3.4. *Let $\delta > 0$. Suppose that $p_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \cdots, n\}$ fulfill*

$$\sum_{i=1}^{2}\sum_{j=1}^{n} p_{ij} \neq 3n.$$

Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\mathfrak{D}f(x_1, x_2)\| \leq \prod_{i=1}^{2}\prod_{j=1}^{n} \|x_{ij}\|^{p_{ij}}\delta$$

for all $x_1, x_2 \in V^n$, then f satisfies (2.3).

In the following corollary, we show that functional equation (2.3) is stable. Since the proof is routine, we include it without proof.

Corollary 3.5. *Given $\delta > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 3n$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}f(x_1, x_2)\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^{\alpha}\delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{C} : V^n \longrightarrow W$ of (2.3) such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \dfrac{\delta}{2^{4n} - 2^{\alpha+n}} \displaystyle\sum_{j=1}^{n} \|x_{1j}\|^{\alpha} & \alpha < 3n \\ \dfrac{\delta}{2^{\alpha+n} - 2^{4n}} \displaystyle\sum_{j=1}^{n} \|x_{1j}\|^{\alpha} & \alpha > 3n \end{cases}$$

for all $x \in V^n$.

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