Fixed Point Theory, 22(2021), No. 1, 3-14
DOI: 10.24193/fpt-ro.2021.1.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

BEST PROXIMITY THEOREMS OF PROXIMAL MULTIFUNCTIONS

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Abstract. Best proximity point theorems for self multifunctions have been proved with different conditions on the space and the considered mappings. In this paper, we prove some best proximity point theorems for a class of generalized multifunctions, namely proximal multifunctions.

Key Words and Phrases: Best proximity point, proximal multifunctions of first kind, proximal multifunctions of second kind, approximatively compact, cyclically Cauchy sequence, fairly Cauchy sequence, fairly complete space, uniform approximation, $T-$approximation, quasi-continuous.
2020 Mathematics Subject Classification: 47H10, 41A65, 90C30.

1. Introduction

Many of real life problems such as the system of linear or algebraic equations, ordinary or partial differential equations, etc can be framed as linear or nonlinear equations of the form $Tx = x$. In this case $x \in X$ is called fixed point of T. The fixed point theorems for contraction mappings on the complete metric spaces are well known and a number of generalizations of these results have been proved. Nadler [15] proved an extension of the fixed point theorem for multifunction contraction. We noted that x is said to be a fixed point of $T : X \to 2^X$ when $x \in Tx$. Several classes of problems like equilibrium problems [5], optimization problems [14], differential inclusions [4] can be modeled and solved using fixed point theorems of multifunctions. But not all maps or multifunctions have fixed point. In this case, one tries to determine an approximate solution x (subject to the condition) that the distance between x and Tx be minimum. Indeed, best proximity point theorems examine the existence of such optimal approximate solutions, known as best proximity points of a map or multifunction. Thus x is the best proximity point of $T : A \to B$ or $T : A \to 2^B$ if

$d(x, Tx) = d(A, B)$. So, a best proximity point theorem is concerned with the global minimazition of the mapping $f(x) = d(x, Tx)$.
We mention that the best proximity point theory is closely related to the following classical result in the best approximation theory.
Theorem 1.1 ([9]) If A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$.
References [3], [7], [8] and [17] have analyzed several classes of contractions for the existence of best the proximity point. Further, best proximity theorems for multifunctions have been proved in [12], [13], [16], [22]. In [18] Sadiq Basha defined proximal contraction mappings and proved the best proximity point for these mappings. Later, it was continued in numerous articles(for example see [1], [20], [21]).
In this paper we prove the best proximity point theorems for some multifunctions we call proximal multifunctions.
Let (X, d) be a metric space and $Y \subseteq X$. We refer to the family of all nonempty closed and bounded subsets of Y as $CB(Y)$. Let us assume throughout this section that A and B are nonempty subsets of a metric space (X, d). We recall the following notations which will be used in the sequel.

$$d(A, B) := \inf\{d(x, y) : x \in A \ \ and \ \ y \in B\},$$

$$A_0 := \{x \in A : d(x, y) = d(A, B) \ \text{ for some } \ y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = d(A, B) \ \text{ for some } \ x \in A\}.$$

Also if A and B are closed, The Hausdorff metric is defined as:

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

Since in sequel we always assume that A_0 is a nonempty set, we need to point out that in [13] sufficient conditions are provided to guarantee that A_0 be a nonempty set.
Definition 1.1 ([18]) The set B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \to d(x, B)$, for some $x \in A$ has a convergent subsequence.
We note that every set is approximatively compact with respect to itself, and that every compact set is approximatively compact.
Definition 1.2 ([18]) A point $x^* \in A$ is said to be the best proximity point of the multifunction $T : A \to 2^B$ when $d(x^*, Tx^*) = d(A, B)$.
Definition 1.3 ([21]) Given a sequence $\{x_n\}$ in A and a sequence $\{y_n\}$ in B, the sequence $\{(x_n, y_n)\}$ in $A \times B$ is said to be a cyclically Cauchy sequence if and only if for every $\epsilon > 0$, there exists a positive integer N such that $d(x_m, y_n) < d(A, B) + \epsilon$, for all $m, n \geq N$.
We note that the sequence $\{x_n\}$ in A is a Cauchy sequence if and only if the sequence $\{(x_n, x_n)\}$ is a cyclically Cauchy sequence in $A \times A$.
Definition 1.4 ([21]) Given a sequence $\{x_n\}$ in A and a sequence $\{y_n\}$ in B, the sequence $\{(x_n, y_n)\}$ in $A \times B$ is said to be a fairly Cauchy sequence if and only if the following conditions are satisfied:

(i) $\{(x_n, y_n)\}$ is a cyclically Cauchy sequence;
(ii) $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Definition 1.5 ([21]) The pair (A, B) is called a fairly complete space if and only if for every fairly Cauchy sequence $\{(x_n, y_n)\}$ in $A \times B$, the sequences $\{x_n\}$ and $\{y_n\}$ are convergent in A and B, respectively.

Definition 1.6 ([19]) A is said to have uniform approximation in B if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B), d(x_1, x_2) < \delta \Rightarrow d(y_1, y_2) < \epsilon$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 1.7 A multifunction $T : A \to 2^B$ is said to have uniform $T-$ approximation in B if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B), d(u_1, u_2) < \delta \Rightarrow H(Tx_1, Tx_2) < \epsilon$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.8 A multifunction $T : A \to 2^B$ is said to be proximal contraction of the first kind if there exists a non-negative number $\alpha < 1$ such that

$$d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B) \Rightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2)$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.9 A multifunction $T : A \to 2^B$ is said to be a proximal contraction of the second kind if and only if there exists a non-negative number $\alpha < 1$ such that

$$d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B) \Rightarrow H(Tu_1, Tu_2) \leq \alpha H(Tx_1, Tx_2)$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.10 A multifunction $T : A \to 2^B$ is said to be a proximally quasi-continuous if

$$d(u_n, Tx_n) = d(u, Tx) = d(A, B), x_n \to x \Rightarrow u_{n_k} \to u$$

for all sequences $\{u_n\}$, $\{x_n\}$ in A, for $x, u \in A$ and for some subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Definition 1.11 A multifunction $T : A \to 2^B$ is said to be a strong proximal contraction of the second kind if and only if the following conditions are satisfied:
(a) T is proximally quasi-continuous,
(b) T is a proximal contraction of the second kind.

2. Main results

Lemma 2.1 Let A and B be nonempty subsets of a metric space such that A_0 and B_0 are nonempty and $T : A \to 2^B$ is a multifunction. Also assume that for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$. Then there exists sequence $\{x_n\}$ in A_0 and $\{y_n\}$ in $Tx_n \cap B_0$ such that for all $n \in \mathbb{N}$, $d(x_{n+1}, Tx_n) = d(A, B) = d(x_{n+1}, y_n)$.

Proof. Let $x_0 \in A_0$. By assumption $T(x_0) \cap B_0 \neq \emptyset$, we obtain that, for any $y \in T(x_0) \cap B_0$ there exists $x \in A$ such that $d(x, y) = d(A, B)$. We claim that there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. To prove it, we know that for any

$x \in A$ we have $d(A,B) \leq d(x,Tx_0)$. Now let $y_0 \in T(x_0) \cap B_0$. Then there exists $x_1 \in A$ such that $d(x_1,y_0) = d(A,B)$ and we can deduce that

$$d(x_1,Tx_0) \leq d(x_1,y_0) = d(A,B) \leq d(x_1,Tx_0).$$

Thus $d(x_1,Tx_0) = d(A,B) = d(x_1,y_0)$. From $d(x_1,y_0) = d(A,B)$ we get $x_1 \in A_0$. By repeating the same process we can make sequence $\{x_n\}$ in A_0 and $\{y_n\}$ in $Tx_n \cap B_0$ such that, for all $n \in \mathbb{N}$, $d(x_{n+1},Tx_n) = d(A,B) = d(x_{n+1},y_n)$.

Theorem 2.2 Consider (X,d) to be a complete metric space and

(i) A,B are nonempty subsets of X and A is closed;
(ii) B is approximatively compact with respect to A;
(iii) A_0,B_0 are nonempty;
(iv) $T : A \to 2^B$ is a proximal contraction of the first kind;
(v) for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$.

Then there exists a unique $x \in A$ such that $d(x,Tx) = d(A,B)$. Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1},Tx_n) = d(A,B)$ is convergent to x.

Proof. Let $x_0 \in A_0$. By using the lemma 2.1 we get sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n) = d(x_{n+2},Tx_{n+1}) = d(A,B).$$

By (iv) , for any $n \in \mathbb{N}$, we get $d(x_{n+2},x_{n+1}) \leq \alpha d(x_{n+1},x_n)$. So we can deduce that $d(x_{n+2},x_{n+1}) \leq \alpha^{n+1} d(x_1,x_0)$. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete and A is closed then there exists $x \in A$ such that $x_n \to x$ as $n \to \infty$. Further, it is easy to prove that

$$\begin{aligned} d(x,B) \leq d(x,Tx_n) &\leq d(x,x_{n+1}) + d(x_{n+1},Tx_n) \\ &= d(x,x_{n+1}) + d(A,B) \\ &\leq d(x,x_{n+1}) + d(x,B). \end{aligned} \tag{2.1}$$

for any $n \in \mathbb{N}$. Thus $d(x,Tx_n) \to d(x,B)$ as $n \to \infty$. Then we may select $y_n \in Tx_n$ such that $d(x,y_n) \to d(x,B)$ as $n \to \infty$. To prove this, since $d(x,Tx_n) \to d(x,B)$ we have

$$\lim_{n\to\infty} \inf_{y \in Tx_n} d(x,y) = d(x,B).$$

Now for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we get

$$|\inf_{y \in Tx_n} d(x,y) - d(x,B)| < \epsilon.$$

Then for some $m \geq N$ there exists no $y \in Tx_m$ such that $|d(x,y) - d(x,B)| < \epsilon$. Otherwise, for each $y \in Tx_m$ we have $|d(x,y) - d(x,B)| \geq \epsilon$. By taking infimum from the both side of the last inequality with respect to y we get

$$|\inf_{y \in Tx_m} d(x,y) - d(x,B)| \geq \epsilon.$$

That is a contradiction. Thus $d(x,y_n) \to d(x,B)$ and by (ii) there exists $y \in B$ such that some subsequence of $\{y_n\}$ is convergent to $y \in B$. Then $d(x,y) = d(x,B)$. Further, by (2.1) we get

$$d(x,B) \leq d(x,x_{n+1}) + d(A,B) \leq d(x,x_{n+1}) + d(x,B).$$

Then $d(x, y) = d(A, B)$ and we can write $x \in A_0$. By (v) and lemma 2.1 there exists $z \in A$ such that $d(z, Tx) = d(A, B)$. Thus for any $n \in \mathbb{N}$ we have

$$d(x_{n+1}, Tx_n) = d(z, Tx) = d(A, B).$$

By (iv) , for any $n \in \mathbb{N}$, we get $d(x_{n+1}, z) \leq \alpha d(x_n, x)$. Since $x_n \to x$ we get $x_{n+1} \to z$. Thus $z = x$ and we get $d(x, Tx) = d(A, B)$. Also if x^* is the other best proximity point of T we get

$$d(x, Tx) = d(x^*, Tx^*) = d(A, B).$$

Then by (iv) we get $d(x, x^*) \leq \alpha d(x, x^*)$. Thus $d(x, x^*) = 0$ and $x = x^*$.

Example 2.1 Consider the Euclidean space $\mathbb{R}^2$. Let us define

$$A := \{(0, y) \mid 0 \leq y \leq 1\}, \quad B := \{(2, y) \mid 0 \leq y \leq 1\}.$$

Consider a multifunction $T : A \to 2^B$ is defined as follow

$$T(0, y) = \{(2, a) \mid 0 \leq a \leq \frac{y}{2}\}.$$

It is clear that $A_0 = A, B_0 = B$ and $T(A_0) \subseteq B_0$. Assume that

$$d(u_1, Tv_1) = d(u_2, Tv_2) = d(A, B) = 2,$$

where $u_1, u_2, v_1, v_2 \in A$. Then we have

$$d(u_1, u_2) = d\left(\frac{1}{2}v_1, \frac{1}{2}v_2\right) = \frac{1}{2}d(v_1, v_2).$$

Thus T is the proximal contraction of the first kind. Since A, B are compact then B is approximatively compact with respect to A. $(0, 0)$ is the unique best proximity point of T.

Example 2.2 Consider the Euclidean space $\mathbb{R}^2$. Let us define

$$A := \{(0, y) \mid 0 \leq y \leq 1\}, \quad B := \{(x, y) \mid 2 \leq x \leq 3, \quad 0 \leq y \leq 1\}.$$

Consider a multifunction $T : A \to 2^B$ is defined as follows

$$T(0, y) = \left\{(x, a) \mid 2 \leq x \leq 3, \quad 0 \leq a \leq \frac{y}{2}\right\}.$$

It is easy to prove that $A_0 = A, B_0 = \{(2, y) \mid 0 \leq y \leq 1\}$. Moreover for any $x \in A_0$ we have $Tx \cap B_0 \neq \emptyset$ and $T(A_0) \nsubseteq B_0$. Assume that

$$d(u_1, Tv_1) = d(u_2, Tv_2) = d(A, B) = 2,$$

where $u_1, u_2, v_1, v_2 \in A$. Then we have

$$d(u_1, u_2) = d\left(\frac{1}{2}v_1, \frac{1}{2}v_2\right) = \frac{1}{2}d(v_1, v_2).$$

Thus T is proximal contraction of the first kind. Since A, B are compact, B is approximatively compact with respect to A. $(0, 0)$ is the unique best proximity point of T.

Example 2.3 Consider the metric space (X, d) where $X = l_\infty$ is the space of essential bounded sequence (with respect the essential norm) and $d = d_\infty$ is the metric that is induced by essential norm([2]). Let us define

$$x_1 = (6, 2, 2, 2, 2, 2, 2, \cdots), \quad x_2 = (2, 2, 6, 2, 2, 2, 2, \cdots),$$

$$u_1 = (2, 2, 2, 4, 2, 2, 2, \cdots), \quad u_2 = (2, 2, 2, 2, 4, 2, 2, \cdots)$$

and

$$A = \{x_1, x_2, u_1, u_2\}.$$

Consider B as the set of all sequences such that all its elements, except for a finite number of them, are zero and the opposite zero elements are equal to 2. For example; $(0, 2, 0, 0, 2, 0, \cdots, 0, \cdots)$ is a member of B. Then we get $d(A, B) = 2$. Let's define a mapping $T : A \to B$ as follows:

$$Tx_1 = (0, 2, 0, 2, 0, 2, 0, 0, \cdots), \quad Tx_2 = (0, 2, 0, 0, 2, 2, 0, 0, \cdots),$$

$$Tu_1 = (2, 0, 2, 0, 0, 2, 0, 0, \cdots), \quad Tu_2 = (0, 2, 2, 0, 0, 2, 0, 0, \cdots).$$

So we have $d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B) = 2$.
By $d(u_1, u_2) = 2$ and $d(x_1, x_2) = 4$ we get $d(u_1, u_2) \leq \alpha d(x_1, x_2)$ where $\alpha = \frac{1}{2}$. Thus T is the proximal contraction of the first kind. It is easy to prove that $A_0 = \{u_1, u_2\}$. Since $B_0 = B$, for any $x \in A$, we get $T(x) \in B_0$. Also A is a closed subset of X. But B is not approximatively compact with respect to A. To prove it, let us define

$$y_n = (2, 2, 2, 2, \underbrace{2, \cdots, 2}_{n}, 0, 0, \cdots).$$

It is obvious that $||u_1 - y_n||_\infty \to d(u_1, B)$ as $n \to \infty$. But $\{y_n\}$ has no convergence subsequence. We note that (X, d) is not a complete metric space and T doesn't have a best proximity point.

Theorem 2.3 Consider (X, d) to be a complete metric space and

(i) A, B are nonempty subsets of X and B is closed;
(ii) A is approximatively compact with respect to B;
(iii) A_0, B_0 are nonempty;
(iv) $T : A \to 2^B$ is a strong proximal contraction of the second kind;
(v) for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$.

Then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$. Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ is convergent to x.

Proof. Let $x_0 \in A_1$. By using the lemma 2.1 we get sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

By (iv) , for any $n \in \mathbb{N}$, we get $H(Tx_{n+2}, Tx_{n+1}) \leq \alpha H(Tx_{n+1}, Tx_n)$ and we can deduce that $H(Tx_{n+2}, Tx_{n+1}) \leq \alpha^{n+1} H(Tx_1, Tx_0)$. By the definition of Hausdorff metric, for any $n \in \mathbb{N}$, there exists $y_n \in Tx_n$ such that

$$d(y_n, y_{n+1}) \leq \alpha^{n+1} H(Tx_1, Tx_0) + k^n$$

for some fixed $k \in (0, 1)$. Thus $\{y_n\}$ is a Cauchy sequence. Since X is complete and B is closed, then there exists $y \in B$ such that $y_n \to y$ as $n \to \infty$. Further, it is easy to prove that

$$d(y_{n+1}, Tx_n) \leq H(Tx_{n+1}, Tx_n) \leq \alpha^{n+1} H(Tx_1, Tx_0).$$

Thus we get

$$\lim_{n\to\infty} d(y_{n+1}, Tx_n) = \lim_{n\to\infty} d(y, Tx_n) = 0. \tag{2.2}$$

Also, we can prove that

$$\begin{aligned} d(y,A) \le d(y,x_{n+1}) &\le d(y,Tx_n) + d(x_{n+1},Tx_n) \\ &= d(y,Tx_n) + d(A,B) \\ &\le d(y,Tx_n) + d(y,A). \end{aligned} \tag{2.3}$$

Therefore $d(y,x_{n+1}) \to d(y,A)$ as $n \to \infty$ and by (ii), $\{x_n\}$ has a convergent subsequence to $x \in A$. Then $d(x,y) = d(A,y)$. By (2.3) we have

$$d(A,B) \le d(y,A) \le d(y,x_{n+1}) \le d(y,Tx_n) + d(x_{n+1},Tx_n).$$

Then

$$d(A,B) \le d(y,x_{n+1}) \le d(y,Tx_n) + d(A,B).$$

Thus $d(x,y) = d(A,B)$ and so we have $x \in A_0$. Since $Tx \cap B_0 \neq \emptyset$, there exists $z \in A$ such that $d(z,Tx) = d(A,B)$. Moreover $\{x_n\}$ has a convergent subsequence like $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. So we have

$$d(x_{n_k+1},Tx_{n_k}) = d(z,Tx) = d(A,B)$$

and by (iv), $\{x_{n_k}\}_k$ has convergence subsequence to z. Thus $z = x$ and we get $d(x,Tx) = d(A,B)$.

Example 2.4 Consider the Euclidean space $\mathbb{R}^2$. Let us define

$$A := \{(-1,y) \mid -1 \le y \le 1\}, \quad B := \{(x,y) \mid 1 \le x \le 2, \quad -1 \le y \le 1\}.$$

Let a multifunction $T : A \to 2^B$ be defined as follows

$$T(-1,y) = \begin{cases} \{(x,1) \mid 1 \le x \le 2\}, & y \quad \text{is rational} \\ \{(x,-1) \mid 1 \le x \le 2\}, & y \quad \text{is not rational.} \end{cases}$$

It is clear that $A_0 = A, B_0 = \{(1,y) \mid -1 \le y \le 1\}$. Moreover for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$. Assume that $d(u_1,Tv_1) = d(u_2,Tv_2) = d(A,B) = 2$ when $u_1,u_2,v_1,v_2 \in A$. We consider the following cases :

Case 1. $v_1 = (-1,y_1)$ and $v_2 = (-1,y_2)$ such that y_1,y_2 are rational. In this case we should have $u_1 = u_2 = (-1,1)$. Then $H(Tu_1,Tu_2) = 0$ and $d(u_1,u_2) = 0$.

Case 2. $v_1 = (-1,y_1)$ and $v_2 = (-1,y_2)$ such that y_1 is rational and y_2 is not rational. In this case we should have $u_1 = (-1,1)$ and $u_2 = (-1,-1)$. Then $H(Tu_1,Tu_2) = 0$ and $d(u_1,u_2) = 2$.

Case 3. $v_1 = (-1,y_1)$ and $v_2 = (-1,y_2)$ such that y_1,y_2 are not rational. In this case we should have $u_1 = u_2 = (-1,-1)$. Then $H(Tu_1,Tu_2) = 0$ and $d(u_1,u_2) = 0$.

Thus T is a proximal contraction of the second kind. Case 2 shows that T is not a proximal contraction of the first kind. Since A,B are compact then B is approximatively compact with respect to A. $(-1,1)$ is the best proximity point of T.

Theorem 2.4 Consider (X,d) to be a complete metric space and

(i) A,B are nonempty closed subsets of X;
(ii) A_0,B_0 are nonempty;
(iii) $T : A \to 2^B$ is a proximal contraction of the first and second kind;
(iv) for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$.

Then there exists $x \in A$ such that $d(x,Tx) = d(A,B)$. Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1},Tx_n) = d(A,B)$ is convergent to x.

Proof. Let $x_0 \in A_1$. By using the lemma 2.1 we get the sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

Similar to the theorem 2.3, it can be shown that, $\{x_n\} \subseteq A$ is a Cauchy sequence and since X is complete and A is closed then there exists $x \in A$ such that $x_n \to x$ as $n \to \infty$. Also, there exists $y_n \in Tx_n$ such that $\{y_n\} \subseteq B$ is a Cauchy sequence and there exists $y \in B$ such that $y_n \to y$ as $n \to \infty$. By (2.1) of theorem 2.3 we get $d(x, Tx_n) \to d(x, B)$, $d(x, Tx_n) \to d(A, B)$ and By (2.2) of theorem 2.3 we get $d(y_{n+1}, Tx_n) \to 0$, $d(y, Tx_n) \to 0$. Also, it is easy to prove that

$$d(A, B) \leq d(x_{n+1}, y_n) \leq d(x_{n+1}, Tx_n) + d(y_{n+1}, Tx_n)$$

and so

$$d(A, B) \leq d(x_{n+1}, y_n) \leq d(A, B) + d(y_{n+1}, Tx_n).$$

Then $d(x_{n+1}, y_n) \to d(x, y) = d(A, B)$ as $n \to \infty$. Thus $x \in A_0$ and there exists $z \in B_0$ such that

$$d(z, Tx) = d(x_{n+1}, Tx_n) = d(A, B).$$

Since T is the proximal contraction of the first kind we have $d(z, x_{n+1}) \leq \alpha d(x_n, x)$. So $x_n \to z$ as $n \to \infty$. Thus we get $z = x$ and the proof is complete.

Example 2.5 Consider the Euclidean space $\mathbb{R}^2$. Let us define

$$A := \{(0, y)|\ 0 \leq y \leq 1\}.$$

For $y \in [0, 1] \cap \mathbb{Q}$ let us define

$$B_y := \{(x, y) \mid 1 \leq x \leq 2\},$$

and for $y \in [0, 1] \cap \mathbb{Q}^c$

$$B_y^c := \{(x, y) \mid 1 < x \leq 2\}.$$

Also let us consider

$$B_1 = \bigcup_{y \in [0,1] \cap \mathbb{Q}} B_y, \quad B_2 = \bigcup_{y \in [0,1] \cap \mathbb{Q}^c} B_y^c, \quad B = B_1 \cup B_2.$$

Let a multifunction $T : A \to 2^B$ be defined as follows

$$T(0, y) = \begin{cases} \left\{(x, \frac{y}{2}) \mid 1 \leq x \leq 2\right\}, & y \text{ is rational} \\ (x, \frac{y}{2}) \mid 1 < x \leq 2\}, & y \text{ is not rational.} \end{cases}$$

We can easily prove that $A_0 = \{(0, y) \mid y \in [0, 1] \cap \mathbb{Q}\}, B_0 = \{(1, y) \mid y \in [0, 1] \cap \mathbb{Q}\}$. Moreover for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$. Also it is easy to check that T is the proximal contraction of the first and second kind. But B is not approximatively compact with respect to A. $(0, 0)$ is the best proximity point of T.

Example 2.6 Consider the Euclidean space $\mathbb{R}^2$. Let us define

$$A_1 := \{a_n = (\frac{1}{n}, 1)\}_n, \ \ A_2 := ([0, 1] \cap \mathbb{Q}^c) \times \{1\}, \ \ A_3 = \{(0, 1)\},$$

$$B_1 := (\mathbb{Q} \cap [0, 1]) \times [-1, -2], \ \ B_2 := (\mathbb{Q}^c \cap [0, 1]) \times (-1, -2], \ \ B_3 := (2, 4) \times [-1, -2]$$

and

$$A := A_1 \cup A_2 \cup A_3, \quad B := B_1 \cup B_2 \cup B_3.$$

Let a multifunction $T : A \to 2^B$ be defined as follows

$$T(x,1) = \begin{cases} \left\{\frac{1}{\sqrt{2}n}\right\} \times (-1,-2], & x = \frac{1}{n} \\ \{2x+2\} \times [-1,-2], & x \text{ is not rational} \\ \{0\} \times [-1,-2], & x = 0. \end{cases}$$

It's clear that $A_0 = A_1 \cup A_3$, $B_0 = \{(\frac{1}{n},-1)\}_n \cup \{(0,-1)\}$. A, B are not closed.
We have

$$d\left(\left(\frac{1}{\sqrt{2}},1\right),(b_n,-1)\right) \to d\left(\left(\frac{1}{\sqrt{2}},1\right),B\right).$$

where $\{b_n\}_n \subseteq (0,1)$ is a sequence with rational elements such that it is convergent to $\frac{1}{\sqrt{2}}$ as $n \to \infty$. But $\{(b_n,-1)\}_n$ has no convergence subsequence in B. Thus B is not approximatively compact with respect to A. Now assume that $x_1 = a_4$ and $x_2 = a_2$.
We have

$$Tx_1 = \left\{\frac{1}{4\sqrt{2}}\right\} \times (-1,-2] \text{ and } Tx_2 = \left\{\frac{1}{2\sqrt{2}}\right\} \times (-1,-2].$$

So for $u_1 = \left(\frac{1}{4\sqrt{2}},1\right)$ and $u_2 = \left(\frac{1}{2\sqrt{2}},1\right)$ in A we have

$$d(u_1,Tx_1) = d(u_2,Tx_2) = d(A,B) = 2.$$

But it is easy to check that $H(Tx_1,Tx_2) = \frac{1}{4}$ and $H(Tu_1,Tu_2) = \frac{1}{2\sqrt{2}}$. Then T is not proximal contraction of the second kind. It is easy to check that T is proximal contraction of the first kind. Also for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$. $(0,1)$ is the best proximity point of T.
In next theorems we replace approximatively compact condition with others conditions.
Theorem 2.5 Consider (X,d) to be a complete metric space and
 (i) A,B are nonempty closed subsets of X;
 (ii) A_0,B_0 are nonempty;
 (iii) (A,B) is a fairly complete space;
 (iv) A has uniform $T-$approximation in B;
 (v) $T : A \to CB(B)$ is a proximal contraction of the first kind;
 (vi) for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$.
Then there exists $x \in A$ such that $d(x,Tx) = d(A,B)$. Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1},Tx_n) = d(A,B)$ is convergent to x.
Proof. Let $x_0 \in A_1$. By using the proof of lemma 2.1 we get sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0 ($y_n \in Tx_n$) such that

$$d(x_{n+1},y_n) = d(x_{n+1},Tx_n) = d(A,B)$$

and

$$d(x_{n+1},Tx_n) = d(x_{n+2},Tx_{n+1}) = d(A,B).$$

for any $n \in \mathbb{N}$. Thus by (v), for any $n \in \mathbb{N}$, we get $d(x_{n+2},x_{n+1}) \leq \alpha d(x_{n+1},x_n)$ and so we can deduce that

$$d(x_{n+2},x_{n+1}) \leq \alpha^{n+1} d(x_1,x_0). \tag{2.4}$$

Thus $\{x_n\}$ is a Cauchy sequence. By (2.4) and (iv), for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x_{n+1}, y_n) = d(x_{n+2}, y_{n+1}) = d(A, B), d(x_{n+1}, x_{n+2}) < \delta$$
$$\Longrightarrow H(Tx_n, Tx_{n+1}) < \epsilon.$$

Considering the fact that the values of T are nonempty, closed and bounded subsets of B and by the definition of Hausdorff metric we can write

$$d(y_n, y_{n+1}) \leq d(y_n, Tx_{n+1}) + H(Tx_n, Tx_{n+1}) + d(y_{n+1}, Tx_n) \leq 3\epsilon.$$

Then $\{y_n\}$ is a Cauchy sequence. Also we have

$$\begin{aligned} d(y_n, x_n) &\leq d(y_n, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq d(A, B) + \alpha^n d(x_1, x_0). \end{aligned}$$

On the other hand, For $m, n > N$, we have

$$d(x_m, y_n) \leq d(x_m, x_n) + d(x_n, y_n).$$

Therefore, (x_n, y_n) is a cyclically Cauchy sequence. By (iii) there exists $x \in A$ and $y \in B$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Thus we have

$$d(x, y) = \lim_{n\to\infty} d(x_{n+1}, y_n) = d(A, B).$$

Then $x \in A_0$ and by (vi) there exists $z \in A$ such that $d(z, Tx) = d(A, B)$.
Also for any $n \in \mathbb{N}$ we have $d(x_{n+1}, Tx_n) = d(A, B)$. By (v) we can deduce that $d(x_{n+1}, z) \leq \alpha d(x_n, z)$. So $x_n \to z$ as $n \to \infty$. Thus we get $z = x$ and the proof is complete.

Theorem 2.6 Consider (X, d) to be a complete metric space and

(i) A, B are nonempty closed subsets of X;
(ii) A_0, B_0 are nonempty;
(iii) (A, B) is a fairly complete space;
(iv) B has uniform approximation in A;
(v) $T : A \to CB(B)$ is a strong proximal contraction of the second kind;
(vi) for any $x \in A_0$ we have $T(x) \cap B_0 \neq \emptyset$.

Then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$. Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$ is convergent to x.

Proof. Let $x_0 \in A_1$. By using the proof of lemma 2.1 we get sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0 $(y_n \in Tx_n)$ such that

$$d(x_{n+1}, y_n) = d(x_{n+1}, Tx_n) = d(A, B)$$

and

$$d(x_{n+1}, Tx_n) = d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

for any $n \in \mathbb{N}$. Thus by (v) , for any $n \in \mathbb{N}$, we get

$$H(Tx_{n+2}, Tx_{n+1}) \leq \alpha d(Tx_{n+1}, Tx_n)$$

and so we can deduce that

$$H(Tx_{n+2}, Tx_{n+1}) \leq \alpha^{n+1} H(Tx_1, Tx_0). \tag{2.5}$$

Considering the fact that the values of T are nonempty, closed and bounded subsets of B, the definition of Hausdorff metric and (2.5) we can write

$$d(y_n, y_{n+1}) \le d(y_n, Tx_{n+1}) + H(Tx_n, Tx_{n+1}) + d(y_{n+1}, Tx_n) \le 3\epsilon. \tag{2.6}$$

By (2.5) and (iv) for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x_{n+1}, y_n) = d(x_{n+2}, y_{n+1}) = d(A, B), d(y_n, y_{n+1}) < \delta \Longrightarrow d(x_n, x_{n+1}) < \epsilon.$$

So $\{x_n\}$ is a Cauchy sequence. Also by (2.5) we get

$$\begin{aligned} d(y_n, x_n) &\le d(x_n, Tx_{n-1}) + d(Tx_{n-1}, y_n) \\ &\le d(A, B) + H(Tx_{n-1}, Tx_n) \\ &\le d(A, B) + \alpha^n H(Tx_1, Tx_0). \end{aligned}$$

On the other hand for $m, n > N$ we have

$$d(x_m, y_n) \le d(x_m, x_n) + d(x_n, y_n).$$

Thus, (x_n, y_n) is a cyclically Cauchy sequence. Also by (2.6) we can deduce that $\{y_n\}$ is a Cauchy sequence. Therefore by (iii) there exists $x \in A$ and $y \in B$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Thus we have

$$d(x, y) = \lim_{n\to\infty} d(x_{n+1}, y_n) = d(A, B).$$

Then $x \in A_0$ and by (vi) there exists $z \in A$ such that $d(z, Tx) = d(A, B)$. Also for any $n \in \mathbb{N}$ we have $d(x_{n+1}, Tx_n) = d(A, B)$. By (v), T is proximally quasi-continuous and we have $x_n \to x$ (as $n \to \infty$). Then there exists the subsequence $\{x_{n_k+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_k+1} \to z$ as $k \to \infty$. Thus we get $z = x$ and the proof is complete.

Acknowledgment. The authors thankful to the anonymous referees for the useful suggestions and remarks that contributed to the improvement of the manuscript.

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Received: October 18, 2018; Accepted: March 27, 2020.