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A NOTE ON BEST PROXIMITY POINT FOR S-CYCLIC MAPPINGS

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Abstract. In this note, we introduce the new class of $S-cyclic$ mappings and prove some best proximity point thereof. Since $S-cyclic$ mappings generalize the ordinary cyclic mappings, our results extend and improve some best proximity results for cyclic mappings.
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1. Introduction

Let A, B be nonempty subsets of a normed linear space X, a mapping $T : A \cup B \longrightarrow A \cup B$ is said to be cyclic (*resp.* noncyclic) if $T(A) \subseteq B$ and $T(B) \subseteq A$ (*resp.* $T(A) \subseteq A$ and $T(B) \subseteq B$). Clearly, when A, B are disjoint, the fixed point problem for the cyclic case will possess no solution. Then, it is natural to wonder whether or no the mapping T assures existence of, if T is cyclic, a best proximity point, *i.e.* a point $x \in A \cup B$ such that $\|x - Tx\| = dist(A, B)$, or, if T is noncyclic, a best proximity pair, *i.e.* a pair $(x, y) \in A \times B$ such that x and y are fixed point of T and $\|x - y\| = dist(A, B)$. In solving this problem, the authors of [2] introduced a new class of mappings.

Definition 1.1 ([2]) Retaining the same notations, T is said to be relatively nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in A$ and $y \in B$.

The assumption of classical nonexpansiveness is stronger than the one of relatively nonexpansiveness. In fact, a relatively nonexpansive mapping needn't even be continuous.

Afterwards, relatively nonexpansive mappings were also generalized in [3].

Definition 1.2 ([3]) Retaining the same notations, T is said to be relatively u-continuous if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Tx - Ty\| < \varepsilon + dist(A, B)$ whenever $\|x - y\| < \delta + dist(A, B)$, for all $x \in A$ and $y \in B$.

As for results related to these two class of mappings. Here, we state some noteworthy theorems.

Theorem 1.3 ([2]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X. Let $T : A \cup B \longrightarrow A \cup B$ be a cyclic relatively nonexpansive mapping. Then T has a best proximity point.*

Theorem 1.4 ([2]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X. Let $T : A \cup B \longrightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping. Then T has a best proximity pair.*

In [5], the authors established an extended version of the latter.

Theorem 1.5 ([5]) *Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X. Let $T : A \cup B \longrightarrow A \cup B$ be a noncyclic relatively u-continuous mapping. Then T has a best proximity pair.*

All the same, an identical result to the previous one was proven for the cyclic case.

Theorem 1.6 ([3]) *Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X. Let $T : A \cup B \longrightarrow A \cup B$ be a cyclic relatively u-continuous mapping. Then T has a best proximity point.*

Very recently, the current authors introduced the class of tricyclic mappings and best proximity points thereof. A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be tricyclic provided that $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$, where A, B, C are nonempty subsets of a metric space (X, d), and a best proximity point of T is a point $x \in A \cup B \cup C$ such that $D\left(x, Tx, T^2x\right) = D(A, B, C)$ where the mapping $D : X \times X \times X \longrightarrow [0, +\infty)$ is defined by $D(x, y, z) := d(x, y) + d(y, z) + d(z, x)$ and $D(A, B, C) := \inf\{D(x, y, z) : x \in A,\ y \in B \text{ and } z \in C\}$. For detailed informations, we refer to [7, 8].

In this paper, we introduce an interesting new class of mappings we shall call $S - cyclic$. In a manner of speaking, the class of $S - cyclic$ mappings is situated somewhere between the class of tricyclic mappings and the class of cyclic ones, more precisely, it is somewhat included in the class of tricyclic mappings but wider than the one of cyclic mappings for every $S - cyclic$ mapping can particularly be a cyclic mapping. Consequently, our results are not only new but extend and reinforce some results of the literature.

2. Preliminaries

Before proceeding to our results, we fix some definitions and notations that will subsequently be used. Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d), then the proximal pair (A_0, B_0) of (A, B) is given by:

$$\begin{aligned} A_0 &: = \{x \in A : d(x, y') = dist(A, B) \text{ for some } y' \in B\}, \\ B_0 &: = \{y \in B : d(x', y) = dist(A, B) \text{ for some } x' \in A\}. \end{aligned}$$

While the proximal triad (A_{00}, B_{00}, C_{00}) of (A, B, C) is defined by:

$$\begin{aligned} A_{00} &= \{x \in A : D(x, y', z'') = D(A, B, C) \text{ for some } y' \in B \text{ and } z'' \in C\}, \\ B_{00} &= \{y \in B : D(x'', y, z') = D(A, B, C) \text{ for some } x'' \in A \text{ and } z' \in C\}, \\ C_{00} &= \{z \in C : D(x', y'', z) = D(A, B, C) \text{ for some } x' \in A \text{ and } y'' \in B\}. \end{aligned}$$

Note that if (A, B, C) is a triad of nonempty, bounded, closed and convex subsets of a Banach space, then so is the triad (A_{00}, B_{00}, C_{00}), and that

$$D(A, B, C) = D(A_{00}, B_{00}, C_{00}).$$

We also point out that in the special case where $C = A$, we have $A_{00} = A_0$ and $B_{00} = B_0$.

In [6], the authors introduced the notion of P-property for pairs. Herein, we define the P-property for triads and we introduce the notion of $P^{\maltese}$-property for both pairs and triads.

Definition 2.1 Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \varnothing$. The pair (A, B) is said to have the P-property if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2).$$

And, we shall say that (A, B) has the $P^{\maltese}$-property if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{array}\right\} \Longrightarrow d(x_1, y_2) = d(x_2, y_1),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

For example, it was shown in [1, 6] that the P-property is satisfied if A, B are nonempty, bounded, closed and convex subsets of a uniformly convex Banach space. Now, we prove that the $P^{\maltese}$-property is fulfilled whenever A, B are nonempty, convex subsets of a Hilbert space. The next lemma is crucial to our claim.

Lemma 2.2 ([4])*Let (A, B) be a pair of nonempty closed convex subsets of a real Hilbert space and let $x, y \in A_0$. Then $\|x - P_B(y)\| = \|y - P_B(x)\|$.*

Proposition 2.3 *Any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H has the $P^{\maltese}$-property.*

Proof. Let $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ such that $\|x_1 - y_1\| = \|x_2 - y_2\| = dist(A, B)$. Since the metric projection operator on a Hilbert space is unique, $y_1 = P_B(x_1)$ and $y_2 = P_B(x_2)$. Now, we apply the previous lemma.

Definition 2.4 Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d) with A_{00} is nonempty. We say that (A, B, C) has the P-property if

$$\left.\begin{array}{l} D(x_1, y_1, z_1) = D(A, B, C) \\ D(x_2, y_2, z_2) = D(A, B, C) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2) = d(z_1, z_2).$$

We also shall say that (A, B, C) has the $P^{\maltese}$-property provided that

$$\left.\begin{array}{l} D(x_1, y_1, z_1) = D(A, B, C) \\ D(x_2, y_2, z_2) = D(A, B, C) \end{array}\right\} \Longrightarrow d(x_1, y_2) = d(y_1, z_2) \text{ and } d(x_2, y_1) = d(y_2, z_1),$$

where $x_1, x_2 \in A_{00}$; $y_1, y_2 \in B_{00}$ and $z_1, z_2 \in C_{00}$.

Here is a simple example of a triad which has the P-property and the $P^{✠}$-property at once.

Example 2.5 Let X be $\mathbb{R}^2$ endowed with its Euclidean distance. Consider

$$A = \{0\} \times [0,1], \; B = \{1\} \times [0,1] \text{ and } C = \{2\} \times [0,1].$$

Then clearly (A, B, C) has both the P-property and $P^{✠}$-property.

Note that in the special case where $A = C$, we have

$$D(A, B, C) = D(A, B, A) = 2dist(A, B).$$

Then we necessarily have

$$x_1 = z_1, x_2 = z_2 \text{ and } d(x_1, y_1) = d(x_2, y_2) = dist(A, B).$$

Thus, the traid (A, B, A) has the P-property and the $P^{✠}$-property if the pair (A, B) does.

Definition 2.6 Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \longrightarrow B \cup C$ is said to be

- S-cyclic if, $T(A) \subseteq B$ and $T(B) \subseteq C$.
- S-cyclic relatively nonexpansive if T is S-cyclic and

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x \in A \text{ and } y \in B.$$

- S-cyclic relatively u-continuous if T is S-cyclic and for each $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\|x - y\| < \delta + dist(A, B) \Longrightarrow \|Tx - Ty\| < \varepsilon + dist(B, C), \text{ for all } x \in A \text{ and } y \in B.$$

In the coming two sections, we show that the existence of a best proximity point for a S-cyclic mapping can be obtained from the existence of a best proximity pair for a noncyclic mapping. Hence, we obtain an already proved best proximity point result for usual cyclic mappings and extend another one.

3. S-cyclic relatively nonexpansive mappings

We begin our argument with the following lemma.

Lemma 3.1 *Let (A, B, C) be a triad of nonempty, closed subsets of a metric space (X, d) such that A_{00} is nonempty. Assume that the triad (A, B, C) has both the P-property and $P^{✠}$-property. Then there exists a bijective, S-cyclic isometry $g : A_{00} \cup B_{00} \longrightarrow B_{00} \cup C_{00}$ such that $D(x, gx, g^2x) = D(g^{-1}y, y, gy) = D(A, B, C)$ for all $x \in A_{00}$ and $y \in B_{00}$.*

Proof. Let $x \in A_{00}$, then there exists a couple $(y', z) \in B_{00} \times C_{00}$ such that $D(x, y', z) = D(A, B, C)$. Taking into consideration that (A, B, C) has the P-property, we get the uniqueness of the couple (y', z). Now, for all $y \in B_{00}$, there exists a unique $(x', z') \in A_{00} \times C_{00}$ such that $D(x', y, z') = D(A, B, C)$. Consider the S-cyclic mapping $g : A_{00} \cup B_{00} \longrightarrow B_{00} \cup C_{00}$ such that, for all $x \in A_{00}$ and $y \in B_{00}$,

$$D(x, gx, z) = D(x', y, gy) = D(A, B, C) \text{ for some } x\prime \in A_{00} \text{ and } z \in C_{00}.$$

Clearly g is well defined. Take $x \in A_{00}$, we have

$$D(x, gx, z) = D\left(x', gx, g^2x\right) = D(A, B, C) \text{ for some } x\prime \in A_{00} \text{ and } z \in C_{00}.$$

The P-property of (A, B, C) implies that $x = x'$ and $g^2x = z$. Hence

$$D\left(x, gx, g^2x\right) = D(A, B, C) \text{ for all } x \in A_{00}.$$

Moreover, g is an isometry. Indeed, let $x_1, x_2 \in A_{00} \cup B_{00}$. If $x_1, x_2 \in A_{00}$,

$$\left.\begin{array}{r} D\left(x_1, gx_1, g^2x_1\right) = D(A, B, C) \\ D\left(x_2, gx_2, g^2x_2\right) = D(A, B, C) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(gx_1, gx_2).$$

The same goes for the case where $x_1, x_2 \in B_{00}$. Now, if $x_1 \in A_{00}$ and $x_2 \in B_{00}$,

$$\left.\begin{array}{r} D\left(x_1, gx_1, g^2x_1\right) = D(A, B, C) \\ D\left(g^{-1}x_2, x_2, gx_2\right) = D(A, B, C) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(gx_1, gx_2).$$

That is, g is an isometry.

Now, we state our main result.

Theorem 3.2 *Let (A, B, C) be a triad of nonempty, bounded, closed and convex subsets of an uniformly convex Banach space. Assume (A, B, C) has both the P-property and $P^{✠}$-property and $D(A, B, C) = dist(A, B) + dist(B, C) + dist(C, A)$. Let $T : A \cup B \longrightarrow B \cup C$ be a S-cyclic relatively nonexpansive mapping such that $T(A_{00}) \subseteq B_{00}$, $T(B_{00}) \subseteq C_{00}$. Then T has a best proximity point.*

Proof. Consider the bijective, S-cyclic isometry $g : A_{00} \cup B_{00} \longrightarrow B_{00} \cup C_{00}$ as in the previous lemma. As $T(A_{00}) \subseteq B_{00}$, $T(B_{00}) \subseteq C_{00}$, the mapping $g^{-1}T : A_{00} \cup B_{00} \longrightarrow A_{00} \cup B_{00}$ is noncylic. We also have

$$\left\|g^{-1}Tx - g^{-1}Ty\right\| = \|Tx - Ty\| \leq \|x - y\|,$$

for all $x \in A_{00}$ and $y \in B_{00}$. Thus $g^{-1}T$ has a best proximity pair [2]. Suppose that $(x, y) \in A_{00} \times B_{00}$ is a best proximity pair of $g^{-1}T$; that is $g^{-1}Tx = x$, $g^{-1}Ty = y$ and $\|x - y\| = dist(A_{00}, B_{00}) = dist(A, B)$. On the other hand we have

$$\begin{aligned} \|x - gx\| + \left\|gx - g^2x\right\| + \left\|g^2x - x\right\| &= D\left(x, gx, g^2x\right) \\ &= D(A, B, C) \\ &= dist(A, B) + dist(B, C) + dist(C, A), \end{aligned}$$

which implies that $\|x - gx\| = dist(A, B)$. Taking into account that (A, B) has the P-property, we obtain $gx = y$. And since $Tx = gx$ and $Ty = gy$, we have $T^2x = g^2x$. Then

$$D\left(x, Tx, T^2x\right) = D\left(x, gx, g^2x\right) = D(A, B, C),$$

which finishes the proof.

Example 3.3 Consider the space $X = \mathbb{R}^2$ along with its Euclidean norm. Let

$$A = \{0\} \times \left[0, \frac{1}{3}\right], \ B = \{1\} \times \left[0, \frac{1}{3}\right] \text{ and } C = \{2\} \times \left[0, \frac{1}{3}\right].$$

Then $D(A, B, C) = 4 = dist(A, B) + dist(B, C) + dist(C, A)$ and the triad (A, B, C) satisfies both the P-property and $P^{✠}$-property. Let $T : A \cup B \longrightarrow B \cup C$ be a mapping defined as $T(0, x) = \left(1, x^2\right)$ and $T(1, y) = \left(2, y^2\right)$ for all $x, y \in \left[0, \frac{1}{3}\right]$. It is easy to verify that T is a S-cyclic relatively nonexpansive mapping. Hence by the previous theorem, T has a best proximity point.

As a corollary of the aforestated theorem, we obtain the next result which, except for the $P^{\maltese}$-property, has the same assumptions as the one in [2] with a yet stronger conclusion though.

Theorem 3.4 *Let (A,B) be a pair of nonempty, bounded, closed and convex subsets with the $P^{\maltese}$-property in an uniformly convex Banach space. Let $T : A\cup B \longrightarrow A\cup B$ be a cyclic relatively nonexpansive mapping. Then T has a best proximity point which is a fixed point for T^2.*

Proof. On one hand, (A,B,A) is a triad of nonempty, bounded, closed and convex subsets that verifies both the P-property and $P^{\maltese}$-property and $D(A,B,A) = 2dist(A,B)$. On the other, $T : A\cup B \longrightarrow B\cup A$ is plainly a S-cyclic relatively nonexpansive mapping such that $T(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$. Now, the previous theorem guarentees the existence of a point $x \in A$ such that

$$D\left(x,Tx,T^2x\right) = 2dist(A,B).$$

Therefore, we necessarily get

$$x = T^2x \text{ and } d(x,Tx) = dist(A,B).$$

And that's the desired result.

Example 3.5 Retaining the same space as in the previous example. Take

$$A = [0,1]\times[0,1],\ B = [2,3]\times[0,1]$$

and fix $(x_1,y_1) \in A$ and $(x_2,y_2) \in B$. Let $T : A\cup B \longrightarrow A\cup B$ be defined as

$$T(x_1,y_1) = \left(2,\frac{y_1}{y_1+1}\right) \text{ and } T(x_2,y_2) = \left(1,\frac{y_2}{y_2+1}\right).$$

Since

$$\left(\frac{y_1}{y_1+1}-\frac{y_2}{y_2+1}\right)^2 = \frac{(y_1-y_2)^2}{(y_1+1)^2(y_2+1)^2} \le (y_1-y_2)^2 \text{ and } 1 \le (x_1-x_2)^2.$$

T is a cyclic relatively nonexpansive mapping. Obviously, $(0,0)$ is a best proximity and fixed point for T and T^2 respectively.

4. S-cyclic relatively u-continuous mappings

We start this section by mentioning a useful proposition.

Proposition 4.1 [3, *Proposition 3.1*] *Let A,B be nonempty subsets of a normed linear space X and let $T : A\cup B \longrightarrow A\cup B$ be a cyclic relatively u-continuous mapping. Then $T(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$.*

Now, here is the main result of this section.

Theorem 4.2 *Let (A, B, C) be a triad of nonempty, compact and convex subsets of a strictly convex Banach space. Assume (A, B, C) has both the P-property and $P^{\maltese}$-property, (A, B) has the P-property and*

$$D(A, B, C) = dist(A, B) + dist(B, C) + dist(C, A).$$

Let $T : A \cup B \longrightarrow B \cup C$ be a S-cyclic relatively u-continuous mapping such that $T(A_{00}) \subseteq B_{00}$, $T(B_{00}) \subseteq C_{00}$. Then T has a best proximity point.

Proof. Proceeding as in the proof of Theorem 13 and using the fact that every noncyclic relatively u-continuous mapping defined on compact and convex subsets of a strictly convex Banach space has a best proximity pair [5], we get the wished for result.

Once again, as a direct consequence of the previous theorem, we acquire a result which is, apart from the P and $P^{\maltese}$-property, with the same suppositions as the one proven in [3], but with a fixed point conclusion too.

Theorem 4.3 *Let (A, B) be a nonempty, compact and convex pair with the P and $P^{\maltese}$-property in a strictly convex Banach space X. Let $T : A \cup B \longrightarrow A \cup B$ be a cyclic relatively u-continuous mapping. Then T has a best proximity point which is a fixed point for T^2.*

Proof. Clearly (A, B, A) is a triad of nonempty, compact and convex subsets that fulfills both the P-property and $P^{\maltese}$-property and $D(A, B, A) = 2dist(A, B)$. Taking into account the Proposition 3.1 in [3], $T : A \cup B \longrightarrow B \cup A$ is plainly a S-cyclic relatively u-continuous mapping such that $T(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$. Now, the triad (A, B, A) fulfills the conditions of the previous theorem.

Example 4.4 As in the previous example, we consider $X = \left(\mathbb{R}^2, \|.\|_2\right)$. Let

$$A = \{0\} \times [0, 1] \text{ and } B = \{1\} \times [0, 1].$$

Note that $dist(A, B) = 1$. Define $T : A \cup B \longrightarrow A \cup B$ by $T(0, x) = (1, \sqrt{x})$ and $T(1, x) = (0, \sqrt{x})$ for all $x \in [0, 1]$. Then T is a cyclic relatively u-continuous mapping. Indeed, let $\varepsilon > 0$ and $x, y \in [0, 1]$, the uniform continuity of $t \longmapsto \sqrt{t}$ assures the existence of some $\delta > 0$ such that $|x - y| < \delta$ implies $\left|\sqrt{x} - \sqrt{y}\right| < \varepsilon$. Now,

$$\|(0, x) - (0, y)\| = \sqrt{1 + |x - y|^2} < \sqrt{1 + \delta^2}$$

implies

$$\|T(0, x) - T(0, y)\| = \sqrt{1 + \left|\sqrt{x} - \sqrt{y}\right|^2} < \sqrt{1 + \varepsilon^2} < \varepsilon + 1.$$

Clearly,$(0, 0)$ is a best proximity point for T and a fixed point for T^2in the same time.

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