Fixed Point Theory, 21(2020), No. 2, 791-804
DOI: 10.24193/fpt-ro.2020.2.56
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT RESULTS FOR w-CONTRACTIONS

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Dedicated to Professor Stojan Radenović on the occasion of his 70th birthday.

Abstract. We work in the setting of metric spaces endowed with a w_0-distance. Thanks to two suitable families of functions, we introduce a new type of contraction which we call w-contraction. We use the w-contractions in order to establish new and more general results of existence and uniqueness of fixed point. In particular, we stress that as applications of our main result, we get the existence and uniqueness of fixed point for cyclic mappings and mappings that satisfy a contractive condition of integral type.

Key Words and Phrases: Fixed point, metric space, w_0-distance, cyclic mapping, contractive condition of integral type.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The metric fixed point theory is an useful tools to deal with various mathematical problems. The Banach contraction principle (see [2]) is at the base of this theory. Such principle was generalized in different directions, see for example [1, 3, 6, 11, 12, 13, 16, 17].

In particular, we recall that Jleli *et al.* used an appropriate family of functions in order to introduce a new type of contraction, called (F, φ)-contraction (see [5]). They obtained results of existence and uniqueness of fixed point in the setting of metric spaces by using the (F, φ)-contractions (see also [18]). Khojasteh *et al.* introduced the concept of $\mathcal{Z}$-contraction (see [7]). The $\mathcal{Z}$-contractions are nonlinear contractions defined by using a specific function, called simulation function. We stress that the existence and uniqueness of fixed points for $\mathcal{Z}$-contraction mappings was proved in [7]. Furthermore, we remind that Samet *et al.* (see [16]) and Vetro and Vetro (see [17]) showed the existence and uniqueness of fixed point which belongs to the zero-set of a given function.

In this paper, we work in the setting of metric spaces endowed with a w_0-distance. Thanks to two suitable families of functions, we introduce a new type of contraction which we call w-contraction. We use the w-contractions in order to establish new and more general results of existence and uniqueness of fixed point. In particular, we stress that as applications of our main result (see Theorem 3.3), we get the existence and uniqueness of fixed point for cyclic mappings (see Theorem 4.3) and mappings that satisfy a contractive condition of integral type (see Theorems 5.4 and 5.5). In addition, we point out that our main result allows to deduce, as particular cases, some of the most known results of fixed point in the existing literature (see Corollaries 6.1, 6.2, 6.3 and 6.4). Finally, following [4] we give another result of existence and uniqueness of fixed point which involves cyclic contractions by weakening the closure assumption of Theorem 4.3 (see Theorem 4.4). We remark that such a closure assumption is usually supposed in the literature.

2. Preliminaries

In this section, we recall some definitions and notations that we will use throughout the paper. Precisely, we remind the notion of w_0-distance, and in addition, we present two families of functions which we will use in order to define the w-contractions.

The notion of w-distance in a metric space was introduced by Kada *et al.* (see [6]) in order to obtain new and more general fixed point results. Such definition was recently revised by Kostić *et al.* in [9]. Exactly, they supposed the lower semicontinuity with respect to both variables and so introduced the w_0-distance.

Definition 2.1. Let (X, d) be a metric space. A function $w : X \times X \to [0, +\infty[$ is called a w_0-distance on X if the following three conditions are verified.

- (w_1) $w(u, z) \leq w(u, v) + w(v, z)$ for all $u, v, z \in X$;
- (w_2) for any $u \in X$ the functions $w(u, \cdot), w(\cdot, u) : X \to [0, +\infty[$ are lower semicontinuous;
- (w_3) for any $\varepsilon > 0$ there exists $\delta > 0$ such that $w(u, z) \leq \delta$ and $w(u, v) \leq \delta$ imply $d(v, z) \leq \varepsilon$.

We stress that the notion of w_0-distance introduced by Kostić *et al.* is less general than one of w-distance given in [6]. However, the notion of w_0-distance is more general than one of metric.

For convenience of the reader we recall the following result of [6] which gives the principal properties of a w-distance.

Lemma 2.2 (See [6]). *Let (X, d) be a metric space and let w be a w-distance on X. Let $\{u_m\}$ and $\{v_m\}$ be sequences in X, let $\{\alpha_m\}$ and $\{\beta_m\}$ be sequences in $[0, +\infty[$ converging to 0 and let $u, v, z \in X$. Then the following hold:*

- *(i) If $w(u_m, v) \leq \alpha_m$ and $w(u_m, z) \leq \beta_m$ for any $m \in \mathbb{N}$, then $v = z$. In particular, if $w(u, v) = 0$ and $w(u, z) = 0$, then $v = z$.*
- *(ii) If $w(u_m, v_m) \leq \alpha_m$ and $w(u_m, z) \leq \beta_m$ for any $m \in \mathbb{N}$, then v_m converges to z.*
- *(iii) If $w(u_k, u_m) \leq \alpha_m$, for any $k, m \in \mathbb{N}$ with $k > m$, then $\{u_m\}$ is a Cauchy sequence.*
- *(iv) If $w(v, u_m) \leq \alpha_m$, for any $m \in \mathbb{N}$, then $\{u_m\}$ is a Cauchy sequence.*

In the sequel, given a metric space (X, d) and a w_0-distance w on X, we will denote by $\mu : X \times X \to [0, +\infty[$ the function defined by

$$\mu(u, v) = \max\{w(u, v), w(v, u)\} \quad \text{for any } u, v \in X. \tag{2.1}$$

It is easy to infer that the function μ verifies the following properties:

- $\mu(u,v) = 0 \Rightarrow u = v$ for any $u,v \in X$;
- μ is symmetric, that is, $\mu(u,v) = \mu(v,u)$ for any $u,v \in X$;
- μ satisfies the triangle inequality, that is, $\mu(u,z) \leq \mu(u,v) + \mu(v,z)$ for any $u,v,z \in X$.

Furthermore, the function $\mu : X \times X \to [0,+\infty[$ defined by (2.1) is such that

$$\mu(u,z) \leq \liminf_{m\to+\infty} \mu(u,u_m) \quad \text{whenever } u_m \to z \text{ as } m \to +\infty. \tag{2.2}$$

This means that μ is lower semicontinuous with respect to the second variable. Hence, taking into account that μ is symmetric, we deduce that

$$\mu(z,u) \leq \liminf_{m\to+\infty} \mu(u_m,u) \quad \text{whenever } u_m \to z \text{ as } m \to +\infty, \tag{2.3}$$

that is, μ is also lower semicontinuous with respect to the first variable.

In this paper, we use a contractive notion which involves two families of functions, called $\mathcal{H}$ and $\mathcal{S}$. Precisely, $\mathcal{H}$ denotes the family of functions $H : [0,+\infty[^3 \to [0,+\infty[$ satisfying the following conditions (see [5]):

(H_1) $\max\{\delta,\theta\} \leq H(\delta,\theta,\lambda)$, for all $\delta,\theta,\lambda \in [0,+\infty[$;
(H_2) $H(0,0,0) = 0$;
(H_3) H is continuous.

$\mathcal{S}$ denotes the family of functions $S : [0,+\infty[^2 \to \mathbb{R}$ satisfying the following conditions (see [1, 7]):

(S_1) $S(\delta,\theta) < \theta - \delta$ for all $\delta,\theta > 0$;
(S_2) if $\{\delta_m\}$, $\{\theta_m\}$ are sequences in $]0,+\infty[$ such that

$$\lim_{m\to+\infty} \delta_m = \lim_{m\to+\infty} \theta_m = \ell \in]0,+\infty[$$

then $\limsup_{m\to+\infty} S(\delta_m,\theta_m) < 0$.

For completeness we remark that both the families $\mathcal{H}$ and $\mathcal{S}$ were used in order to establish new and more general results of existence and uniqueness of fixed point (see [5] and [1, 7, 9], respectively). Furthermore, Jleli *et al.* used the family $\mathcal{H}$ in order to introduce a new type of contraction (see Definition 2.4 of [5]).

We notice that the following are examples of functions $H : [0,+\infty[^3 \to [0,+\infty[$ belonging to the family $\mathcal{H}$:

(i) $H(\delta,\theta,\lambda) = \delta + \theta + \lambda$, for all $\delta,\theta,\lambda \in [0,+\infty[$;
(ii) $H(\delta,\theta,\lambda) = \max\{\delta,\theta\} + \lambda$, for all $\delta,\theta,\lambda \in [0,+\infty[$.

In addition, the following are examples of functions $S : [0,+\infty[^2 \to \mathbb{R}$ belonging to the family $\mathcal{S}$:

(i) $S(\delta,\theta) = \sigma\,\theta - \delta$ for all $\delta,\theta \in [0,+\infty[$ where $\sigma \in [0,1[$;
(ii) $S(\delta,\theta) = \theta - \psi(\theta) - \delta$ for all $\delta,\theta \in [0,+\infty[$ where $\psi : [0,+\infty[\to [0,+\infty[$ is a lower semicontinuous function such that $\psi(\theta) = 0$ if and only if $\theta = 0$;
(iii) $S(\delta,\theta) = \theta\,\psi(\theta) - \delta$ for all $\delta,\theta \in [0,+\infty[$ where $\psi : [0,+\infty[\to [0,1[$ is such that $\lim_{\theta\to r^+} \psi(\theta) < 1$ for all $r > 0$.

In the next sections, given $X \neq \emptyset$, $f : X \to X$, $u_0 \in X$ and $u_m = fu_{m-1}$ for all $m \in \mathbb{N}$, we will call $\{u_m\}$ a sequence of Picard starting at u_0.

3. Fixed points for w-contractions

In this section, we state and prove our main result. Firstly, we introduce the notion of w-contraction.

Definition 3.1. Let (X,d) be a metric space and $w : X\times X \to [0,+\infty[$ be a w_0-distance on X. A mapping $f : X \to X$ is a w-contraction if there exist three functions $H \in \mathcal{H}$, $S \in \mathcal{S}$ and $\rho : X \to [0,+\infty[$, such that

$$S(H(\mu(fu,fv),\rho(fu),\rho(fv)),H(\mu(u,v),\rho(u),\rho(v))) \geq 0 \quad \text{for all } u,v\in X. \quad (3.1)$$

Lemma 3.2. *Let (X,d) be a metric space and $w : X\times X \to [0,+\infty[$ be a w_0-distance on X. Assume that the mapping $f : X \to X$ is a w-contraction with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $\rho : X \to [0,+\infty[$, that is,*

$$S(H(\mu(fu,fv),\rho(fu),\rho(fv)),H(\mu(u,v),\rho(u),\rho(v))) \geq 0 \quad \textit{for all } u,v\in X.$$

Then any sequence $\{u_m\}$ of Picard starting at a point $u_0 \in X$ is a Cauchy sequence, whenever $u_{m-1} \neq u_m$ for all $m \in \mathbb{N}$.

Proof. Let u_0 be an arbitrary point in X. Suppose that the sequence $\{u_m\}$ of Picard starting at u_0 is such that $u_{m-1} \neq u_m$ for all $m \in \mathbb{N}$. Firstly, we establish that

$$\lim_{m\to+\infty}\mu(u_{m-1},u_m)=0 \quad \text{and} \quad \lim_{m\to+\infty}\rho(u_m)=0. \quad (3.2)$$

We recall that $\mu(u,v)=0$ implies $u=v$ and so, taking into account that $u_{m-1}\neq u_m$ for all $m\in\mathbb{N}$, we have that $\mu(u_{m-1},u_m)>0$ for all $m\in\mathbb{N}$. Hence using the property (H_1) of the function H, we infer that

$$H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m)) \geq \mu(u_{m-1},u_m) > 0 \quad \text{for all } m\in\mathbb{N}.$$

Using (3.1) with $u=u_{m-1}$ and $v=u_m$ and the property (S_1) of the function S, we obtain

$$\begin{aligned} 0 &\leq S(H(\mu(u_m,u_{m+1}),\rho(u_m),\rho(u_{m+1})),H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m))) \\ &< H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m)) - H(\mu(u_m,u_{m+1}),\rho(u_m),\rho(u_{m+1})) \end{aligned}$$

for all $m\in\mathbb{N}$. The previous inequality shows that

$$H(\mu(u_m,u_{m+1}),\rho(u_m),\rho(u_{m+1})) < H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m)) \text{ for all } m\in\mathbb{N}.$$

This ensures that $\{H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m))\}$ is a decreasing sequence of positive real numbers. So, there exists some $\ell\geq 0$ such that

$$\lim_{m\to+\infty}H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m))=\ell.$$

We claim that $\ell=0$. We assume by contradiction that $\ell>0$. Then using the property (S_2) with

$$\delta_m = H(\mu(u_m,u_{m+1}),\rho(u_m),\rho(u_{m+1}))$$

and

$$\theta_m = H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m)),$$

we obtain

$$0\leq\limsup_{m\to+\infty}S(H(\mu(u_m,u_{m+1}),\rho(u_m),\rho(u_{m+1})),H(\mu(u_{m-1},u_m),\rho(u_{m-1}),\rho(u_m)))<0.$$

Clearly, this is a contradiction and hence we conclude that $\ell = 0$. Next, using the property (H_1) of the function H, we get

$$\max\{\mu(u_{m-1}, u_m), \rho(u_{m-1})\} \le H(\mu(u_{m-1}, u_m), \rho(u_{m-1}), \rho(u_m)) \quad \text{for } m \in \mathbb{N}$$

and hence

$$\lim_{m\to+\infty} \mu(u_{m-1}, u_m) = 0 \quad \text{and} \quad \lim_{m\to+\infty} \rho(u_{m-1}) = 0.$$

Secondly, we prove that $\{u_m\}$ is a Cauchy sequence. In order to show this by Lemma 2.2 (iii), it is sufficient to establish that for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\mu(u_n, u_m) < \varepsilon \quad \text{for all } m > n \ge n(\varepsilon). \tag{3.3}$$

We assume for way of contradiction that (3.3) does not hold. Then, there exist a positive real number ε_0 and two sequences $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k \ge k$ and $\mu(u_{n_k}, u_{m_k}) \ge \varepsilon_0 > \mu(u_{n_k}, u_{m_k-1})$ for all $k \in \mathbb{N}$. The previous inequality and the first limit of (3.2) imply that

$$\lim_{k\to+\infty} \mu(u_{n_k}, u_{m_k}) = \lim_{k\to+\infty} \mu(u_{n_k-1}, u_{m_k-1}) = \varepsilon_0.$$

Now, taking into account that H is a continuous function, we have

$$\begin{aligned}\lim_{k\to+\infty} H(\mu(u_{n_k-1}, u_{m_k-1}), \rho(u_{n_k-1}), \rho(u_{m_k-1})) &= \lim_{k\to+\infty} H(\mu(u_{n_k}, u_{m_k}), \rho(u_{n_k}), \rho(u_{m_k}))\\ &= H(\varepsilon_0, 0, 0) > 0.\end{aligned}$$

Furthermore, we can assume $\mu(u_{n_k-1}, u_{m_k-1}) > 0$ for all $k \in \mathbb{N}$. Consequently, we deduce that

$$H(\mu(u_{n_k-1}, u_{m_k-1}), \rho(u_{n_k-1}), \rho(u_{m_k-1})) \ge \mu(u_{n_k-1}, u_{m_k-1}) > 0 \quad \text{for all } k \in \mathbb{N}.$$

Since we also have $H(\mu(u_{n_k}, u_{m_k}), \rho(u_{n_k}), \rho(u_{m_k})) \ge \mu(u_{n_k}, u_{m_k}) > 0$, we can use the property (S_2) with

$$\delta_k = H(\mu(u_{n_k}, u_{m_k}), \rho(u_{n_k}), \rho(u_{m_k}))$$

and

$$\theta_k = H(\mu(u_{n_k-1}, u_{m_k-1}), \rho(u_{n_k-1}), \rho(u_{m_k-1})).$$

Thus, we get

$$\begin{aligned}0 \le \limsup_{k\to+\infty} S(&H(\mu(u_{n_k}, u_{m_k}), \rho(u_{n_k}), \rho(u_{m_k})),\\ &H(\mu(u_{n_k-1}, u_{m_k-1}), \rho(u_{n_k-1}), \rho(u_{m_k-1}))) < 0.\end{aligned}$$

Obviously, this is out of the question. So, we conclude that for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that (3.3) holds and hence the sequence $\{u_m\}$ is Cauchy. □

Now, we are ready to formulate and prove our first main result.

Theorem 3.3. *Let (X, d) be a complete metric space and $w : X \times X \to [0, +\infty[$ be a w_0-distance on X. Assume that the mapping $f : X \to X$ is a w-contraction with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $\rho : X \to [0, +\infty[$, that is,*

$$S(H(\mu(fu, fv), \rho(fu), \rho(fv)), H(\mu(u, v), \rho(u), \rho(v))) \ge 0 \quad \textit{for all } u, v \in X.$$

If ρ is a lower semicontinuous function, then f has a unique fixed point ξ such that $\rho(\xi) = 0$.

Proof. Firstly, we establish the uniqueness of the fixed point. Suppose that f has two fixed points $\xi, \zeta \in X$ such that $\xi \neq \zeta$. This ensures that

$$H(\mu(\xi,\zeta),\rho(\xi),\rho(\zeta)) \geq \mu(\xi,\zeta) > 0.$$

Using (3.1) with $u = \xi$ and $v = \zeta$ and the property (S_1) of the function S, we infer that

$$\begin{aligned} 0 &\leq S(H(\mu(f\xi, f\zeta), \rho(f\xi), \rho(f\zeta)), H(\mu(\xi,\zeta),\rho(\xi),\rho(\zeta))) \\ &< H(\mu(\xi,\zeta),\rho(\xi),\rho(\zeta)) - H(\mu(\xi,\zeta),\rho(\xi),\rho(\zeta)) = 0. \end{aligned}$$

Clearly, this is a contradiction. Hence, we have $\xi = \zeta$ and further we obtain the claim, that is, the uniqueness of the fixed point.

In order to establish the existence of a fixed point, we consider a point $u_0 \in X$. Let $\{u_m\}$ be a sequence of Picard starting at u_0. We stress that if $u_k = u_{k+1}$ for some $k \in \mathbb{N}$ then $u_k = u_{k+1} = fu_k$, that is, u_k is a fixed point of f. We claim that $\rho(u_k) = 0$. We assume by contradiction that $\rho(u_k) > 0$. Then, by the property (H_1) of the function H, we get

$$0 < \rho(u_k) \leq H(\mu(u_k,u_k),\rho(u_k),\rho(u_k)).$$

Taking into account that $u_m = u_k$ for all $m > k$, $m \in \mathbb{N}$, using (3.1) with

$$u = v = u_k = u_{k+1} = u_{k+2}$$

and the property (S_1) of the function S, we deduce that

$$\begin{aligned} 0 &\leq S(H(\mu(u_{k+1},u_{k+2}),\rho(u_{k+1}),\rho(u_{k+2})), H(\mu(u_k,u_{k+1}),\rho(u_k),\rho(u_{k+1}))) \\ &< H(\mu(u_k,u_{k+1}),\rho(u_k),\rho(u_{k+1})) - H(\mu(u_{k+1},u_{k+2}),\rho(u_{k+1}),\rho(u_{k+2})) \\ &= H(\mu(u_k,u_k),\rho(u_k),\rho(u_k)) - H(\mu(u_k,u_k),\rho(u_k),\rho(u_k)) = 0. \end{aligned}$$

Clearly, the previous inequality gives a contradiction. Hence, it follows that $\rho(u_k) = 0$. So, we conclude that if $u_k = u_{k+1}$ for some $k \in \mathbb{N}$, then u_k is a fixed point of f such that $\rho(u_k) = 0$.

Now, we can suppose that $u_m \neq u_{m+1}$ for every $m \in \mathbb{N}$. By Lemma 3.2, we have that the sequence $\{u_m\}$ is Cauchy. Hence, since (X, d) is complete, there exists some $\xi \in X$ such that

$$\lim_{m\to+\infty} u_m = \xi. \tag{3.4}$$

From the proof of Lemma 3.2, we say that for every $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ such that

$$\mu(u_{m(k)}, u_m) < \frac{1}{k} \quad \text{for all } m > m(k). \tag{3.5}$$

From (3.5), using the semicontinuity of μ with respect to the second variable (see (2.2)), we get

$$\mu(u_{m(k)},\xi) \leq \liminf_{m\to+\infty} \mu(u_{m(k)}, u_m) \leq \frac{1}{k}. \tag{3.6}$$

Consequently, from (3.6) we infer that there exists a subsequence $\{u_{m(k)}\}$ of $\{u_m\}$ such that

$$\lim_{k\to+\infty} \mu(u_{m(k)},\xi) = 0. \tag{3.7}$$

Since μ satisfies the triangle inequality, we have also

$$\lim_{k\to+\infty} \mu(u_{m(k)+1}, \xi) = 0. \tag{3.8}$$

By (3.2), taking into account that ρ is a lower semicontinuous function, we get

$$0 \le \rho(\xi) \le \liminf_{m\to+\infty} \rho(u_m) = 0,$$

that is, $\rho(\xi) = 0$. We assert that ξ is a fixed point of f. Clearly, ξ is a fixed point of f if there exists a subsequence u_{m_j} of u_m such that $u_{m_j} = \xi$ or $fu_{m_j} = f\xi$, for all $j \in \mathbb{N}$. If such a subsequence there is not, we can assume that $u_m \neq \xi$ and $fu_m \neq f\xi$ for all $m \in \mathbb{N}$. So, using (3.1) with $u = u_m$ and $v = \xi$ and the property (S_1) of the function S, we deduce that

$$\begin{aligned} 0 &\le S(H(\mu(fu_m, f\xi), \rho(fu_m), \rho(f\xi)), H(\mu(u_m, \xi), \rho(u_m), \rho(\xi))) \\ &< H(\mu(u_m, \xi), \rho(u_m), \rho(\xi)) - H(\mu(fu_m, f\xi), \rho(fu_m), \rho(f\xi)). \end{aligned}$$

This implies

$$H(\mu(fu_m, f\xi), \rho(fu_m), \rho(f\xi)) < H(\mu(u_m, \xi), \rho(u_m), \rho(\xi)) \quad \text{for all } n \in \mathbb{N}.$$

Next, using the semicontinuity of μ with respect to the first variable (see (2.3)) and further using (3.7) and (3.8), we deduce

$$\begin{aligned} \mu(\xi, f\xi) &\le \liminf_{k\to+\infty} \mu(u_{m(k)+1}, f\xi) = \liminf_{k\to+\infty} \mu(fu_{m(k)}, f\xi) \\ &\le \liminf_{k\to+\infty} H(\mu(fu_{m(k)}, f\xi), \rho(fu_{m(k)}), \rho(f\xi)) \\ &\le \liminf_{k\to+\infty} H(\mu(u_{m(k)}, \xi), \rho(u_{m(k)}), \rho(\xi)) = 0 \end{aligned}$$

(recall that H is continuous and (3.7) holds). Then $\mu(\xi, f\xi) = 0$ and hence $\xi = f\xi$, that is, ξ is a fixed point of f such that $\rho(\xi) = 0$. □

4. Application to cyclic mappings

Here, using the results obtained in Section 3, we deduce a fixed point theorem for cyclic mappings on metric spaces.

We remark that Kirk *et al.* introduced in [8] the following definition (see also [17]).

Definition 4.1. (See [8, 17]) Let (X, d) be a metric space, q be a positive integer and $f : X \to X$ be a mapping. $X = \cup_{i=1}^{q} A_i$ is said a cyclic representation of X with respect to f if

(i) A_i is a nonempty closed set for each $i = 1, 2, \ldots, q$;
(ii) $f(A_i) \subset A_{i+1}$ for each $i = 1, 2, \ldots, q$, where $A_{q+1} = A_1$.

Since then, fixed point theorems involving a cyclic representation of X with respect to a self-mapping f appeared in many articles (see, for example, [4, 10, 17] and their references). Motived by this, we give the following definition.

Definition 4.2. Let (X, d) be a metric space, $w : X \times X \to [0, +\infty[$ be a w_0-distance on X, q be a positive integer, $A_1, A_2, \ldots, A_q$ be nonempty subsets of X and $Y = \cup_{i=1}^{q} A_i$. A mapping $f : Y \to Y$ is said a cyclic w-contraction if

(i) $f(A_i) \subset A_{i+1}$ for each $i = 1, 2, \ldots, q$, where $A_{q+1} = A_1$;
(ii) there exist three functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $\rho : X \to [0, +\infty[$ such that

$$S\left(H(\mu(fu, fv), \rho(fu), \rho(fv)), H(\mu(u, v), \rho(u), \rho(v))\right) \geq 0 \tag{4.1}$$

for every $u \in A_i$, $v \in A_{i+1}$, $i = 1, 2, \ldots, q$.

Now, we are ready to prove the following theorem which is an extension of the Kirk *et al.*'s cyclic fixed point theorems (see [8], Theorems 1.3, 2.3 and 2.4).

Theorem 4.3. *Let (X, d) be a complete metric space, $w : X \times X \to [0, +\infty[$ be a w_0-distance on X, q be a positive integer, $A_1, \ldots, A_q$ be nonempty closed subsets of X, $Y = \cup_{i=1}^q A_i$ and $f : Y \to Y$ be a cyclic w-contraction. If ρ is a lower semicontinuous function, then f has a unique fixed point ξ in Y such that $\rho(\xi) = 0$.*

Proof. Firstly, we assume that $\cap_{i=1}^q A_i \neq \emptyset$. We notice that $f(\cap_{i=1}^q A_i) \subset \cap_{i=1}^q A_i$. Therefore, $f : \cap_{i=1}^q A_i \to \cap_{i=1}^q A_i$ satisfies all the hypotheses of Theorem 3.3, that is, $\cap_{i=1}^q A_i$ is a complete metric space (we recall that A_i is closed for each $i = 1, 2, \ldots, q$), f is a w-contraction on $\cap_{i=1}^q A_i$ and ρ is a semicontinuous function. Hence by Theorem 3.3, we conclude that f has a unique fixed point ξ in $\cap_{i=1}^q A_i$ (and so in Y) such that $\rho(\xi) = 0$.

We observe that the foregoing discussion ensures that in order to obtain the claim it is sufficient to show that $\cap_{i=1}^q A_i \neq \emptyset$. Let $u_1 \in A_1$ and let $\{u_m\}$ be a sequence of Picard starting at u_1. Since $Y = \cup_{i=1}^q A_i$ is a cyclic representation of Y with respect to f, we have that $u_{mq+i} \in A_i$ for all $i = 1, \ldots, q$ and $m \in \mathbb{N} \cup \{0\}$. We note that if $u_k = u_{k+1}$ for some $k \in \mathbb{N}$ then $u_m = u_k$ for all $m \geq k$. Thus, $u_k \in A_i$ for each $i = 1, \ldots, q$ and hence $\cap_{i=1}^q A_i \neq \emptyset$.

Therefore, we suppose that $u_m \neq u_{m+1}$ for every $m \in \mathbb{N}$. By Lemma 3.2, we have that the sequence $\{u_m\}$ is Cauchy. Hence, since (X, d) is complete, there exists some $\xi \in X$ such that

$$\lim_{m \to +\infty} u_m = \xi.$$

Since $u_{mq+i} \to \xi$ as $m \to +\infty$ and the set A_i is closed, for each $i = 1, \ldots, q$, we get that $\xi \in \cap_{i=1}^q A_i \neq \emptyset$. This prove the claim. □

Next, we give another result for cyclic w-contractions which improves one of Theorem 4.3. Precisely, following [4] we weaken the closure assumption on the sets A_i, that is, we only suppose that A_1 is closed. We stress that such a result does not follow from Theorem 3.3.

Theorem 4.4. *Let (X, d) be a complete metric space, $w : X \times X \to [0, +\infty[$ be a w_0-distance on X, q be a positive integer, $A_1, \ldots, A_q$ be nonempty subsets of X, $Y = \cup_{i=1}^q A_i$ and $f : Y \to Y$ be a cyclic w-contraction. If A_1 is closed and ρ is a lower semicontinuous function, then f has a unique fixed point ξ in Y such that $\rho(\xi) = 0$.*

Proof. We notice that in order to prove the uniqueness of the fixed point it is sufficient proceed as in the proof of Theorem 3.3 (we remark that the fixed point of f belongs to $\cap_{i=1}^q A_i$). Therefore, we only establish the existence of a fixed point. Let $u_1 \in A_1$ and let $\{u_m\}$ be a sequence of Picard starting at u_1. Since $f : Y \to Y$ is a cyclic w-contraction, we have that $u_{mq+1} \in A_1$ for all $m \in \mathbb{N} \cup \{0\}$. We stress that if $u_k = u_{k+1}$

for some $k \in \mathbb{N}$, then u_k is a fixed point of f and, further, $\rho(u_k) = 0$ (see the proof of Theorem 3.3). So, we suppose that $u_m \neq u_{m+1}$ for every $m \in \mathbb{N}$. By Lemma 3.2, we have that the sequence $\{u_m\}$ is Cauchy. Hence, since (X, d) is complete, there exists some $\xi \in X$ such that

$$\lim_{m \to +\infty} u_m = \xi.$$

Since $u_{mq+1} \to \xi$ as $m \to +\infty$ and the set A_1 is closed, we have that $\xi \in A_1$. We assert that ξ is a fixed point of f. Clearly, ξ is a fixed point of f if there exists a subsequence $\{u_{m(k)q+2}\}$ of $\{u_{mq+2}\}$ such that $u_{m(k)q+2} = \xi$ or $fu_{m(k)q+2} = f\xi$, for all $k \in \mathbb{N} \cup \{0\}$. If such a subsequence there is not, we can assume that $u_{mq+2} \neq \xi$ and $fu_{mq+2} \neq f\xi$ for all $m \in \mathbb{N} \cup \{0\}$.

Thanks to the proof of Theorem 3.3, we can affirm that there exists a subsequence $\{u_{m(k)q+2}\}$ of $\{u_{mq+2}\}$ such that $\mu(\xi, u_{m(k)q+2}) \to 0$ as $k \to +\infty$. Further, we can affirm that $\rho(u_{m(k)q+2}) \to 0$ as $k \to +\infty$ and, taking into account that ρ is semicontinuous, we have that $\rho(\xi) = 0$. Now, using (4.1) with $u = \xi \in A_1$ and $v = u_{m(k)q+2} \in A_2$ and the property (S_1) of the function S, we deduce that

$$\begin{aligned} 0 &\leq S(H(\mu(f\xi, fu_{m(k)q+2}), \rho(f\xi), \rho(fu_{m(k)q+2})), H(\mu(\xi, u_{m(k)q+2}), \rho(\xi), \rho(u_{m(k)q+2}))) \\ &< H(\mu(\xi, u_{m(k)q+2}), \rho(\xi), \rho(u_{m(k)q+2})) - H(\mu(f\xi, fu_{m(k)q+2}), \rho(f\xi), \rho(fu_{m(k)q+2})). \end{aligned}$$

This implies

$$H(\mu(f\xi, fu_{m(k)q+2}), \rho(f\xi), \rho(fu_{m(k)q+2})) < H(\mu(\xi, u_{m(k)q+2}), \rho(\xi), \rho(u_{m(k)q+2}))$$

for all $k \in \mathbb{N} \cup \{0\}$. Next, using the semicontinuity of μ with respect to the second variable (see (2.2)), we deduce

$$\begin{aligned} \mu(f\xi, \xi) &\leq \liminf_{k \to +\infty} \mu(f\xi, u_{m(k)q+3}) = \liminf_{k \to +\infty} \mu(f\xi, fu_{m(k)q+2}) \\ &\leq \liminf_{k \to +\infty} H(\mu(f\xi, fu_{m(k)q+2}), \rho(f\xi), \rho(fu_{m(k)q+2})) \\ &\leq \liminf_{k \to +\infty} H(\mu(\xi, u_{m(k)q+2}), \rho(\xi), \rho(u_{m(k)q+2})) = H(0,0,0) = 0 \end{aligned}$$

(we recall that H is continuous). Then $\mu(f\xi, \xi) = 0$ and hence $\xi = f\xi$, that is, ξ is a fixed point of f such that $\rho(\xi) = 0$. □

5. Application to contractions of integral type

In this section, we give some fixed point results for mappings which satisfy a contractive condition of integral type. First, we start with some remarks.

Remark 5.1. Let $\psi : [0, +\infty[\to [0, +\infty[$ with $\psi(\theta) < \theta$ for all $\theta > 0$ and $\psi(0) = 0$ be an upper semicontinuous function. Then the function $S : [0, +\infty[\times [0, +\infty[\to [0, +\infty[$ defined by $S(\delta, \theta) = \psi(\theta) - \delta$ belongs to $\mathcal{S}$.

In fact, (S_1) holds because $S(\delta, \theta) = \psi(\theta) - \delta < \theta - \delta$ for each $\delta, \theta > 0$. Regarding (S_2), we consider two sequences $\{\delta_m\}$, $\{\theta_m\}$ in $]0, +\infty[$ such that

$$\lim_{m \to +\infty} \delta_m = \lim_{m \to +\infty} \theta_m = \ell \in]0, +\infty[.$$

The upper semicontinuity of the function ψ ensures that

$$\limsup_{m \to +\infty} S(\delta_m, \theta_m) \leq \limsup_{m \to +\infty} \psi(\theta_m) - \ell \leq \psi(\ell) - \ell < \ell - \ell = 0.$$

Remark 5.2. Let $\psi : [0,+\infty[\to [0,+\infty[$ with $\psi(\theta) > \theta$ for all $\theta > 0$ and $\psi(0) = 0$ be a lower semicontinuous function. Then the function $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\delta,\theta) = \theta - \psi(\delta)$ belongs to $\mathcal{S}$.

We notice that from Remarks 5.1 and 5.2, we can deduce the following.

Remark 5.3. Let $\iota : [0,+\infty[\to [0,+\infty[$ be a function Lebesgue integrable in every interval $[0,\tau]$ with $\tau > 0$, then

(i) $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\delta,\theta) = \int_0^\theta \iota(u)du - \delta$, for all $\delta,\theta \in [0,+\infty[$, belongs to $\mathcal{S}$ if $\int_0^\tau \iota(u)du < \tau$ for all $\tau > 0$;

(ii) $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\delta,\theta) = \theta - \int_0^\delta \iota(u)du$, for all $\delta,\theta \in [0,+\infty[$, belongs to $\mathcal{S}$ if $\int_0^\tau \iota(u)du > \tau$ for all $\tau > 0$.

Now, using Theorem 3.3 and Remark 5.3 (i), we can give a new kind of contractive condition of integral type which ensures the existence and uniqueness of fixed point.

Theorem 5.4. *Let (X,d) be a complete metric space, $w : X\times X \to [0,+\infty[$ be a w_0-distance on X and let $f : X \to X$ be a mapping. Suppose that there exist a function $H \in \mathcal{H}$, a lower semicontinuous function $\rho : X \to [0,+\infty[$ and a function $\iota : [0,+\infty[\to [0,+\infty[$ Lebesgue integrable in every interval $[0,\tau]$, $\tau > 0$, such that*

$$H(\mu(fu,fv),\rho(fu),\rho(fv)) \le \int_0^{H(\mu(u,v),\rho(u),\rho(v))} \iota(t)dt \quad \text{for all } u,v \in X.$$

If $\int_0^\tau \iota(t)dt < \tau$ for all $\tau > 0$, then f has a unique fixed point ξ such that $\rho(\xi) = 0$.

In addition, using Theorem 3.3 and Remark 5.3 (ii), we can easily get the following.

Theorem 5.5. *Let (X,d) be a complete metric space, $w : X\times X \to [0,+\infty[$ be a w_0-distance on X and let $f : X \to X$ be a mapping. Suppose that there exist a function $H \in \mathcal{H}$, a lower semicontinuous function $\rho : X \to [0,+\infty[$ and a function $\iota : [0,+\infty[\to [0,+\infty[$ Lebesgue integrable in every interval $[0,\tau]$, $\tau > 0$, such that*

$$\int_0^{H(\mu(fu,fv),\rho(fu),\rho(fv))} \iota(t)dt \le H(\mu(u,v),\rho(u),\rho(v)) \quad \text{for all } u,v \in X.$$

If $\int_0^\tau \iota(t)dt > \tau$ for all $\tau > 0$, then f has a unique fixed point ξ such that $\rho(\xi) = 0$.

We conclude this section giving an example which motives our study.

Example 5.6. Let $X = [0,\frac{15}{8}]\cup\{2\}$. We consider X endowed with the usual metric $d(u,v) = |u-v|$ for all $u,v \in X$. Furthermore, we endow X with the w_0-distance $w : X\times X \to [0,+\infty[$ defined by $w(u,v) = v$ for all $u,v \in X$. Obviously, (X,d) is a complete metric space and $\mu : X\times X \to [0,+\infty[$ is given by $\mu(u,v) = \max\{u,v\}$ for all $u,v \in X$. Let $f : X \to X$ be the function defined by

$$fu = \begin{cases} \frac{u}{3} & \text{if } u \in [0,\frac{15}{8}], \\ \frac{3}{2} & \text{if } u = 2. \end{cases}$$

Clearly, f satisfies the contractive condition of Theorem 5.5 with respect to the function $H \in \mathcal{H}$ defined by $H(\delta,\theta,\lambda) = \delta+\theta+\lambda$ for all $\delta,\theta,\lambda \in [0,+\infty[$, the lower semicontinuous function $\rho : X \to [0,+\infty[$ defined by $\rho(u) = u$ for all $u \in X$ and the function $\iota : [0,+\infty[\to [0,+\infty[$ given by

$$\iota(t) = 1 + \frac{1}{(t+1)^2} \qquad \text{for all } t \in [0,+\infty[.$$

Let $u, v \in X$, we notice that

$$H(\mu(fu,fv),\rho(fu),\rho(fv)) = 2fv + fu \quad \text{ and } \quad H(\mu(u,v),\rho(u),\rho(v)) = 2v + u.$$

If $u \le v$ and $u, v \in [0, \frac{15}{8}]$, then $2fv + fu \le v$ and hence

$$\begin{aligned}\int_0^{H(\mu(fu,fv),\rho(fu),\rho(fv))} \iota(t)dt &\le \int_0^{v} \iota(t)dt \\ &= \frac{v+2}{v+1} v \le 2v \le 2v + u \\ &= H(\mu(u,v),\rho(u),\rho(v)).\end{aligned}$$

If $u \in [0, \frac{15}{8}]$ and $v = 2$, then $2fv + fu = \frac{9+u}{3}$ and hence

$$\begin{aligned}\int_0^{H(\mu(fu,fv),\rho(fu),\rho(fv))} \iota(t)dt &= \int_0^{\frac{9+u}{3}} \iota(t)dt = \frac{15+u}{12+u}\,\frac{9+u}{3} \\ &\le \frac{5}{4}\,\frac{9+u}{3} \le 4 + u \\ &= H(\mu(u,v),\rho(u),\rho(v)).\end{aligned}$$

If $u = v = 2$, then $2fv + fu = \frac{9}{2}$ and hence

$$\begin{aligned}\int_0^{H(\mu(fu,fv),\rho(fu),\rho(fv))} \iota(t)dt &= \int_0^{\frac{9}{2}} \iota(t)dt \\ &= \frac{13}{11}\,\frac{9}{2} \le 6 \\ &= H(\mu(u,v),\rho(u),\rho(v)).\end{aligned}$$

Since all the conditions of Theorem 5.5 are satisfied, we can affirm that f has a unique fixed point $\xi = 0 = \rho(\xi)$ in X.

We stress that if we choose the w_0-distance $w = d$ and $\rho(u) = 0$ for all $u \in X$, by $d(f0, f2) = 3/2$ and $d(0, 2) = 2$, we deduce that

$$\int_0^{d(f0,f2)} \iota(t)dt = \frac{21}{10} \ge 2 = d(0,2).$$

This implies that Theorem 27 of [15] cannot be used in order to affirm that f has a fixed point with respect to the contractive condition of Theorem 5.5 associated to the function ι. In addition, the previous consideration shows that the function ρ has a decisive role in enlarging the class of self mappings which are w-contractions.

6. Consequences

In this section, using Theorem 3.3 we show that the notion of w-contraction includes different kinds of contractive conditions in the existing literature.

Firstly, we give a result of Jleli *et al.* type (see [5], Theorem 2.1).

Corollary 6.1. *Let (X, d) be a complete metric space, $w : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a mapping. Suppose that there exist $\sigma \in]0, 1[$, a function $H \in \mathcal{H}$ and a lower semicontinuous function $\rho : X \to [0, +\infty[$ such that*

$$H(\mu(fu, fv), \rho(fu), \rho(fv)) \leq \sigma\, H(\mu(u, v), \rho(u), \rho(v)) \quad \text{for all } u, v \in X.$$

Then f has a unique fixed point $\xi \in X$ such that $\rho(\xi) = 0$.

Proof. The claim follows by Theorem 3.3 if we choose $S \in \mathcal{S}$ given by $S(\delta, \theta) = \sigma\,\theta - \delta$ for all $\delta, \theta \geq 0$. □

We notice that we obtain the Banach contraction principle if we take $w = d$, $H(\delta, \theta, \lambda) = \delta + \theta + \lambda$ for all $\delta, \theta, \lambda \in [0, +\infty[$ and $\rho(u) = 0$ for all $u \in X$.

Secondly, we give a result of Rhoades type (see [14]).

Corollary 6.2. *Let (X, d) be a complete metric space, $w : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a mapping. Suppose that there exist a function $H \in \mathcal{H}$ and two lower semicontinuous functions $\psi : [0, +\infty[\to [0, +\infty[$ with $\psi^{-1}(0) = \{0\}$ and $\rho : X \to [0, +\infty[$ such that*

$$H(\mu(fu, fv), \rho(fu), \rho(fv)) \leq H(\mu(u, v), \rho(u), \rho(v)) - \psi(H(\mu(u, v), \rho(u), \rho(v)))$$

for all $u, v \in X$. Then f has a unique fixed point $\xi \in X$ such that $\rho(\xi) = 0$.

Proof. The claim follows by Theorem 3.3 if we choose $S \in \mathcal{S}$ given by

$$S(\delta, \theta) = \theta - \psi(\theta) - \delta,$$

for all $\delta, \theta \geq 0$. □

In addition, we notice that also the following result (see [12]) is an immediate consequence of Theorem 3.3.

Corollary 6.3. *Let (X, d) be a complete metric space, $w : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a mapping. Suppose that there exist a function $H \in \mathcal{H}$, a function $\psi : [0, +\infty[\to [0, 1[$ with $\limsup_{t \to r^+} \psi(t) < 1$ for all $r > 0$ and a lower semicontinuous function $\rho : X \to [0, +\infty[$ such that*

$$H(\mu(fu, fv), \rho(fu), \rho(fv)) \leq \psi(H(\mu(u, v), \rho(u), \rho(v)))\, H(\mu(u, v), \rho(u), \rho(v))$$

for all $u, v \in X$. Then f has a unique fixed point $\xi \in X$ such that $\rho(\xi) = 0$.

Proof. We get the claim by using Theorem 3.3 and taking $S \in \mathcal{S}$ given by

$$S(\delta, \theta) = \theta\, \psi(\theta) - \delta,$$

for all $\delta, \theta \geq 0$. □

Lastly, we give a result of Boyd-Wong type (see [3]).

Corollary 6.4. *Let (X,d) be a complete metric space, $w : X \times X \to [0,+\infty[$ be a w_0-distance on X and let $f : X \to X$ be a mapping. Suppose that there exist a function $H \in \mathcal{H}$, an upper semicontinuous function $\psi : [0,+\infty[\to [0,+\infty[$ with $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ and a lower semicontinuous function $\rho : X \to [0,+\infty[$ such that*

$$H(\mu(fu,fv),\rho(fu),\rho(fv)) \le \psi(H(\mu(u,v),\rho(u),\rho(v))) \quad \text{for all } u,v \in X.$$

Then f has a unique fixed point $\xi \in X$ such that $\rho(\xi) = 0$.

Proof. Again, we can get the claim thanks to Theorem 3.3 by choosing $S \in \mathcal{S}$ given by $S(\delta,\theta) = \psi(\theta) - \delta$, for all $\delta,\theta \ge 0$. □

We point out that if we take $w = d$, $H(\delta,\theta,\lambda) = \delta + \theta + \lambda$ for all $\delta,\theta,\lambda \in [0,+\infty[$ and $\rho(u) = 0$ for all $u \in X$, then we have the Boyd-Wong result.

The following example shows which Theorem 3.3 is a proper generalization in the setting of metric spaces both Banach contraction principle and Boyd-Wong result.

Example 6.5. (See [17], Example 4) Let $X = [0,1]$ and we endow X with the usual metric $d(u,v) = |u-v|$ for all $u,v \in X$. Further, we consider on X the w_0-distance $w : X \times X \to [0,+\infty[$ given by $w(u,v) = v$ for all $u,v \in X$. Clearly, (X,d) is a complete metric space. Fix $\sigma \in [0,1[$ and consider the function $f : X \to X$ defined by

$$fu = \begin{cases} 0 & \text{if } u = 0, \\ \frac{\sigma}{2n} - \sigma\frac{2n-1}{2n}(2nu-1) & \text{if } \frac{1}{2n} \le u \le \frac{1}{2n-1}, \\ \frac{\sigma}{2n} + \sigma\frac{2n+1}{2n}(2nu-1) & \text{if } \frac{1}{2n+1} \le u \le \frac{1}{2n}. \end{cases}$$

We notice that if $\sigma > 3/5$ then f is not a nonexpansive function in the metric space (X,d). In fact, if for odd $n > 1$ we choose $u = \frac{1}{2n-1}$ and $v = \frac{1}{n-1}$, we have

$$d(fu,fv) = \frac{\sigma}{n-1} \quad \text{and} \quad d(u,v) = \frac{n}{(n-1)(2n-1)} \le \frac{3}{5(n-1)}.$$

Hence, if $\sigma > 3/5$ we get that $d(fu,fv) > d(u,v)$. So f is not a nonexpansive function. Therefore, both the Banach contraction principle and Boyd-Wong result cannot be used in order to affirm that f has a fixed point.

Now, if we consider the function $\rho : X \to [0,+\infty[$ defined by $\rho(u) = 0$ for all $u \in X$ and the function $H(\delta,\theta,\lambda) = \delta + \theta + \lambda$ for all $\delta,\theta,\lambda \in [0,+\infty[$, then we have

$$\begin{aligned} H(\mu(fu,fv),\rho(fu),\rho(fv)) &= \mu(fu,fv) \\ &= \max\{fu,fv\} \le \max\{\sigma u,\sigma v\} \\ &= \sigma \max\{u,v\} \\ &= \sigma H(\mu(u,v),\rho(u),\rho(v)), \end{aligned}$$

for all $u,v \in X$. Since all the conditions of Corollary 6.1 are satisfied we get that f has a unique fixed point in X.

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Received: February 28, 2018; Accepted: June 7, 2018.