Fixed Point Theory, 22(2021), No. 1, 59-74
DOI: 10.24193/fpt-ro.2021.1.04
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED DISCS IN QUASI-METRIC SPACES

HASSEN AYDI*, NIHAL TAŞ**, NIHAL YILMAZ ÖZGÜR*** AND NABIL MLAIKI****

*Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication,
H. Sousse 4000, Tunisia
China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
E-mail: hassen.aydi@tdtu.edu.vn

**Balıkesir University, Department of Mathematics, 10145 Balıkesir, Turkey
E-mail: nihaltas@balikesir.edu.tr

***Balıkesir University, Department of Mathematics, 10145 Balıkesir, Turkey
E-mail: nihal@balikesir.edu.tr

****Prince Sultan University, Department of Mathematics and General Sciences,
11586 Riyadh, Saudi Arabia
E-mail: nmlaiki@psu.edu.sa; nmlaiki2012@gmail.com

Abstract. In this paper, we present some results of fixed disc and common fixed disc in quasi-metric spaces, under some very interesting contractions. Obtained results are supported by illustrative examples.
Key Words and Phrases: Fixed disc, common fixed disc, quasi-metric space, contraction.
2020 Mathematics Subject Classification: 54H25, 47H09, 47H10.

1. Introduction and Preliminaries

There are various approaches to generalize the known fixed-point results. One of these approaches is to generalize the used metric spaces such as a quasi-metric space.

First, we remind the reader of the definition of a metric space.

Definition 1.1 Let $\mathcal{X}$ be a nonempty set. A mapping $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is said to be a metric on $\mathcal{X}$ if for all $x, y, z \in \mathcal{X}$, it satisfies the following conditions:
(d_1) $x = y$ if and only if $d(x, y) = 0$;
(d_2) $d(x, y) = d(y, x)$;
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$.
In this case, the pair $(\mathcal{X}, d)$ is called a metric space.

In quasi-metric spaces, the symmetry condition is dropped.

Definition 1.2 Let $\mathcal{X}$ be a nonempty set. A mapping $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is said to be a quasi-metric on $\mathcal{X}$ if for all $x, y, z \in \mathcal{X}$, it satisfies the following conditions:

(q_1) $x = y$ if and only if $q(x, y) = 0$;
(q_2) $q(x, y) \leq q(x, z) + q(z, y)$.
In this case, the pair $(\mathcal{X}, q)$ is called a quasi-metric space.

Notice that every metric is a quasi-metric, but there exist some examples of a quasi-metric which is not a metric (see [12] and [21] for some examples). On the other hand, there are some advantages of quasi-metric spaces with respect to metric spaces as tools in program verification (see [23] and the references therein). So, studying on a quasi-metric space is important for the context of fixed-point theory and generalizations. For more details, we refer the readers to [1, 2, 3, 4, 5, 9]. Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers, and $\mathbb{N}$ represents the set of all positive integers.

Example 1.3 Let $\mathcal{X} = l_1$ be defined by

$$l_1 = \left\{ \{\xi_n\}_{n\geq 1} \subset \mathbb{R} : \sum_{n=1}^{\infty} |\xi_n| < \infty \right\}.$$

Consider $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that

$$q(\eta, \xi) = \sum_{n=1}^{\infty} (\xi_n - \eta_n)^+,$$

where $\alpha^+ := \max\{\alpha, 0\}$ denotes the positive part of a number $\alpha \in \mathbb{R}$, $\xi = \{\xi_n\}$ and $\eta = \{\eta_n\}$ are in $\mathcal{X}$. Note that, $(\mathcal{X}, q)$ is a quasi-metric space. For the topological properties of this space one can consult [11].

Now, we mention the some topological notions related to quasi-metric spaces. We recall convergence and completeness on a quasi-metric space.

Definition 1.4 Let $(\mathcal{X}, q)$ be a quasi-metric space, $\{\xi_n\}$ be a sequence in $\mathcal{X}$ and $\xi \in \mathcal{X}$. The sequence $\{\xi_n\}$ converges to ξ if and only if

$$\lim_{n\to\infty} q(\xi_n, \xi) = \lim_{n\to\infty} q(\xi, \xi_n) = 0. \tag{1.1}$$

Remark 1.5 In a quasi-metric space $(\mathcal{X}, q)$, the limit for a convergent sequence is unique. Also, if $\xi_n \to \xi$, we have for all $y \in \mathcal{X}$

$$\lim_{n\to\infty} q(\xi_n, \eta) = q(\xi, \eta) \text{ and } \lim_{n\to\infty} q(\eta, \xi_n) = q(\eta, \xi).$$

In fact, $\xi_n \to \xi$ and $\eta_n \to \eta \Rightarrow q(\xi_n, \eta_n) \to q(\xi, \eta)$.

Definition 1.6 [22] Let $(\mathcal{X}, q)$ be a quasi-metric space and $\{\xi_n\}$ be a sequence in $\mathcal{X}$. We say that $\{\xi_n\}$ is right K-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $q(\xi_n, \xi_m) < \varepsilon$ for all $n \geq m > N$.

Definition 1.7 [22] Let $(\mathcal{X}, q)$ be a quasi-metric space and $\{\xi_n\}$ be a sequence in $\mathcal{X}$. We say that $\{\xi_n\}$ is left K-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $q(\xi_n, \xi_m) < \varepsilon$ for all $m \geq n > N$.

Definition 1.8 Let $(\mathcal{X}, q)$ be a quasi-metric space and $\{\xi_n\}$ be a sequence in $\mathcal{X}$. We say that $\{\xi_n\}$ is Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $q(\xi_n, \xi_m) < \varepsilon$ for all $m, n > N$.

Remark 1.9 A sequence $\{\xi_n\}$ in a quasi-metric space is Cauchy if and only if it is right K-Cauchy and left K-Cauchy.

Definition 1.10 [24] Let $(\mathcal{X}, q)$ be a quasi-metric space.
(1) $(\mathcal{X}, q)$ is said left-complete if and only if each left K-Cauchy sequence in $\mathcal{X}$ is convergent.
(2) $(\mathcal{X}, q)$ is said right-complete if and only if each right K-Cauchy sequence in X is convergent.
(3) $(\mathcal{X}, q)$ is said complete if and only if each Cauchy sequence in $\mathcal{X}$ is convergent.

Remark 1.11 If q is a quasi-metric on X, then $\overline{q}(x, y) = q(y, x)$ for all $x, y \in X$ is another quasi-metric, called the conjugate of q and $q^s(x, y) = \max\{q(x, y), \overline{q}(x, y)\}$ for all $x, y \in X$ is a metric on X. Moreover, we have

(1) $\xi_n \to_q \xi \quad \Leftrightarrow \quad \lim_{n\to\infty} q(\xi, \xi_n) = 0$;
(2) $\xi_n \to_{\overline{q}} \xi \quad \Leftrightarrow \quad \lim_{n\to\infty} \overline{q}(\xi, \xi_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} q(\xi_n, \xi) = 0$.

Also, note that

$$\begin{aligned} \xi_n \to_{q^s} \xi \Leftrightarrow \quad & \lim_{n\to\infty} q(\xi, \xi_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} q(\xi_n, \xi) = 0 \\ \Leftrightarrow \quad & \xi_n \to_q \xi \quad \text{and} \quad \xi_n \to_{\overline{q}} \xi. \end{aligned}$$

Hence, $\xi_n \to_q \xi \quad \text{implies} \quad \xi_n \to_{q^s} \xi$.

Lemma 1.12 Let $(\mathcal{X}, q)$ be a quasi-metric space and $T : \mathcal{X} \to \mathcal{X}$ be a self-mapping. Suppose that T is continuous at $\xi \in \mathcal{X}$. Then for each sequence $\{\xi_n\}$ in $\mathcal{X}$ such that $\xi_n \to \xi$, we have $T\xi_n \to T\xi$, that is,

$$\lim_{n\to\infty} q(T\xi_n, T\xi) = \lim_{n\to\infty} q(T\xi, T\xi_n) = 0.$$

Using the above notions, it has been studied several fixed-point theorems using various approaches and techniques (see [10], [12], [21], [23], [28] and the references therein).

In this paper, we provide some results on fixed-discs for different contractive mappings in the class of quasi-metric spaces. To do this, we use some new techniques and modify some known contractive conditions. As an application, we give a common fixed-disc theorem. The obtained results are supported by several examples.

2. Main results

A recent approach used to generalize the known fixed-point results is to consider the geometric properties of fixed points when the number of fixed points is not unique. In this context, a new approach, the so called the fixed-circle problem, has been studied in metric spaces via different contractive conditions (see [15], [18], [19], [20] and [25] for more details). In some of these studies, fixed-disc results have been appeared simultaneously (see [6], and [17]).

At first, we recall the following definitions from [10].

Let (X, q) be a quasi-metric space, $x_0 \in X$ and $r > 0$. The upper closed ball of radius r centered x_0 and the lower closed ball of radius r centered x_0 are defined by,

$$\overline{B^+}(x_0, r) = \{x \in X : q(x, x_0) \leq r\}$$

and

$$\overline{B^-}(x_0, r) = \{x \in X : q(x_0, x) \le r\},$$

respectively. Now, we define the notions of a circle and a disc on a quasi-metric space (X, q) as follows: Let $r \ge 0$ and $x_0 \in X$. The circle $C^q_{x_0,r}$ and the disc $D^q_{x_0,r}$ are

$$C^q_{x_0,r} = \{x \in X : q(x_0, x) = q(x, x_0) = r\}$$

and

$$D^q_{x_0,r} = \overline{B^+}(x_0, r) \cap \overline{B^-}(x_0, r) = \{x \in X : q(x_0, x) \le r \text{ and } q(x, x_0) \le r\}.$$

We note that the disc $D^q_{x_0,r}$ is, in fact, the closed ball with respect to the associated metric q^s. Indeed, we have

$$q(x_0, x) \le r \text{ and } q(x, x_0) \le r \Leftrightarrow \max\{q(x_0, x), q(x, x_0)\} \le r \Leftrightarrow q^s(x, x_0) \le r.$$

Let (X, q) be a quasi-metric space and T be a self-mapping on X. To obtain some fixed-disc results, we define new contractive conditions using the following number

$$r = \inf\{q(x, Tx) \mid \quad x \in X, \quad Tx \ne x\}. \tag{2.1}$$

2.1. **Quasi-type F_q-contractions.** In [27], Wardowski defined a new class of functions as follows.

Definition 2.1 [27] Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ such that

(F_1) F is strictly increasing;

(F_2) For each sequence $\{\alpha_n\}$ in $(0, \infty)$, the following holds

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Using this class of functions, we give the following definition.

Definition 2.2 Let (X, q) be a quasi-metric space, T a self-mapping on X and $F \in \mathbb{F}$. Then T is said to be a quasi-F_q-contraction if there exist $t > 0$ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Rightarrow t + F(q(x, Tx)) \le F(q(x_0, x)), \tag{2.2}$$

for each $x \in X$.

Let $Fix(T)$ be the fixed-point set of T. In the following theorem, we see that $Fix(T)$ contains a disc.

Theorem 2.3 Let (X, q) be a quasi-metric space, T be a quasi-F_q-contraction with $x_0 \in X$ on X and r defined as in (2.1). Then we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. At first, we show that x_0 is a fixed point of T. Assume that $q(x_0, Tx_0) > 0$. By the quasi-F_q-contractive property of T we deduce that

$$t + F(q(x_0, Tx_0)) \le F(q(x_0, x_0)),$$

whence $F(q(x_0, Tx_0)) < F(0)$, which leads to a contradiction given the fact that F is strictly increasing. Thus, we get $Tx_0 = x_0$.

If $r = 0$ then we get $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$ and clearly, T fixes the center of the disc $D^q_{x_0,r}$ and the whole disc $D^q_{x_0,r}$.

Let $r > 0$ and $x \in \overline{B^-}(x_0, r)$ with $Tx \neq x$. By the definition of r, we have $r \leq q(x, Tx)$. Because of the quasi-F_q-contractive property, there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that

$$t + F(q(x, Tx)) \leq F(q(x_0, x)) \leq F(r) \leq F(q(x, Tx)),$$

for all $x \in X$. This is a contradiction with $t > 0$. Hence it should be $Tx = x$, hence $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Now, we introduce a new rational type contractive condition.

Definition 2.4 Let (X, q) be a quasi-metric space, T a self-mapping on X and $F \in \mathbb{F}$. Then T is said to be quasi-F_q-rational contraction if there exist $t > 0$ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Rightarrow t + F(q(x, Tx)) \leq F(M^q_R(x_0, x)), \tag{2.3}$$

for all $x \in X$, where

$$M^q_R(x, y) = \max\left\{ \begin{array}{c} q(x, y), q(x, Tx), q(y, Ty), \\ \dfrac{q(x, Tx)q(y, Ty)}{1 + q(x, y)}, \dfrac{q(x, Tx)q(y, Ty)}{1 + q(Tx, Ty)} \end{array} \right\}.$$

Theorem 2.5 Let (X, q) be a quasi-metric space, T a quasi-F_q-rational contraction self-mapping with $x_0 \in X$ on X, $Tx_0 = x_0$ and r defined as in (2.1). Then we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. Suppose that $r = 0$. So we have $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$. Using the hypothesis $Tx_0 = x_0$, T fixes the disc $D^q_{x_0,r}$.

Let $r > 0$ and $x \in \overline{B^-}(x_0, r)$ with $Tx \neq x$. By the definition of r, we have $r \leq q(x, Tx)$. Because of the quasi-F_q-rational contractive property, there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that

$$t + F(q(x, Tx)) \leq F(M^q_R(x_0, x)),$$

for all $x \in X$. Then we get

$$\begin{aligned} t + F(q(x, Tx)) &\leq F(M^q_R(x_0, x)) \\ &= F\left(\max\left\{ \begin{array}{c} q(x_0, x), q(x_0, Tx_0), q(x, Tx), \\ \dfrac{q(x_0, Tx_0)q(x, Tx)}{1 + q(x_0, x)}, \dfrac{q(x_0, Tx_0)q(x, Tx)}{1 + q(Tx_0, Tx)} \end{array} \right\} \right) \\ &\leq F(\max\{r, q(x, Tx)\}) = F(q(x, Tx)), \end{aligned}$$

a contradiction. Hence it should be $Tx = x$. Consequently, $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

2.2. **Quasi-α-x_0-contractive type mappings.** First, we present the definition of an x_0-contractive mapping in quasi-metric spaces.

Definition 2.6 Let (X, q) be a quasi-metric space, T a self-mapping on X and $0 < k < 1$. Then T is said to be a quasi-x_0-contractive mapping if there exist $x_0 \in X$ such that

$$q(x, Tx) \leq kq(x_0, x), \tag{2.4}$$

for every $x \in X$.

Clearly, x_0 is always a fixed point of T in Definition 2.6. Now, we show that if T is a quasi-x_0-contractive mapping, then $Fix(T)$ contains a disc.

Theorem 2.7 Let (X, q) be a quasi-metric space, T a quasi-x_0-contractive self-mapping with $x_0 \in X$ on X and r defined as in (2.1). Then we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. In the case $r = 0$, it is clear that $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$ is a fixed disc of T.

Suppose that $r > 0$. Let $x \in \overline{B^-}(x_0, r)$ be such that $Tx \neq x$. By the definition of r, we have $r \leq q(x, Tx)$. On the other hand, using the quasi-x_0-contractive property of T, we obtain

$$0 < q(x, Tx) \leq kq(x_0, x) \leq kr < r,$$

which leads us to a contradiction. Thus, $Tx = x$ for every $x \in \overline{B^-}(x_0, r)$, that is, $\overline{B^-}(x_0, r) \subseteq Fix(T)$. In particular, T fixes the disc $D^q_{x_0,r}$.

Now, we define the concept of quasi-α-x_0-contractive self-mappings in a quasi-metric spaces.

Definition 2.8 Let T be a self mapping on a quasi-metric space (X, q). Then T is said to be a quasi-α-x_0-contractive self-mapping if there exist a function $\alpha : X \times X \to (0, \infty)$, $0 < k < 1$ and $x_0 \in X$ such that

$$\alpha(x_0, Tx)q(x, Tx) \leq kq(x_0, x), \tag{2.5}$$

for all $x \in X$.

We recall α-x_0-admissible maps as follows:

Definition 2.9 [6] Let X be a non-empty set. Given a function $\alpha : X \times X \to (0, \infty)$ and $x_0 \in X$. Then T is said to be an α-x_0-admissible if for every $x \in X$,

$$\alpha(x_0, x) \geq 1 \quad \Rightarrow \quad \alpha(x_0, Tx) \geq 1.$$

Theorem 2.10 Let (X, q) be a quasi-metric space, T a quasi-α-x_0-contractive self-mapping with $x_0 \in X$ on X and r defined as in (2.1). Assume that T is α-x_0-admissible and $\alpha(x_0, x) \geq 1$ for all $x \in \overline{B^-}(x_0, r)$. Then we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. By the definition of a quasi-α-x_0-contractive self-mapping, it is easy to see that x_0 is always a fixed point of T. Therefore, if $r = 0$ then we have $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$ and the proof follows.

Suppose that $r > 0$. Let $x \in \overline{B^-}(x_0, r)$ such that $Tx \neq x$. By the definition of r, we have $r \leq q(x, Tx)$. On the other hand, we have $\alpha(x_0, x) \geq 1$. Using the α-x_0-admissible property and the quasi-α-x_0-contractive property of T, we find

$$0 < q(x, Tx) \leq \alpha(x_0, Tx)q(x, Tx) \leq kq(x_0, x) \leq kr < r,$$

which leads us to a contradiction. Thus, $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

The concept of a quasi-F^α_q-contractive mapping is defined as follows.

Definition 2.11 Let (X, q) be a quasi-metric space, T a self-mapping on X and $F \in \mathbb{F}$. Then T is called a quasi-F^α_q-contraction if there exist $t > 0$, a function $\alpha : X \times X \to (0, \infty)$ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Rightarrow t + \alpha(x_0, Tx)F(q(x, Tx)) \leq F(q(x_0, x)), \tag{2.6}$$

for all $x \in X$.

Theorem 2.12 Let (X, q) be a quasi-metric space, T a quasi-F^α_q-contractive self-mapping with $x_0 \in X$ and r defined as in (2.1). Suppose that T is α-x_0-admissible and $\alpha(x_0, x) \geq 1$ for all $x \in \overline{B^-}(x_0, r)$. Then we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. At first, using the quasi-F^α_q-contractive property, one can easily deduce that $Tx_0 = x_0$. Hence we have $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$ if $r = 0$. Clearly, T fixes the disc $D^q_{x_0,r}$.

Assume that $r > 0$. Let $x \in \overline{B^-}(x_0, r)$ where $Tx \neq x$. Therefore, by the definition of r, we have $r \leq q(x, Tx)$. On the other hand, we have $\alpha(x_0, x) \geq 1$ and T is α-x_0-admissible. So, using the quasi-F^α_q-contractive property of T, we deduce

$$F(q(x, Tx)) < t + \alpha(x_0, Tx)F(q(x, Tx)) \leq F(q(x_0, x)) \leq F(r) \leq F(q(x, Tx)).$$

Thus, by the fact that F is strictly increasing and $t > 0$ we get a contradiction. Hence, we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Definition 2.13 Let (X, q) be a quasi-metric space, T a self-mapping on X and $F \in \mathbb{F}$. Then T is called a Ćirić type quasi-F_q-contraction if there exist $t > 0$ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Longrightarrow t + \alpha(x_0, Tx)F(q(x, Tx)) \leq F(M^q_C(x_0, x)), \tag{2.7}$$

for all $x \in X$, where

$$M^q_C(x, y) = \max\left\{q(x, y), q(x, Tx), q(y, Ty), \frac{q(x, Ty) + q(y, Tx)}{2}\right\}. \tag{2.8}$$

Proposition 2.14 Let (X, q) be a metric space. If T is a Ćirić type quasi-F_q-contraction with $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then we have $Tx_0 = x_0$.

Proof. Assume that $Tx_0 \neq x_0$. From the definition of a Ćirić type quasi-F_q-contraction, we get

$$\begin{aligned} q(x_0, Tx_0) &> 0 \Longrightarrow t + \alpha(x_0, Tx_0)F(q(x_0, Tx_0)) \leq F(M_C^q(x_0, x_0)) \\ &= F\left(\max\left\{ \begin{array}{c} q(x_0, x_0), q(x_0, Tx_0), q(x_0, Tx_0), \\ \dfrac{q(x_0, Tx_0) + q(x_0, Tx_0)}{2} \end{array} \right\}\right) \\ &= F(q(x_0, Tx_0)), \end{aligned}$$

which is a contradiction since $t > 0$. Then we have $Tx_0 = x_0$.

We give a generalization of Theorem 2.12.

Theorem 2.15 Let (X, q) be a quasi-metric space, T a Ćirić type quasi-F_q-contraction with $x_0 \in X$ and r defined as in (2.1). Assume that T is α-x_0-admissible and if for every $x \in D^q_{x_0,r}$, we have $q(x_0, Tx) \leq r$. Then T fixes the disc $D^q_{x_0,r}$.

Proof. If $r = 0$, clearly $D^q_{x_0,r} = \{x_0\}$ is a fixed-disc (point) by Proposition 2.14.

Assume that $r > 0$. Let $x \in D^q_{x_0,r}$. By the definition of r, we have $q(x, Tx) \geq r$. So using the Ćirić type quasi-F_q-contractive property and the fact that T is α-x_0-admissible and F is increasing, we get

$$\begin{aligned} F(q(x, Tx)) &< \alpha(x_0, Tx)F(q(x, Tx)) + t \leq F(M_C^q(x_0, x)) \\ &= F\left(\max\left\{q(x_0, x), q(x_0, Tx_0), q(x, Tx), \frac{q(x_0, Tx) + q(x, Tx_0)}{2}\right\}\right) \\ &\leq F(\max\{r, q(x, Tx), 0, r\}) \leq F(q(x, Tx)), \end{aligned}$$

which leads to a contradiction. Therefore, $q(x, Tx) = 0$ and so $Tx = x$. Hence, T fixes the disc $D^q_{x_0,r}$.

2.3. **Quasi-α-φ-x_0-contractive type mappings.** At first, we recall the notion of a (c)-comparison functions [8] (see also [13]).

Definition 2.16 [8] A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a (c)-comparison function if

$(i)_\varphi$ φ is increasing;

$(ii)_\varphi$ There exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} \upsilon_k$ such that

$$\varphi^{k+1}(t) \leq a\varphi^k(t) + \upsilon_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}_+$.

The class of (c)-comparison functions will be denoted by Ψ_c.

Lemma 2.17 [8] If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function, then the followings hold:

(i) φ is a comparison function;

(ii) $\varphi(t) < t$ for any $t \in \mathbb{R}_+$;

(iii) φ is continuous at 0;

(iv) the series $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$.

Now, using Definition 2.16 and Lemma 2.17, we introduce two new contractions and obtain two new fixed-disc theorems as follows:

Definition 2.18 Let (X, q) be a quasi-metric space and T a self-mapping on X. Then T is said to be a quasi-α-φ-x_0-contraction if there exist $\alpha : X \times X \to (0, \infty)$, $\varphi \in \Psi_c$ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Longrightarrow \alpha(x_0, Tx)q(x, Tx) \leq \varphi\left(q(x_0, Tx)\right),$$

for each $x \in X$.

Theorem 2.19 Let (X, q) be a quasi-metric space, T a quasi-α-φ-x_0-contractive self-mapping with $x_0 \in X$ and r defined as in (2.1). Assume that T is α-x_0-admissible. If $\alpha(x_0, x) \geq 1$ for $x \in \overline{B^-}(x_0, r)$ and $0 < q(x_0, Tx) \leq r$ for $x \in \overline{B^-}(x_0, r) - \{x_0\}$, then we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. Using the quasi-α-φ-x_0-contractive property, we have $Tx_0 = x_0$. Indeed, we assume $Tx_0 \neq x_0$, that is, $q(x_0, Tx_0) > 0$. Then using the condition (ii) in Lemma 2.17 and α-x_0-admissibility, we get

$$\alpha(x_0, Tx_0)q(x_0, Tx_0) \leq \varphi\left(q(x_0, Tx_0)\right) < q(x_0, Tx_0),$$

a contradiction. It should be $Tx_0 = x_0$.

Suppose that $r = 0$. In this case, $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$ and the proof follows.

Now we suppose that $r > 0$ and $x \in \overline{B^-}(x_0, r) - \{x_0\}$ such that $x \neq Tx$. Using the definition of r, we have $r \leq q(x, Tx)$. By the hypothesis, we known $\alpha(x_0, x) \geq 1$. From the quasi-α-φ-x_0-contractive property and α-x_0-admissibility, we get

$$\alpha(x_0, Tx)q(x, Tx) \leq \varphi\left(q(x_0, Tx)\right) < q(x_0, Tx) \leq r,$$

a contradiction. Therefore, $Tx = x$, that is, $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Using the number $M^q_C(x, y)$ defined as in (2.8), we define the following new contraction.

Definition 2.20 Let (X, q) be a quasi-metric space and T a self-mapping on X. Then T is said to be a Ćirić type quasi-α-φ-x_0-contraction if there exist $\alpha : X \times X \to (0, \infty)$, $\varphi \in \Psi_c$ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Longrightarrow \alpha(x_0, Tx)q(x, Tx) \leq \varphi\left(M^q_C(x_0, x)\right),$$

for each $x \in X$.

Theorem 2.21 Let (X, q) be a quasi-metric space, T a Ćirić type quasi-α-φ-x_0-contractive self-mapping with $x_0 \in X$ and r defined as in (2.1). Assume that T is α-x_0-admissible. If $\alpha(x_0, x) \geq 1$ and $q(x_0, Tx) \leq r$ for $x \in D^q_{x_0,r}$, then T fixes the disc $D^q_{x_0,r}$.

Proof. Using the hypothesis, we have $Tx_0 = x_0$. Indeed, we assume $Tx_0 \neq x_0$, that is, $q(x_0, Tx_0) > 0$. Then using the condition (ii) in Lemma 2.17 and α-x_0-admissibility, we get

$$M_C^q(x_0,x_0)=\max\left\{\begin{array}{c} q(x_0,x_0),q(x_0,Tx_0),q(x_0,Tx_0),\\ \dfrac{q(x_0,Tx_0)+q(x_0,Tx_0)}{2}\end{array}\right\}=q(x_0,Tx_0)$$

and

$$\alpha(x_0,Tx_0)q(x_0,Tx_0)\leq\varphi\left(M_C^q(x_0,x_0)\right)<q(x_0,Tx_0),$$

a contradiction. It should be $Tx_0=x_0$.

Let $r=0$. In this case, we have $D^q_{x_0,r}=\{x_0\}$.

Now we suppose that $r>0$ and $x\in D^q_{x_0,r}-\{x_0\}$ such that $x\neq Tx$. Using the definition of r, we have $r\leq q(x,Tx)$. By the hypothesis, we known $\alpha(x_0,x)\geq 1$. By the Ćirić type quasi-α-φ-x_0-contractive property and α-x_0-admissibility, we get

$$M_C^q(x_0,x)=\max\left\{\begin{array}{c} q(x_0,x),q(x_0,Tx_0),q(x,Tx),\\ \dfrac{q(x_0,Tx)+q(x,Tx_0)}{2}\end{array}\right\}\leq q(x,Tx)$$

and

$$\alpha(x_0,Tx)q(x,Tx)\leq\varphi\left(M_C^q(x,x_0)\right)<q(x,Tx),$$

a contradiction. Therefore, $Tx=x$, that is, $D^q_{x_0,r}$ is a fixed disc of T.

2.4. **Quasi-α-ψ-φ-x_0-contractive type mappings.** We recall the notion of an altering distance function.

Definition 2.22 [14] A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the followings hold:

(i) ψ is continuous and nondecreasing;

(ii) $\psi(t)=0$ if and only if $t=0$.

Using this definition, we present two new contractive conditions and two new fixed-disc results.

Definition 2.23 Let (X,q) be a quasi-metric space and T a self-mapping on X. Then T is said to be a quasi-α-ψ-φ-x_0-contraction if there exist $\alpha:X\times X\to(0,\infty)$, two altering distance functions ψ, φ and $x_0\in X$ such that

$$q(x,Tx)>0\Longrightarrow\alpha(x_0,Tx)\psi\left(q(x,Tx)\right)\leq\psi\left(q(x_0,x)\right)-\varphi\left(q(x_0,x)\right),$$

for each $x\in X$.

Theorem 2.24 Let (X,q) be a quasi-metric space, T a quasi-α-ψ-φ-x_0-contractive self-mapping with $x_0\in X$ and r defined as in (2.1). Assume that T is α-x_0-admissible. If $\alpha(x_0,x)\geq 1$ for $x\in\overline{B^-}(x_0,r)$, then we have $\overline{B^-}(x_0,r)\subseteq Fix(T)$, in particular T fixes the disc $D^q_{x_0,r}$.

Proof. Using the quasi-α-ψ-φ-x_0-contractive property, we have $Tx_0=x_0$. Indeed, we assume $Tx_0\neq x_0$, that is, $q(x_0,Tx_0)>0$. Then using the condition (ii) in Definition 2.22 and α-x_0-admissibility, we get

$$\begin{array}{rcl}\alpha(x_0,Tx_0)\psi\left(q(x_0,Tx_0)\right) & \leq & \psi\left(q(x_0,x_0)\right)-\varphi\left(q(x_0,x_0)\right)\\ & = & \psi(0)-\varphi(0)=0,\end{array}$$

a contradiction. It should be $Tx_0 = x_0$.

Suppose that $r = 0$. In this case, we get $\overline{B^-}(x_0, r) = D^q_{x_0,r} = \{x_0\}$.

Now, we suppose that $r > 0$ and $x \in \overline{B^-}(x_0, r) - \{x_0\}$ such that $x \neq Tx$. Using the definition of r, we have $r \leq q(x, Tx)$. By the hypothesis, we know $\alpha(x_0, x) \geq 1$. By the quasi-α-ψ-φ-x_0-contractive property and α-x_0-admissibility, we get

$$\begin{aligned} \alpha(x_0, Tx)\psi(q(x, Tx)) &\leq \psi(q(x_0, x)) - \varphi(q(x_0, x)) \\ &= \psi(r) - \varphi(r) < \psi(r), \end{aligned}$$

a contradiction. Therefore $Tx = x$, that is, $\overline{B^-}(x_0, r) \subseteq Fix(T)$. In particular, T fixes the disc $D^q_{x_0,r}$.

We define the following contraction using the number $M^q_C(x, y)$ defined as in (2.8).

Definition 2.25 Let (X, q) be a quasi-metric space and T a self-mapping on X. Then T is said to be a Ćirić type quasi-α-ψ-φ-x_0-contraction if there exist $\alpha : X \times X \to (0, \infty)$, two altering distance functions ψ, φ and $x_0 \in X$ such that

$$q(x, Tx) > 0 \Longrightarrow \alpha(x_0, Tx)\psi(q(x, Tx)) \leq \psi(M^q_C(x_0, x)) - \varphi(M^q_C(x_0, x)),$$

for each $x \in X$.

Theorem 2.26 Let (X, q) be a quasi-metric space, T a Ćirić type quasi-α-ψ-φ-x_0-contractive self-mapping with $x_0 \in X$ and r defined as in (2.1). Assume that T is α-x_0-admissible. If $\alpha(x_0, x) \geq 1$ and $q(x_0, Tx) \leq r$ for $x \in D^q_{x_0,r}$, then T fixes the disc $D^q_{x_0,r}$.

Proof. Using the hypothesis, we have $Tx_0 = x_0$. Indeed, we assume $Tx_0 \neq x_0$, that is, $q(x_0, Tx_0) > 0$ and we get

$$\begin{aligned} \alpha(x_0, Tx_0)\psi(q(x_0, Tx_0)) &\leq \psi(M^q_C(x_0, x_0)) - \varphi(M^q_C(x_0, x_0)) \\ &= \psi(q(x_0, Tx_0)) - \varphi(q(x_0, Tx_0)) \\ &< \psi(q(x_0, Tx_0)), \end{aligned}$$

a contradiction. It should be $Tx_0 = x_0$.

Let $r = 0$. In this case, we have $D^q_{x_0,r} = \{x_0\}$ and the proof follows.

Now, we suppose that $r > 0$ and $x \in D^q_{x_0,r} - \{x_0\}$ such that $x \neq Tx$. Using the definition of r, we have $r \leq q(x, Tx)$. By the hypothesis, we know that $\alpha(x_0, x) \geq 1$. From the Ćirić type quasi-α-ψ-φ-x_0-contractive property and α-x_0-admissibility, we get

$$\begin{aligned} \alpha(x_0, Tx)\psi(q(x, Tx)) &\leq \psi(M^q_C(x_0, x)) - \varphi(M^q_C(x_0, x)) \\ &< \psi(q(x, Tx)), \end{aligned}$$

a contradiction. Therefore, $Tx = x$, that is, $D^q_{x_0,r}$ is a fixed disc of T.

2.5. **Some comparisons and remarks.** In this section, we give some relationships between the above contractions. We also provide some illustrative examples.

Let us take the function $\alpha : X \times X \to (0, \infty)$ as $\alpha(x, y) = 1$ for all $(x, y) \in X \times X$ in Definition 2.8 and Definition 2.11. Then the notions of a quasi-x_0-contraction and a quasi-α-x_0-contraction coincide. Similarly, the concepts of a quasi-F_d-contraction and a quasi-F_d^{α}-contraction coincide.

Now if we consider the function $\alpha : X \times X \to (0, \infty)$ as $\alpha(x, y) \in (0, 1]$ for all $(x, y) \in X \times X$, then every quasi-x_0-contraction is a quasi-α-x_0-contraction. Indeed, we get

$$q(x, Tx) > 0 \Longrightarrow \alpha(x_0, Tx)q(x, Tx) \leq q(x, Tx) \leq kq(x_0, x),$$

for each $x \in X$. On the other hand, if we take the function $\alpha : X \times X \to (0, \infty)$ as $\alpha(x, y) \in [1, \infty)$ for all $(x, y) \in X \times X$, then every quasi-α-x_0-contraction is a quasi-x_0-contraction. Indeed, we have

$$q(x, Tx) > 0 \Longrightarrow q(x, Tx) \leq \alpha(x_0, Tx)q(x, Tx) \leq kq(x_0, x),$$

for each $x \in X$.

Using the above approach, we see that if we consider the function $\alpha : X \times X \to (0, \infty)$ as $\alpha(x, y) \in (0, 1]$ for all $(x, y) \in X \times X$, then every quasi-F_d-contraction is a quasi-F_d^{α}-contraction. Also, if we take the function $\alpha : X \times X \to (0, \infty)$ as $\alpha(x, y) \in [1, \infty)$ for all $(x, y) \in X \times X$, then every quasi-F_d^{α}-contraction is a quasi-F_d-contraction.

Notice that the radius r of the fixed disc is independent from the choice of the center x_0 in all of the obtained theorems (see the following examples). Moreover, all of the proved fixed-disc results can be considered as fixed-circle results.

The defined contractive conditions in previous subsections are modified from some classical contractions used to obtain fixed-point theorems on metric or some generalized metric spaces. For example, the notion of a quasi-x_0-contractive mapping is given by modifying the Banach's contraction principle [7]. For an another example, the notion of a quasi-F_d-contraction is modified using the notion of an F-contraction [27].

Now, we give some examples to show the validity of our obtained results.

Consider $X = \{0, 1, 2\}$. Given the function $q : X \times X \to [0, \infty)$ as

x / y	0	1	2
0	0	1	2
1	2	0	1
2	2	1	0

Then the function q is a quasi-metric, but it is not a metric because of $q(0, 1) = 1 \neq q(1, 0) = 2$. Let us define the self-mapping $T : X \to X$ by

$$T : \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

We obtain

$$r = \inf\{q(x, Tx) \mid Tx \neq x, x \in X\} = 1$$

and the disc

$$D^q_{1,1} = \{x \in X : q(x,1) \leq 1 \text{ and } q(1,x) \leq 1\} = \{1,2\}.$$

• The self-mapping T is a quasi-F_d-contraction with $t = \ln 2$, $F = \ln x$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.3.

• The self-mapping T is a quasi-F_d-rational contraction with $t = \ln 2$, $F = \ln x$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.5.

• The self-mapping T is a quasi-x_0-contractive mapping with $k = \frac{1}{2}$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.7.

• The self-mapping T is a quasi-α-x_0-contraction with $k = \frac{1}{2}$, $\alpha(x,y) = 1$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.10.

• The self-mapping T is a quasi-F^{α}_d-contraction with $t = \ln 2$, $F = \ln x$, $\alpha(x,y) = 1$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.12.

• The self-mapping T is a Ćirić type quasi-F_d-contraction with $t = \ln 2$, $F = \ln x$, $\alpha(x,y) = 4$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.15.

• The self-mapping T is a Ćirić type quasi-α-φ-x_0-contraction with $\varphi(t) = \frac{t}{2}$ $(t \geq 0)$, $\alpha(x,y) = 1$ and $x_0 = 1$. Then T satisfies the conditions of Theorem 2.21.

• The self-mapping T is a quasi-α-ψ-φ-x_0-contraction and a Ćirić type quasi-α-ψ-φ-x_0-contraction with $\alpha(x,y) = 1$, $x_0 = 1$, $\psi(t) = t$ and

$$\varphi(t) = \begin{cases} \dfrac{t\sqrt{t}}{1+\sqrt{t}} & , \quad t \in [0,1] \\ \dfrac{t}{2} & , \quad t > 1, \end{cases}$$

where ψ and φ are two altering distance functions given in [16]. Then T satisfies the conditions of Theorem 2.24 and Theorem 2.26.

Consequently, T fixes the disc $D^q_{1,1}$.

Let us define the self-mapping $S : X \to X$ as

$$S : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

We obtain

$$r = \inf\{q(x, Sx) \mid Sx \neq x, x \in X\} = 1$$

and the disc

$$D^q_{0,1} = \{x \in X : q(x,0) \leq 1 \text{ and } q(0,x) \leq 1\} = \{0\}.$$

The self-mapping S is a quasi-α-φ-x_0-contraction with $\varphi(t) = \frac{t}{2}$ $(t \geq 0)$, $\alpha(x,y) = 1$ and $x_0 = 1$. Then S satisfies the conditions of Theorem 2.19. Hence S fixes the disc $D^q_{0,1}$.

Finally, we note that the converse statements of the obtained theorems are not true everwhen. To obtain an example of this situation for Theorem 2.3, let us consider the quasi-metric $q(x,y)$ defined by

$$q(x,y) = \begin{cases} x - y & ; \quad x \geq y \\ 1 & ; \quad x < y, \end{cases}$$

for all $x, y \in X = \mathbb{R}$ given in [12]. If we consider the self-mapping $H : \mathbb{R} \to \mathbb{R}$ defined by

$$Hx = \begin{cases} x+3 & ; \quad |x| > 2 \\ x & ; \quad |x| \leq 2, \end{cases}$$

H is not a quasi-F_d-contraction for any $F \in \mathbb{F}$, $t > 0$ and $x_0 = 0$. But, the disc $D^q_{0,1} = [-1, 1]$ is fixed by H.

3. An application: A common fixed-disc theorem

Let (X, q) be a quasi-metric space, $T, S : X \to X$ be two self-mappings and $D^q_{x_0,r}$ be a disc on X. If $Tx = Sx = x$ for all $x \in D^q_{x_0,r}$, then the disc $D^q_{x_0,r}$ is called as the common fixed disc of the pair (T, S).

Following [17] and [26], we modify the number $M^q_C(x, y)$ defined in (2.8) as follows:

$$M^q_{T,S}(x,y) = \max\left\{q(Tx,Sy), q(Tx,Sx), q(Ty,Sy), \frac{q(Tx,Sy)+q(Ty,Sx)}{2}\right\}.$$

To obtain a common fixed-disc theorem, we define the following number:

$$\mu^q = \inf\left\{q(Tx,Sx) : x \in X, Tx \neq Sx\right\}.$$

In the following theorem, we use the numbers $M^q_{T,S}(x, y)$, r which is defined in (2.1), μ^q and ρ defined by

$$\rho = \min\{r, \mu^q\}.$$

Theorem 3.1 Let (X, q) be a quasi-metric space, $T, S : X \to X$ two self-mappings and T an α-x_0-admissible map. Assume that there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that

$$q(Tx,Sx) > 0 \Longrightarrow t + \alpha(x_0,Tx)F(q(Tx,Sx)) \leq F(M^q_{T,S}(x,x_0)),$$

for each $x \in X$ and

$$\alpha(x_0,x) \geq 1,\ q(Tx,x_0) \leq \rho,\ q(x_0,Sx) \leq \rho,$$

for each $x \in D^q_{x_0,\rho}$. If T is a quasi-F_q-contraction with $x_0 \in X$ and r (or S is a quasi-F_q-contraction with $x_0 \in X$ and r), then $D^q_{x_0,\rho}$ is a common fixed disc of the pair (T, S) in X.

Proof. Let $x = x_0$. If $q(Tx_0, Sx_0) > 0$ then we have

$$M^q_{T,S}(x_0,x_0) = \max\left\{\begin{array}{c} q(Tx_0,Sx_0), q(Tx_0,Sx_0), q(Tx_0,Sx_0), \\ \dfrac{q(Tx_0,Sx_0)+q(Tx_0,Sx_0)}{2} \end{array}\right\} = q(Tx_0,Sx_0)$$

and

$$\begin{aligned} t + \alpha(x_0,Tx_0)F(q(Tx_0,Sx_0)) &\leq F(M^q_{T,S}(x_0,x_0)) = F(q(Tx_0,Sx_0)) \\ &\Longrightarrow t \leq (1-\alpha(x_0,Tx_0))\,F(q(Tx_0,Sx_0)), \end{aligned}$$

a contradiction with $t > 0$. Therefore $Tx_0 = Sx_0$, that is, x_0 is a coincidence point of the pair (T, S). If T is a quasi-F_q-contraction (or S is a quasi-F_q-contraction) then using Theorem 2.3, we have $Tx_0 = x_0$ (or $Sx_0 = x_0$) and hence $Tx_0 = Sx_0 = x_0$.

Now if $\rho = 0$, then clearly $D^q_{x_0,\rho} = \{x_0\}$ and this disc is a common fixed disc of the pair (T, S).

Let $\rho > 0$ and $x \in D^q_{x_0,\rho}$. Assume that $Tx \neq Sx$, that is, $q(Tx, Sx) > 0$. Using the hypothesis, α-x_0-admissibility of T and the definition of ρ, we get

$$\begin{aligned} M^q_{T,S}(x, x_0) &= \max\left\{q(Tx, Sx_0), q(Tx, Sx), q(Tx_0, Sx_0), \frac{q(Tx, Sx_0) + q(Tx_0, Sx)}{2}\right\} \\ &= \max\left\{q(Tx, x_0), q(Tx, Sx), \frac{q(Tx, x_0) + q(x_0, Sx)}{2}\right\} \\ &\leq q(Tx, Sx) \end{aligned}$$

and so

$$\begin{aligned} t + \alpha(x_0, Tx)F(q(Tx, Sx)) \quad &\leq \quad F(M^q_{T,S}(x, x_0)) \leq F(q(Tx, Sx)) \\ &\Longrightarrow \quad t \leq (1 - \alpha(x_0, Tx))F(q(Tx, Sx)), \end{aligned}$$

a contradiction with $t > 0$. We have found that x is a coincidence point of the pair (T, S), that is, $Tx = Sx$. If T (or S) is a quasi-F_q-contraction, then by Theorem 2.3, we have $Tx = x$ (or $Sx = x$) and hence $Tx = Sx = x$. Consequently, $D^q_{x_0,\rho}$ is a common fixed disc of the pair (T, S).

Acknowledgement. The authors would like to thank the anonymous referee for her/his comments that helped us improve this article. The second and third authors were supported by Balýkesir University Research Grant no: 2018/021. The fourth author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

References

[1] H. Afshari, S. Kalantari, H. Aydi, *Fixed point results for generalized $\alpha - \psi$-Suzuki-contractions in quasi-b-metric-like spaces*, Asian-Eur. J. Math., **11**(2018), no. 1, 1850012 (12 pages).

[2] H.H. Alsulami, E. Karapınar, F. Khojasteh, A.F. Roldán-López-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces*, Discrete Dyn. Nat. Soc., (2014), Article ID 269286, 10 pages.

[3] H. Aydi, *α-implicit contractive pair of mappings on quasi b-metric spaces and an application to integral equations*, J. Nonlinear Convex Anal., **17**(2016), no. 12, 2417-2433.

[4] H. Aydi, A. Felhi, E. Karapınar, F.A. Alojail, *Fixed points on quasi-metric spaces via simulation functions and consequences*, J. Math. Anal., **9**(2018), no. 2, 10-24.

[5] H. Aydi, M. Jellali, E. Karapınar, *On fixed point results for α-implicit contractions in quasi-metric spaces and consequences*, Nonlinear Anal., Model. Control, **21**(2016), no. 1, 40-56.

[6] H. Aydi, N. Taş, N.Y. Özgür, N. Mlaiki, *Fixed-discs in rectangular metric spaces*, Symmetry, **11**(2019), no. 2, 294.

[7] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fundam. Math., **2**(1922), 133-181.

[8] V. Berinde, *Contracţii Generalizate şi Aplicaţii*, **22**, Editura Cub Press, Baia Mare, 1997.

[9] N. Bilgili, E. Karapinar, B. Samet, *Generalized $\alpha - \psi$ contractive mappings in quasi metric spaces and related fixed point theorems*, J. Inequal. Appl., **2014**, 2014:36.

[10] S.A. Chen, W. Li, D. Zou, S.B. Chen, *Fixed point theorems in quasi-metric spaces,* Proc. of the Sixth International Conference on Machine Learning and Cybernetics, Hong Kong, 19-22 August 2007, 2499-2504.

[11] Şt. Cobzaş, *Functional Analysis in Asymmetric Normed Spaces*, Frontiers in Mathematics, Birkhäuser/Springer Basel, 2013. x+219 pp.
[12] N.V. Dung, *Remarks on quasi-metric spaces*, Miskolc Math. Notes, **15**(2014), no. 2, 401-422.
[13] E. Karapınar, S. Czerwik, H. Aydi, (α, ψ)*-Meir-Keeler contraction mappings in generalized b-metric spaces*, J. Funct. Spaces, **2018**(2018), art. ID 3264620, 4 pages.
[14] M. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., **30**(1984), 1-9.
[15] N. Mlaiki, N. Taş, N.Y. Özgür, *On the fixed-circle problem and Khan type contractions*, Axioms, **7**(2018), 80.
[16] A. Mukheimer, α-ψ-ϕ*-contractive mappings in ordered partial b-metric spaces*, J. Nonlinear Sci. Appl., **7**(2014), no. 3, 168-179.
[17] N.Y. Özgür, *Fixed-disc results via simulation functions*, Turkish J. Math. **43**(2019), no. 6, 2794-2805.
[18] N.Y. Özgür, N. Taş, *Some fixed-circle theorems on metric spaces*, Bull. Malays. Math. Sci. Soc., **42**(2019), no. 4, 1433-1449.
[19] N.Y. Özgür, N. Taş, *Some fixed-circle theorems and discontinuity at fixed circle*, AIP Conference Proceedings, **1926**, 020048 (2018).
[20] R.P. Pant, N.Y. Özgür, N. Taş, *On discontinuity problem at fixed point*, Bull. Malays. Math. Sci. Soc., **43**(2020), no. 1, 499-517.
[21] V.Ć. Rajić, *Some remarks on G-metric spaces and fixed point theorems*, Int. J. Anal. Appl., **5**(2014), no. 1, 102-114.
[22] I.L. Reilly, P.V. Subrahmanyam, M.K. Vamanamurthy, *Cauchy sequences in quasi-pseudo-metric spaces*, Monatsh. Math., **93**(1982), 127-140.
[23] N. Shahzad, O. Valero, *Fixed point theorems in quasi-metric spaces and the specialization partial order*, Fixed Point Theory, **19**(2018), no. 2, 733-749.
[24] M.B. Smyth, *Completeness of quasi-uniform and syntopological spaces*, J. London Math. Soc., **49**(1994), 385-400.
[25] N. Taş, N.Y. Özgür, N. Mlaiki, *New types of* F_C*-contractions and the fixed-circle problem*, Mathematics, **6**(2018), no. 10, 188.
[26] A. Tomar, R. Sharma, *Some coincidence and common fixed point theorems concerning* F*-contraction and applications*, J. Int. Math. Virtual Inst., **8**(2018), 181-198.
[27] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012**, 2012:94.
[28] W.A. Wilson, *On quasi-metric spaces*, Amer. J. Math., **53**(1931), no. 3, 675-684.

Received: January 10, 2019; Accepted: May 8, 2019.