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ON THE ARONSZAJN PROPERTY FOR FRACTIONAL NEUTRAL EVOLUTION EQUATIONS WITH INFINITE DELAY ON HALF-LINE

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Abstract. We establish an R_δ structure theorem for the fixed point set of the Krasnosel'skii type operator on Fréchet space. Applying this result, a topological structure for the set of all mild solutions of fractional neutral evolution equations with infinite delay on half-line is investigated. We show that the solution set is an R_δ-set.
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1. Introduction

It is well know that, in 1890, Peano proved that the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)), \quad 0 < t \le a, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, has local solutions although the uniqueness property did not hold in general. This observation became a motivation for studying the structure of solution set, Sol, for problem (1.1). Peano himself proved that, in the case of $n = 1$, the set

$$Sol(t) = \{x(t) : x \in Sol\}$$

is nonempty, compact and connected in the standard topology of the real line, for t in some neighborhood of 0. In 1923, Kneser generalized this result into the case of arbitrary n. Next, in 1928, Hukuhara proved that Sol is a continuum in the Banach space of continuous functions with sup norm.

In many cases, the solution sets for differential problems often correspond with the fixed point sets of operators in suitable function spaces. By establishing the theorem for R_δ structure of the fixed point set of compact operator, Aronszajn [3] proved that Sol is an R_δ-set. So Sol is acyclic. The analogous result was obtained for upper-Carathéodory inclusions by De Blasi and Myjak in [12]. For more details, historical remarks and related references, see [2].

In 1955, Krasnosel'skii [25] proved a fixed point theorem motivated by an observation that the inversion of a perturbed differential operator may yield the sum of compact and Banach contraction operators. His theorem actually combines both the Banach contraction principle and the Schauder fixed point theorem, and is useful in establishing existence theorems for perturbed operator equations. Since then there have appeared a large number of papers contributing generalizations or modifications of the Krasnosel'skii fixed point theorem and their applications. One of the main features of such generalizations is the adopting of generalized forms of the Banach principle or the Schauder theorem. One of the most impressive generalizations of the Krasnosel'skii theorem was given by Hoa and Schmitt [21] in 1994. However, until now there are not results for the topological structure of the fixed point set of Krasnosel'skii type operators on Fréchet spaces even with the Banach contraction case. In 1993, Kubacek [27] established a Fréchet version for Aronszajn theorem. He proved the R_δ property for the fixed point of operator satisfied either the Palais-Smale condition or compactness. Notice that the Palais-Smale condition is very interested in the variant principle.

We recall that (see [8]).

Definition 1.1. Let M be a metric space.

(i) M is called an absolute retract if each continuous map $f : B \to M$, where B is a closed subset of some topological space N, possesses a continuous extension over N.

(ii) M is called an R_δ-set if it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Note that any R_δ-set is a nonempty compact connected space. On the other hand, it is acyclic with respect to the Čech homology functor with rational coefficients, i.e. it has the same homology as the one point space. It may be not be a singleton but, from the point of view of algebraic topology, it's equivalent to a point (see [18]).

In this paper we establish the theorem on the R_δ property of fixed point set for Krasnosel'skii type of the form $\mathcal{B}+\mathcal{Q}$ on Fréchet space, where $\mathcal{B}$ is contraction and $\mathcal{Q}$ is Palais-Smale (see Theorem 3.3 and Theorem 3.4). This theorem is the improvement of the results on the Fréchet space in [27]. Then, by using this result, we not only obtain the existence result but also the R_δ property for mild solution set of the fractional neutral evolution equation with infinite delay of the form

$$\begin{cases} D^q\left[D^q x(t) - h(t, x(t), x_t)\right] = Ax(t) + f\left(t, x(t), x_t\right), \, t > 0, \\ x_0(t) = \varphi(t), \, t \in (-\infty, 0], \\ D^q x(0) = \xi. \end{cases} \tag{1.2}$$

where D^q is the Caputo fractional derivative of order $q \in (\frac{1}{2}, 1)$, the histories $x_t : (-\infty, 0] \to E$ defined by $x_t(s) = x(t+s)$ for all $s \in (-\infty, 0]$, and the functions $h, f : \mathbb{R}_+ \times E \times \mathcal{BM}_g \to E$ are given functions satisfying some conditions specified later and $\mathcal{BM}_g$ is a phase space defined in Section 3. Throughout this paper E denote a Banach space endowed with the norm $|\cdot|$. Suppose that $A : D(A) \subset E \to E$ is the infinitesimal generator of a cosine family $\{C(t)\}_{t\geq 0}$ on E and $\varphi : (-\infty, 0] \to E$ is a function belonging to $\mathcal{BM}_g$.

Notice that the problem

$$\begin{cases} D^q \left[x(t) - h(t, x(t), x_t)\right] = Ax(t) + f\left(t, x(t), x_t\right), \, 0 < t \leq a, \\ x_0(t) + g\left(x_{t_1}, x_{t_2}, ..., x_{t_n}\right)(t) = \varphi(t), \, t \in [-r, 0], \end{cases} \tag{1.3}$$

on a compact interval with finite delay has been studied by Zhou and Jiao [40] but φ only belongs to $C([-r, 0], E)$ which is the space of continuous functions from $[-r, 0]$ into E, A is the infinitesimal generator of a compact semigroup. They only proved an existence result by using the Krasnosell'skii fixed point theorem.

We notice that the existence result for the second-order neutral functional differential equations have been extensively studied in recent years by using various fixed point theorems when the corresponding cosine family $C(t)(t \geq 0)$ is compact. In such case, it follows that underlying space must be finite-dimensional, therefore severely weaken the applicability of the existence result. Futhermore, if the sine family $S(t)$ is compact then A is compact. These are restrictions in the application. Thus, there naturally arises an equation:

"Is there any chance to solve this problem without this compact condition on $C(t), S(t)$ and f".

In this paper, we will remove the compactness on $C(t), S(t)$ and f. By our new Aronszajn type theorem combined with the conditions involving with noncompactness measure, we proved the R_δ property for mild solution set of the equation (1.2).

We already known that there is also not any result for the existence for the mild solution of this fractional neutral evolution equation on Fréchet space with infinite delay even with the compactness of $C(t), S(t)$ or f. Our work can be considered as a contribution to this nascent fields.

For more results on fractional differential equation, we refer the interested reader to [1, 5, 6, 13, 14, 15, 16, 24, 32, 33, 36, 40] and on topological structure of solution sets we refer to [34, 9, 10, 11, 17, 18, 19, 26, 27, 29, 39].

The organization of this paper is as follows: In section 2, we introduce some preliminaries and assumptions. In section 3, we establish the theorems about topological structure for fixed point set of Krasnosel'skii type operator in Fréchet. Finally, the R_δ property for solution set of the equation (1.2) is proved in section 4.

2. Preliminaries and assumptions

For fractional calculus we recall that if $x : [0, +\infty) \subset \mathbb{R} \to E$ then

Definition 2.1. We have

(i) the fractional integral of order $\alpha > 0$ with the lower limit zero for x is defined as

$$I^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,\ t > 0,$$

and

(ii) the Caputo fractional derivative of order $\alpha > 0$ for the function x is defined by

$$D^{\alpha}x(t) := I^{n-\alpha}x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds.$$

We provide that the right-hand sides of above equalites are pointwise defined on $[0,\infty)$. We also note that the integrals in these definitions are taken in Bochner's sense. We consider a closed linear operator A densely defined in a Banach space E.

Definition 2.2. A bounded linear operator family $\{C(t)\}_{t\in\mathbb{R}}$ is called the cosine family of A if the following conditions are satisfied:

(i) $C(t)$ is strongly continuous for $t \geq 0$ and $C(0) = I$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

Definition 2.3. The associated sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)xds,$$

for $x \in E, t \in \mathbb{R}$.

Definition 2.4. The cosine family $\{C(t)\}_{t\in\mathbb{R}}$ is called uniformly bounded if there is constant $M \geq 1$ such that

$$|C(t)| \leq M, \forall t \in \mathbb{R}. \tag{2.4}$$

Remark 2.5. If the cosine family $\{C(t)\}_{t\in\mathbb{R}}$ is uniformly bounded then $S(t)$ is equicontinuous i.e. $\|S(t_1) - S(t_2)\| \to 0$ as $t_1 \to t_2$.

An operator A is said to belong to $\mathcal{C}$ if A is the infinitesimal generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ satisfying (2.4). Let $R(\lambda, A) = (\lambda I - A)^{-1}$. Then we have

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t}C(t)xdt,$$

$$R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t}S(t)xdt,$$

for all $\lambda > 0, x \in E$.

From the Proposition 2.1 and 2.2 in [35], we have.

Lemma 2.6. *Let $\{C(t) : t \in \mathbb{R}\}$ be a cosine family in E with infinitesimal generator A. Then the following assertions are true*

(i) $S(t+\tau) = S(t)C(\tau) + C(t)S(\tau)$, for all $t, \tau \in \mathbb{R}$,

(ii) $\frac{d}{dt}S(t)x = C(t)x$, $\frac{d}{dt}C(t)x = AS(t)x$, for all $x \in E, t \in \mathbb{R}$,

(iii) If $x \in E$ then $S(t)S(\tau)x \in D(A)$ and $AS(t)S(\tau)x = S(t)AS(\tau)x$ for all $t, \tau \in \mathbb{R}$.

We have the following corollary

Corollary 2.7. *Let $\{C(t) : t \in \mathbb{R}\}$ be a cosine family in E with infinitesimal generator A. We have*

$$C(t+s) = C(t)C(s) + S(t)AS(s), \forall t, s \in \mathbb{R}.$$

Now assume that F is a Fréchet space whose topology is given by the family of seminorms $\{p_n : n \in \mathbb{N}\}$. Let Ψ be a family of real functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which are nondecreasing, right continuous, and satisfy $\varphi(t) < t$ for $t > 0$. The following definitions are taken from [30].

A map $\mathcal{B} : F \to F$ is called

(i) a Banach contraction (or k_n-contraction) if there is $\{k_n\}_{n=1}^{\infty}$ be a sequence in $[0, 1)$ such that

$$p_n(\mathcal{B}(x) - \mathcal{B}(y)) \leq k_n p_n(x - y),$$

for all $x, y \in F, n \in \mathbb{N}$.

(ii) a Boyd-Wong contraction if, for each $n \in \mathbb{N}$, there exists $\varphi_n \in \Psi$ such that

$$p_n(\mathcal{B}(x) - \mathcal{B}(y)) \leq \varphi_n[p_n(x - y)],$$

for all $x, y \in F$.

(iii) a Meir-Keeler contraction if for each $n \in \mathbb{N}$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq p_n(x - y) < \varepsilon + \delta \Rightarrow p_n(\mathcal{B}(x) - \mathcal{B}(y)) < \varepsilon.$$

(iv) a Hoa-Schmitt contraction if for any $a \in F$ and $n \in \mathbb{N}$, there exists $k_a \in \mathbb{Z}_+$ with the following property: for any $\varepsilon > 0$, there exists $r \in \mathbb{N}$ and $\delta > 0$ (δ is independent with a) such that for $x, y \in F, \alpha_a^n(x, y) < \varepsilon + \delta$ implies

$$\alpha_a^n(\mathcal{B}_a^r(x) - \mathcal{B}_a^r(y)) < \varepsilon,$$

where $\mathcal{B}_a(x) := \mathcal{B}(x) + a$ and $\alpha_a^n(x, y) := \max\{p_n(\mathcal{B}_a^i(x) - \mathcal{B}_a^j(x)) : i, j = 0, 1, 2, ..., k_a\}$.

Remark 2.8. Note that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

The following lemma is an important result for Hoa-Schmitt contraction operator. By Theorem 1 in [21], we have

Lemma 2.9. *Let $\mathcal{B} : F \to F$ be such that*

(i) $\mathcal{B}$ is uniformly continuous, that is, for $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $p_n(x - y) < \delta$ implies $p_n(\mathcal{B}(x) - \mathcal{B}(y)) < \varepsilon$,

(ii) $\mathcal{B}$ is a Hoa-Schmitt contraction.

Then $(I - \mathcal{B})^{-1}$ is well defined and uniformly continuous on F.

The following is step 1 in the proof of Theorem 1 in [21].

Lemma 2.10. *Under the hypothesis of Lemma 2.9, for any $a \in F$, the operator $\mathcal{B}$ admits a unique fixed point being $(I - \mathcal{B})^{-1}$, and the iterated sequence $\{\mathcal{B}_a^n(x)\}_n$ converges to $(I - \mathcal{B})^{-1}(a)$, for all $x \in F$.*

Definition 2.11. The operator $T : F \to F$ is called by Palais-Smale if T is continuous and every sequence $\{x_n\}_n$ such that $\lim\limits_{n\to\infty} (I - T)(x_n) = 0$ contains a convergent subsequence.

The Hausdorff measure of noncompactness β is defined each bounded subsets Ω of Banach space E by

$$\beta(\Omega) = \inf \Big\{\varepsilon > 0 : \Omega \text{ can be covered by a finite number of balls of radius smaller than } \varepsilon\Big\}.$$

It is well known that β enjoys the following properties: For all bounded subsets $\Omega, \Omega_1, \Omega_2$ of the Banach space E. We have:

(1) $\beta(\Omega) = 0$ if only if Ω is relatively compact in E.
(2) $\beta(\Omega_1) \leq \beta(\Omega_2)$ when $\Omega_1 \subset \Omega_2$.
(3) $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$.
(4) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.
(5) $\beta(\Omega_1 \cup \Omega_2) \leq \max\{\beta(\Omega_1), \beta(\Omega_2)\}$.
(6) $\beta(\Omega) = \beta(\overline{\Omega}) = \beta(co\Omega) = \beta(\overline{co}\Omega)$, where $co(A)$ and $\overline{co}(A)$ are the convex hull and the closed convex hull of A respectively.
(7) $\beta(\lambda\Omega) = |\lambda|\beta(\Omega)$ for all $\lambda \in \mathbb{R}$.
(8) If the map $Q : D(Q) \subset E \to X$ is Lipschitz continuous with constant k, then $\beta_X(Q(\Omega)) \leq k\beta(\Omega)$ for all any bounded subset $\Omega \subset D(Q)$, where X is a Banach space and β_X is a noncompactness measure of Hausdorff in X.

Since no confusion may occur, we denote by $\beta(\cdot)$ the Hausdorff measure of noncompactness on both the bounded sets of E and $C([a, b], E)$ which is the space of all countinuous function $u : [a, b] \to E$. By [7] and [20], we have the following Lemmas

Lemma 2.12. *Let E be a Banach space and let $D \subset C([a, b], E)$ be bounded and equicontinuous. Then $\beta(D(t))$ is continuous on $[a, b]$, and $\beta(D) = \max\limits_{[a,b]} \beta(D(t))$.*

Lemma 2.13. *Let E be a Banach space, and let $D = \{u_n\} \subset L^1([a, b], E)$ and there exist $\eta \in L^1([a, b], \mathbb{R}_+)$ such that $\|u_n(t)\| \leq \eta(t)$ a.e. $t \in [a, b]$. Then $\beta(D(t))$ is the Lebesgue integral on $[a, b]$, and*

$$\beta\Big(\Big\{ \int\limits_{[a,b]} u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2 \int\limits_{[a,b]} \beta(D(t)).$$

3. Topological structure for fixed point set of Krasnosel'skii type operators

First, we introduce a characteristic of the R_δ set. By Lemma 11 in [27] we have

Lemma 3.1. *Let X be metric space and $\{A_n\}$ a sequence of compact absolute retracts in X. M is a non-empty subset of X such that*

(i) $M \subset A_n, \forall n \in \mathbb{N}$,

(ii) for each neighbourhood V of M in X there exists a $n_0 \in \mathbb{N}$ such that $A_n \subset V$ for each $n > n_0$,

Then M is R_δ.

By the above lemma, Aronszajn proved a result concerning the topological structure of the fixed point set of the compact operators on Banach space. Next, the excellent detection of Aronszajn is given. The following Lemma is the Corolarry of Theorem 1.2 in [27].

Lemma 3.2. *Let F be a Fréchet space whose topology is given by the family of increasing semi-norms $\{p_n : n \in \mathbb{N}\}, X \subset F$. Let $T : X \to F$ be Palais-Smale operator. Suppose there exists a sequence of continuous maps $T_n : X \to F$ such that*

(i) $p_n (T_n(x) - T(x)) \leq \frac{1}{n}$, for all $n \in \mathbb{N}$, $x \in X$,

(ii) $\forall n \in \mathbb{N}$, if $h \in X$ satisfying $p_n(h) \leq \frac{1}{n}$ then the equation $x = T_n (x) + h$ has an unique solution.

Then $Fix(T)$ is an R_δ set.

Next we shall prove two important results of this paper.

Theorem 3.3. *Let F be a Fréchet space whose topology is given by the family of increasing semi-norms $\{| \cdot |_n : n \in \mathbb{N}\}$ and $\mathcal{B}, \mathcal{Q} : F \to F$ be two operators. Assume that*

(i) $\mathcal{B}$ is Hoa-Schmitt contraction,

(ii) $\mathcal{Q}$ is continuous such that there is a sequence of continuous operators $\mathcal{Q}_n : F \to F$ $(n = 1, 2, ...)$ such that

$$\sup_{x\in F} |\mathcal{Q}_n (x) - \mathcal{Q} (x)|_n \leq \frac{1}{n},$$

(iii) $\mathcal{B} + \mathcal{Q}$ is Palais-Smale.

(iv) for every $h \in F$ with $|h|_n < \frac{1}{n}$ the equation $x = \mathcal{B}(x - h) + \mathcal{Q}_n(x) + h$ has an unique solution.

Then $Fix(\mathcal{B} + \mathcal{Q})$ is an R_δ set.

Proof. It follows from the assumption (i) that the operator $I - \mathcal{B}$ is invertible and its inverse is uniformly continuous on F. We define the continuous operator $T : F \to F$ by

$$T(x) = (I - \mathcal{B})^{-1} \mathcal{Q}(x), \text{ for } x \in F.$$

Let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} (I - T) (x_n) = 0$. By $I - \mathcal{B}$ is continuous, we have

$$\lim_{n\to\infty} (x_n - \mathcal{B}(x_n) - \mathcal{Q}(x_n)) = 0.$$

We proved that T is Palais-Smale by the Palais-Smale condition of $\mathcal{B} + \mathcal{Q}$.

Now we put $T_n = (I - \mathcal{B})^{-1} \mathcal{Q}_n$. Then T_n is continuous. On the other hand, since $(I - \mathcal{B})^{-1}$ is uniformly continuous on F, for each $n \in \mathbb{N}$, there exists $\delta_n \in \left(0, \frac{1}{2}\right)$ and $\delta_n \to 0$ such that for all $x, y \in F$ and $|x - y|_n < \delta_n$ we have

$$\left|(I - \mathcal{B})^{-1} (x) - (I - \mathcal{B})^{-1} (y)\right|_n < \frac{1}{n}. \tag{3.5}$$

Since the assumption (ii) there exists a subsequence of $\{\mathcal{Q}_n\}$ which still denoted by $\mathcal{Q}_n$ such that

$$\sup_{x\in F} |\mathcal{Q}_n(x) - \mathcal{Q}(x)|_n < \delta_n. \tag{3.6}$$

Combining (3.5) and (3.6) we deduce $|T_n(x) - T(x)|_n < \frac{1}{n}$, for all $x \in F$. Finally it is noted that, for $y \in F$ satisfying $|y|_n < \frac{1}{n}$, the equation $x = T_n(x) + y$ is equivalent to the equation

$$x = \mathcal{B}(x - y) + \mathcal{Q}_n(x) + y. \tag{3.7}$$

Therefore, by applying Lemma 3.2 the proof of Theorem is completely proved. □

Next, by using the above theorem, we will prove the following important theorem. This theorem gives a technique to prove the R_δ property for the fixed point set of Krasnosel'skii type operator in Fréchet space of all continuous functions on $\mathbb{R}_+$.

Theorem 3.4. *Let $X = C(\mathbb{R}_+; E)$ be the Fréchet space of all continuous functions on $\mathbb{R}_+$ taking values in a Banach space $(E; |\cdot|)$ and endowed with the family of increasing semi-norms $\{|\cdot|_n : n \in \mathbb{N}\}$, where $|x|_n = \sup_{t\in[0,n]} |x(t)|$. Let $\mathcal{B} : X \to X$ be Hoa-Schmitt contraction, $\mathcal{Q} : X \to X$ be a continuous operator such that $\mathcal{B} + \mathcal{Q}$ is Palais-Smale.*
Assume that there exist $t_0 \in [a, b]$, $e_0 \in E$ satisfying the following conditions

(i) $\mathcal{Q}(x)(t_0) = e_0$ *for all* $x \in X$,
(ii) $\mathcal{Q}(X)$ *is equicontinuous.*
(iii) *for any* $\varepsilon > 0$, *if* $x|_{I_\varepsilon} = y|_{I_\varepsilon}$ *then* $\mathcal{B}(x)|_{I_\varepsilon} = \mathcal{B}(y)|_{I_\varepsilon}$ *and* $\mathcal{Q}(x)|_{I_\varepsilon} = \mathcal{Q}(y)|_{I_\varepsilon}$, *where*

$$I_\varepsilon = \mathbb{R}_+ \cap [t_0 - \varepsilon, t_0 + \varepsilon],$$

Then $Fix(\mathcal{B} + \mathcal{Q})$ is an R_δ-set.

Proof. For each $n \in \mathbb{N}$, we define $b_n : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$b_n(t) = \begin{cases} t_0 & \text{if} \quad |t - t_0| \le 1/n, \\ t - \dfrac{1}{n|t - t_0|}(t - t_0) & \text{if} \quad |t - t_0| \ge 1/n, \end{cases}$$

and consider the operator $\mathcal{Q}_n : X \to X$ defined by

$$\mathcal{Q}_n(x)(t) = \mathcal{Q}(x)(b_n(t)),\ t \in [a, b].$$

First, we will prove that

$$\lim_{n\to\infty} \sup_{x\in X} |\mathcal{Q}_n(x) - \mathcal{Q}(x)|_m = 0.$$

It is not difficult to see that $\mathcal{Q}_n$ is continuous. On the other hand, thanks to the locally equicontinuous of $\mathcal{Q}(X)$ we deduce that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, m) > 0$ such that

$$|\mathcal{Q}(x)(t_1) - \mathcal{Q}(x)(t_2)| < \varepsilon, \tag{3.8}$$

for all $x \in X$ and for all $t_1, t_2 \in [0, m]$ satisfying $|t_1 - t_2| < \delta$. Now we choose $n_0 = n_0(\varepsilon, m) \in \mathbb{N}$ such that $\frac{1}{n_0} < \delta(\varepsilon, m)$. Since $|b_n(t) - t| < \frac{1}{n}$, for all $t \in [0, m]$ we deduce that

$$|\mathcal{Q}_n(x)(t) - \mathcal{Q}(x)(t)| = |\mathcal{Q}(x)(b_n(t)) - B(x)(t)| < \varepsilon$$

for all $x \in X$, $t \in [0, m]$ and for all $n \geq n_0$. This implies

$$\sup_{x \in X} |\mathcal{Q}_n(x) - \mathcal{Q}(x)|_m \leq \varepsilon, \, \forall n \geq n_0.$$

Next, it's necessary to note that, for $y \in X$ such that $|y|_n < \frac{1}{n}$, the perturbed equation $x = \mathcal{B}(x - y) + \mathcal{Q}_n(x) + y$ is equivalent to the equation $x = \mathcal{T}_n(x) + y$, where

$$\mathcal{T}_n = (I - \mathcal{B})^{-1} \mathcal{Q}_n, \, n \in \mathbb{N}.$$

We set $E_i = \left\{t \in \mathbb{R}_+ : \frac{i-1}{n} \leq |t - t_0| \leq \frac{i}{n}\right\}$.
Let $x \in X$ and $t \in E_1$ arbitrary. It follows from assumption (i) that

$$\mathcal{Q}_n(x)(t) = \mathcal{Q}(x)\left(b_n(t)\right) = \mathcal{Q}(x)(t_0) = e_0.$$

And by using (ii), this implies

$$\mathcal{B}_{\mathcal{Q}_n(x)}(0)(t) = \mathcal{B}(0)(t) + \mathcal{Q}_n(x)(t) = \mathcal{B}(0)(t) + e_0 := e(t).$$

Then we have

$$\mathcal{B}^2_{\mathcal{Q}_n(x)}(0)(t) = \mathcal{B}\left(\mathcal{B}_{\mathcal{Q}_n(x)}(0)\right)(t) + \mathcal{Q}_n(x)(t) = \mathcal{B}(e)(t) + e_0 = \mathcal{B}_{e_0}(e)(t),$$

where $e_0(t) := e_0$ for all $t \in \mathbb{R}_+$. Similarly, it is easy to obtain the equality

$$\mathcal{B}^{m+1}_{\mathcal{Q}_n(x)}(0)(t) = \mathcal{B}^m_{e_0}(e)(t).$$

So by passing to the limit as $m \to \infty$

$$(I - \mathcal{B})^{-1} \mathcal{Q}_n(x)(t) = (I - \mathcal{B})^{-1} (e_0)(t), \tag{3.9}$$

for all $x \in X$ and $t \in E_1$. Now let $k \in \{2, 3, 4, ...\}$ and let $x, z \in X$ satisfying the condition

$$x(t) = z(t), \; \forall t \in E_{k-1}.$$

We shall prove that, for all $m \in \mathbb{N}$,

$$\mathcal{B}^m_{\mathcal{Q}_n(x)}(e_0)\Big|_{E_k} = \mathcal{B}^m_{\mathcal{Q}_n(z)}(e_0)\Big|_{E_k}.$$

Indeed, we first note that if $t \in E_k$ then $b_n(t) \in E_{k-1}$ by $|b_n(t) - t_0| = |t - t_0| - 1/n$. Hence,

$$\mathcal{Q}_n(x)(t) = \mathcal{Q}(x)\left(b_n(t)\right) = \mathcal{Q}(z)(b_n(t)) = \mathcal{Q}_n(z)(t)$$

for all $t \in E_k$. This implies that

$$\mathcal{B}_{\mathcal{Q}_n(x)}(e_0)(t) = \mathcal{B}_{\mathcal{Q}_n(z)}(e_0)(t), \; \forall t \in E_k.$$

A simple inductive argument implies that

$$\mathcal{B}^m_{\mathcal{Q}_n(x)}(e_0)(t) = \mathcal{B}^m_{\mathcal{Q}_n(z)}(e_0)(t), \; \forall t \in E_k.$$

By passing to the limit as $m \to \infty$ we obtain

$$(I - \mathcal{B})^{-1} \mathcal{Q}_n(x)(t) = (I - \mathcal{B})^{-1} \mathcal{Q}_n(z)(t),$$

for all $t \in E_k$. So we have proved that

$$x|_{E_{k-1}} = z|_{E_{k-1}} \Longrightarrow \left.(\mathcal{T}_n x)\right|_{E_k} = \left.(\mathcal{T}_n z)\right|_{E_k}. \tag{3.10}$$

Next, we will prove that $I-\mathcal{T}_n$ is injective for all $n\in\mathbb{N}$. In fact, assume that for some $x,z\in X$ we have $x-z=\mathcal{T}_n(x)-\mathcal{T}_n(z)$. This implies that

$$x-z=(I-\mathcal{B})^{-1}\,\mathcal{Q}_n(x)-(I-\mathcal{B})^{-1}\,\mathcal{Q}_n(z).$$

By combining (3.9) and (3.10) we deduce that $x=z$.
Choose $y\in X$ and look for an $x\in X$ such that $x-\mathcal{T}_n(x)=y$.
As E_1 is a bounded set and $y\in X$, the set $\{y(t)+(I-\mathcal{B})^{-1}(e_0)(t):t\in E_1\}$ is bounded. E_1 is a closed subset of $\mathbb{R}_+$, so by the Dugundji extension theorem there exists a bounded continuous map $x_1:\mathbb{R}_+\to X$ such that

$$x|_{E_1}=y|_{E_1}+(I-\mathcal{B})^{-1}(e_0)|_{E_1}.$$

By (3.9), E_1 and E_2 are closed subsets of $\mathbb{R}_+$, the map x_1 is continuous on E_1, the map $y+\mathcal{T}_n(x_1)$ is continuous on E_2 and for $t\in E_1\cap E_2$(i.e. $|t-t_0|=\frac{1}{n}$) we have

$$y(t)+\mathcal{T}_n(x_1)(t)=y(t)+(I-\mathcal{B})^{-1}(e_0)(t),$$

so the map $\overline{x_2}$ defined by

$$\overline{x_2}=\begin{cases}x_1\left(t\right),t\in E_1\\ y\left(t\right)+\mathcal{T}_n\left(x_1\right)\left(t\right),t\in E_2\end{cases}$$

is continuous on $E_1\cup E_2$ and its range is bounded. Similarly, due to the Dugundji extension theorem there exists a bounded continuous map $x_2:\mathbb{R}_+\to X$ such that $x_2|_{E_1\cup E_2}=\overline{x_2}$. Proceeding by induction we can construct a sequence $\{x_m\}_{m=0}^{\infty}$ of bounded continuous maps such that $x_0=(I-\mathcal{B})^{-1}(e_0)$ and

$$x_{m+1}|_{E_m}=x_m|_{E_m} \tag{3.11}$$

$$x_m|_{E_m}=y|_{E_m}+(\mathcal{T}_n x_{m-1})|_{E_m} \tag{3.12}$$

Due to (3.11), there exists an $x\in X$, $x=\lim\limits_{m\to\infty}x_m$, and for $t\in E_m$ we have

$$x_m\left(t\right)=x_{m+1}\left(t\right)=\cdots=x\left(t\right),$$

so by (3.12) and (3.10), for $t\in E_m, m>1$, we have

$$x\left(t\right)=x_m\left(t\right)=y\left(t\right)+\mathcal{T}_n\left(x_{m-1}\right)\left(t\right)=y\left(t\right)+\mathcal{T}_n x\left(t\right),$$

i.e.

$$x-\mathcal{T}_n(x)=y,$$

the validity of the last equality for $t\in E_1$ being a consequence of the definition of the map x_1. The proof is completed. □

4. Structure of the solution set on half-line

4.1. **Notations.** We introduction the basic definitions

• *The functional spaces:* Now we define the abstract phase space $\mathcal{BM}_g$, which has been used in [38]. Assume that $g:(\,-\infty,0]\to(0,+\infty)$ is a continuous function with

$m = \int\limits_{-\infty}^{0} g\,(t)\,dt < +\infty$. For any $a > 0$ we define $\mathcal{BM}(a) = \{\psi : [\,-a, 0] \to X$ such that $\psi(t)$ is bounded and measurable $\}$, and equip the space $\mathcal{BM}(a)$ with the norm

$$\|\psi\|_{[\,-a,0]} = \sup_{s\in[\,-a,0]} |\psi\,(s)|\,.$$

Let us define $\mathcal{BM}_g = \Big\{\psi : (\,-\infty, 0] \to E$: such that for any $c > 0$, $\psi|_{[\,-c,0]} \in \mathcal{BM}(c)$ and $\int\limits_{-\infty}^{0} g(s)\|\psi\|_{[s,0]}ds < +\infty\Big\}$. If $\mathcal{BM}_g$ is endowed with the norm

$$\|\psi\|_{\mathcal{BM}_g} = \int\limits_{-\infty}^{0} g\,(s)\,\|\psi\|_{[s,0]}ds,$$

then it is clear that $(\mathcal{BM}_g, \|\cdot\|_{\mathcal{BM}_g})$ is a Banach space.

Now we consider the space

$$\mathcal{AM}_g = \Big\{x : \mathbb{R} \to E \text{ :such that}\, x|_{[0,+\infty)} \in C\,([0, +\infty)\,, E)\,, x_0 \in \mathcal{BM}_g\Big\}.$$

Set $\{\|\cdot\|_n\}_n$ be a seminorm family in $\mathcal{AM}_g$ defined by

$$\|x\|_n = \|x_0\|_{\mathcal{BM}_g} + \sup\{|x\,(s)| : s \in [0, n]\}\,.$$

Proving similar to [38], we have

Lemma 4.1. *Assume that $x \in \mathcal{AM}_g$, then for $t \in \mathbb{R}_+, x_t \in \mathcal{BM}_g$. Moreover*

$$m|x(t)| \le \|x_t\|_{\mathcal{BM}_g} \le \|x_0\|_{\mathcal{BM}_g} + m \sup_{s\in[0,t]} |x\,(s)|\,,$$

and

$$\|x_\theta - y_\theta\|_{\mathcal{BM}_g} \le 2\|x - y\|_n$$

for all $x, y \in \mathcal{AM}_g$ and $\theta \in [0, n]$.

Note that $F = \Big\{y \in \mathcal{AM}_g : y_0 = 0 \in \mathcal{BM}_h\Big\}$ is a Fréchet space endowed with the seminorm $\{p_n\}$ defined by

$$p_n(y) = \sup_{s\in[0,n]} |y\,(s)|\,.$$

- *Other notations:* We define
  - ⋄ For each $\varphi \in \mathcal{BM}_g$, we put

$$\widehat{\varphi}\,(t) = \begin{cases} \varphi\,(t)\,, t \in (-\infty, 0]\,, \\ C_q(t)\varphi(0) + S_q(t)[\xi - h(0, \varphi(0), \varphi)], t \in \mathbb{R}_+, \end{cases}$$

    then $\widehat{\varphi} \in \mathcal{AM}_g$.
  - ⋄ For $t \in \mathbb{R}_+$ and $x \in F$ we shall denote by Φ the map

$$(t, x) \mapsto \Phi(t, x) = (t, x(t) + \widehat{\varphi}(t), x_t + \widehat{\varphi}_t).$$

4.2. **Some preliminaries and hypothesis.** Before giving the definition of mild solution of (1.2), we first prove the following lemma.

Lemma 4.2. *If (1.2) holds, then we have*

$$\begin{aligned}x(t) &= \int_0^\infty \phi_q(\theta)C(t^q\theta)\varphi(0)d\theta + \int_0^\infty \phi_q(\theta)S(t^q\theta)[\xi - h(0,\varphi(0),\varphi)]d\theta \\ &\quad + q\int_0^t\int_0^\infty \theta(t-s)^{q-1}\phi_q(\theta)C[(t-s)^q\theta]h(s,x(s),x_s)d\theta ds \\ &\quad + q\int_0^t\int_0^\infty \theta(t-s)^{q-1}\phi_q(\theta)S[(t-s)^q\theta]f(s,x(s),x_s)d\theta ds.\end{aligned}$$

where ϕ_q is a probability density function defined on $(0,\infty)$, that is $\phi_q(\theta) \geq 0$ and

$$\int_0^\infty \phi_q(\theta)d\theta = 1.$$

Proof. Put $\rho(t) = h(t,x(t),x_t), g(t) = f(t,x(t),x_t)$. Observe that

$$x(t) = \varphi(0) + [\xi - \rho(0)]\int_0^t \frac{(t-s)^{q-1}}{\Gamma q}ds + I^q\rho(t) + I^{2q}[Ax(t) + g(t)].$$

Let $\lambda > 0$. Applying Laplace transforms

$$\widehat{x}(\lambda) = \int_0^\infty e^{-\lambda s}x\,(s)\,ds, \widehat{\rho}(\lambda) = \int_0^\infty e^{-\lambda s}\rho(s)ds,$$

and

$$\widehat{g}\,(\lambda) = \int_0^\infty e^{-\lambda s}g(s)ds$$

we have

$$\widehat{x}(\lambda) = \frac{1}{\lambda}\varphi(0) + \frac{1}{\lambda^{q+1}}[\xi - \rho(0)] + \frac{1}{\lambda^q}\widehat{\rho}(\lambda) + \frac{1}{\lambda^{2q}}[A\widehat{x}(\lambda) + \widehat{g}(\lambda)].$$

This implies that

$$\begin{aligned}\widehat{x}(\lambda) &= \lambda^{2q-1}R(\lambda^{2q},A)\varphi(0) + \lambda^{q-1}R(\lambda^{2q},A)[\xi - \rho(0)] + \lambda^q R(\lambda^{2q},A)\widehat{\rho}(\lambda) \\ &\quad + R(\lambda^{2q},A)\widehat{g}(\lambda) = I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Note that

$$I_1 = \lambda^{2q-1} R\left(\lambda^{2q}, A\right) \varphi(0) = \lambda^{q-1} \int_0^t e^{-\lambda^q t} C(t) \varphi(0)\, dt$$

$$= \int_0^\infty q(\lambda t)^{q-1} e^{-(\lambda t)^q} C(t^q) \varphi(0)\, dt = \int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}\left[e^{-(\lambda t)^q}\right] C(t^q) \varphi(0)\, dt.$$

Furthermore, we have

$$e^{-r^q} = \int_0^\infty e^{-r\theta} \psi_q(\theta) d\theta,$$

where

$$\psi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \theta \in (0, \infty)$$

for all $q \in (0,1)$. Thus we have

$$I_1 = \int_0^\infty \int_0^\infty \theta \psi_q(\theta) e^{\lambda t\theta} C(t^q) \varphi(0)\, d\theta dt = \int_0^\infty e^{\lambda t} \int_0^\infty \psi_q(\theta) C\left(\frac{t^q}{\theta^q}\right) \varphi(0)\, d\theta dt$$

We also have

$$I_2 = \lambda^{q-1} R\left(\lambda^{2q}, A\right) [\xi - \rho(0)] = \lambda^{q-1} \int_0^\infty e^{-\lambda^q t} S(t) [\xi - \rho(0)]\, dt$$

$$= \lambda^{q-1} \int_0^\infty q t^{q-1} e^{-(\lambda t)^q} S(t^q) [\xi - \rho(0)]\, dt$$

$$= \lambda^{-1} \int_0^\infty -\frac{d}{dt}\left[e^{-(\lambda t)^q}\right] S(t^q) [\xi - \rho(0)]\, dt.$$

Thus, we obtain

$$I_2 = \int_0^\infty \int_0^\infty \theta \psi_q(\theta) e^{-\lambda t\theta} S(t^q) [\xi - \rho(0)]\, d\theta dt$$

$$= \int_0^\infty e^{-\lambda t} \int_0^\infty \psi_q(\theta) S\left(\frac{t^q}{\theta^q}\right) [\xi - \rho(0)]\, d\theta dt.$$

Next, we get

$$I_3 + I_4 = \lambda^q R\left(\lambda^{2q}, A\right) \widehat{\rho}(\lambda) + R\left(\lambda^{2q}, A\right) \widehat{g}(\lambda)$$

$$= \int_0^\infty e^{-\lambda^q t} C(t) \widehat{\rho}(\lambda)\, dt + \int_0^\infty e^{-\lambda^q t} S(t) \widehat{g}(\lambda)\, dt.$$

Moreover,

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^q t} C(t)\widehat{\rho}(\lambda)\,dt \\
&= \int_0^\infty \int_0^\infty q t^{q-1} e^{-(\lambda t)^q} C(t^q)\, e^{-\lambda s} h(s, x(s), x_s)\, ds dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty q\psi_q(\theta)\, e^{-\lambda(t+s)} C\left(\frac{t^q}{\theta^q}\right) \frac{t^{q-1}}{\theta^q} h(s, x(s), x_s)\, d\theta ds dt \\
&= \int_0^\infty e^{-\lambda t} \left[q \int_0^t \int_0^\infty \psi_q(\theta)\, C\left[\frac{(t-s)^q}{\theta^q}\right] h(s, x(s), x_s)\, \frac{(t-s)^{q-1}}{\theta^q} d\theta ds \right] dt.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^q t} S(t)\widehat{\rho}(\lambda)\,dt \\
&= \int_0^\infty e^{-\lambda t} \left[q \int_0^t \int_0^\infty \psi_q(\theta)\, S\left[\frac{(t-s)^q}{\theta^q}\right] f(s, x(s), x_s)\, \frac{(t-s)^{q-1}}{\theta^q} d\theta ds \right] dt.
\end{aligned}$$

Now we can invert the last Laplace transform to get

$$\begin{aligned}
x(t) &= \int_0^\infty \psi_q(\theta) C\left(\frac{t^q}{\theta^q}\right) \varphi(0)\, d\theta + \int_0^\infty \psi_q(\theta) S\left(\frac{t^q}{\theta^q}\right) \varphi(0)\, d\theta \\
&+ q \int_0^t \int_0^\infty \psi_q(\theta)\, C\left[\frac{(t-s)^q}{\theta^q}\right] h(s, x(s), x_s)\, \frac{(t-s)^q}{\theta^q} d\theta ds \\
&+ q \int_0^t \int_0^\infty \psi_q(\theta)\, S\left[\frac{(t-s)^q}{\theta^q}\right] f(s, x(s), x_s)\, \frac{(t-s)^q}{\theta^q} d\theta ds \\
&= \int_0^\infty \phi_q(\theta) C(t^q\theta)\, \varphi(0)\, d\theta + \int_0^\infty \phi_q(\theta) S(t^q\theta)\, [\xi - \rho(0)]\, d\theta \\
&+ q \int_0^t \int_0^\infty \theta (t-s)^{q-1} \phi_q(\theta)\, C\, [(t-s)^q\theta] h(s, x(s), x_s)\, d\theta ds \\
&+ q \int_0^t \int_0^\infty \theta (t-s)^{q-1} \phi_q(\theta)\, S\, [(t-s)^q\theta] f(s, x(s), x_s)\, d\theta ds,
\end{aligned}$$

where $\phi_q(\theta) = \frac{1}{q}\theta^{-1-1/q}\psi_q\left(\theta^{-1/q}\right)$ is the probability density function defined on $(0,\infty)$. This completes the proof. $\square$

For any $x \in E$. Define operators $\{S_q(t)\}_{t\in\mathbb{R}}, \{C_q(t)\}_{t\in\mathbb{R}}, \{\widehat{S}_q(t)\}_{t\in\mathbb{R}}$, and $\{\widehat{C}_q(t)\}_{t\in\mathbb{R}}$ by

$$S_q(t)x = \int_0^\infty \phi_q(\theta) S(t^q\theta)x d\theta,$$

$$\widehat{S}_q(t)x = q\int_0^\infty \theta\phi_q(\theta) S(t^q\theta)x d\theta,$$

$$C_q(t)x = \int_0^\infty \phi_q(\theta) C(t^q\theta)x d\theta,$$

$$\widehat{C}_q(t)x = q\int_0^\infty \theta\phi_q(\theta) C(t^q\theta)x d\theta.$$

Due to Lemma 4.2, we give the following definition of the mild solution of (1.2).

Definition 4.3. By the mild solution of problem (1.2) we mean that the function $x \in \mathcal{AM}_h$ which satisfies

$$\begin{cases} x(t) &= C_q(t)\varphi(0) + S_q(t)[\xi - h(0,\varphi(0),\varphi)] \\ & \quad + \int_0^t (t-s)^{q-1}\widehat{C}_q(t-s)h(s,x(s),x_s)ds \\ & \quad + \int_0^t (t-s)^{q-1}\widehat{S}_q(t-s)f(s,x(s),x_s)ds, \quad t>0, \\ x_0 &= \varphi \end{cases} \tag{4.13}$$

Let $u = x + \widehat{\varphi}$. It is easy to see that u satisfies (4.13) if only if $x \in F$ and

$$x(t) = \int_0^t (t-s)^{q-1}\widehat{C}_q(t-s)h(s,x(s)+\widehat{\varphi}(s),x_s+\widehat{\varphi}_s)ds$$
$$+ \int_0^t t-s)^{q-1}\widehat{S}_q(t-s)f(s,x(s+\widehat{\varphi}(s),x_s+\widehat{\varphi}_s)ds.$$

Moreover, $x \mapsto u$ is an isometry.

Remark 4.4. Throughout this section the norm on a product space $M_1\times M_2\times\cdots\times M_k$ is always denoted by $\|\cdot\|$ and is defined by

$$\|(m_1,m_2,...,m_k)\| = \sum_{i=1}^k \|m_i\|_i,$$

where $\|\cdot\|_i$ is the norm on M_i.

Next we have the following lemma. It is proof is straightforward and we will omit them.

Lemma 4.5. *The map Φ is continuous from $\mathbb{R}_+ \times F$ into $\mathbb{R}_+ \times E \times \mathcal{BM}_g$. Moreover, we have*

$$\|\Phi(t,x)\| \le t + (m+1)p_n(x) + \mathbf{Q},$$

for $(t,x) \in [0,n] \times F$ and for all $n \in \mathbb{N}$, where $\mathbf{Q} = (m+1)M|\psi(0)| + \|\psi\|_{\mathcal{BM}_g}$.

We prove the following lemmas relative to operators $\{S_q(t)\}_{t\in\mathbb{R}}, \{C_q(t)\}_{t\in\mathbb{R}}$, $\{\widehat{S}_q(t)\}_{t\in\mathbb{R}}$, and $\{\widehat{C}_q(t)\}_{t\in\mathbb{R}}$ before we proceed further.

Lemma 4.6. *For any fixed $t \ge 0$, $S_q(t), C_q(t), \widehat{S}_q(t)$ and $\widehat{C}_q(t)$ are linear and bounded operators. Moverover, we have*

$$|S_q(t)x| \le \frac{Mt^q}{\Gamma(1+q)}|x|, |\widehat{S}_q(t)x| \le \frac{qMt^q\Gamma(3)}{\Gamma(1+2q)}|x|,$$

and

$$|C_q(t)x| \le M|x|\int_0^\infty \phi_q(\theta)d\theta = M|x|, |\widehat{C}_q(t)x| \le \frac{qM}{\Gamma(1+q)}|x|.$$

Proof. For any fixed $t \ge 0$, since $C(t), S(t)$ are linear operators, it is easy to see that $S_q(t), C_q(t), \widehat{S}_q(t)$ and $\widehat{C}_q(t)$ are also linear operators. For $\zeta \in [0,1]$, according to [28], direct calculation gives that

$$\int_0^\infty \frac{1}{\theta^\zeta}\psi_q(\theta)d\theta = \frac{\Gamma\left(1+\frac{\zeta}{q}\right)}{\Gamma(1+\zeta)}.$$

Then we have

$$\int_0^\infty \theta^\zeta\phi_q(\theta)d\theta = \int_0^\infty \frac{1}{\theta^{q\zeta}}\psi_q(\theta)\,d\theta = \frac{\Gamma(1+\zeta)}{\Gamma(1+q\zeta)}.$$

In the case $\eta = 1$, we have

$$\int_0^\infty \theta\phi_q(\theta)d\theta = \int_0^\infty \frac{1}{\theta^q}\psi_q(\theta)\,d\theta = \frac{1}{\Gamma(1+q)}.$$

For any $x \in E$, we have

$$|S_q(t)x| = \Big|\int_0^\infty \phi_q(\theta)S(t^q\theta)\,xd\theta\Big| \le t^qM|x|\int_0^\infty \theta\phi_q(\theta)d\theta = \frac{Mt^q}{\Gamma(1+q)}|x|,$$

$$|\widehat{S}_q(t)x| = \Big|q\theta\int_0^\infty \phi_q(\theta)S(t^q\theta)\,xd\theta\Big| \le qt^qM|x|\int_0^\infty \theta^2\phi_q(\theta)d\theta = \frac{qMt^q\Gamma(3)}{\Gamma(1+2q)}|x|.$$

Moreover, we have

$$|C_q(t)x| = \Big|\int_0^\infty \phi_q(\theta) C(t^q\theta)\, x d\theta\Big| \le M|x| \int_0^\infty \phi_q(\theta) d\theta = M|x|,$$

$$|\widehat{C}_q(t)x| = \Big|q\theta \int_0^\infty \phi_q(\theta) C(t^q\theta)\, x d\theta\Big| \le qM|x| \int_0^\infty \theta\phi_q(\theta) d\theta = \frac{qM}{\Gamma(1+q)}|x|. \qquad \square$$

Lemma 4.7. *Operators $S_q(t), C_q(t), \widehat{S}_q(t)$ and $\widehat{C}_q(t)$ are strongly continuous, which means that for $\forall x \in E$ and $0 \le t' < t''$, we have*

$$|S_q(t'')x - S_q(t')x| \to 0, |\widehat{S}_q(t'')x - \widehat{S}_q(t')x| \to 0,$$

and

$$|C_q(t'')x - C_q(t')x| \to 0, |\widehat{C}_q(t'')x - \widehat{C}_q(t')x| \to 0,$$

as $t' \to t''$.

Proof. $\forall x \in E$ and $0 \le t' < t''$, we get that

$$\left|\widehat{C}_q(t)x - \widehat{C}_q(t')x\right| \le q \int_0^\infty \theta\phi_q(\theta) \left|\left\{C\left((t'')^q\theta\right) - C\left((t')^q\theta\right)\right\} x\right| d\theta.$$

According to the strong continuity of $C(t)$, we have $\widehat{C}_q(t)$ is strongly continuous. Using a similar method, we complete the proof. $\square$

In order to study the topological structure of mild solution set for problem (1.2) we make the following assumptions

(H1) $A \in \mathcal{C}$ generates a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ such that $0 \in \rho(A)$ and $\{C(t)\}_{t\in\mathbb{R}}$ is equicontinuous.

(H2) The function $f : \mathbb{R}_+ \times E \times \mathcal{BM}_g \to E$ satisfies the following conditions:
  - (i) the map $t \mapsto f(t,x,y)$ is measurable, for all $(x,y) \in E \times \mathcal{BM}_g$,
  - (ii) the map $(x,y) \mapsto f(t,x,y)$ is continuous for a.e. $t \in \mathbb{R}_+$,
  - (iii) there exists a constant $q_1 \in [0,q)$ such that for each $C > 0$, there is a nonnegative function $r \in L^{1/q_1}(\mathbb{R}_+)$ such that $|f(t,x,y)| \le r(t)$ for a.e. $t \in \mathbb{R}_+$ and for all $(x,y) \in E \times \mathcal{BM}_g$,
  - (iv) for every bounded subsets $D \subset \mathcal{BM}_g, K \subset E$, there exists a position function $k_n \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\beta(f(t,K,D)) \le k(t)\Big[\sup_{\theta\in(-\infty,0]} \beta(D(\theta)) + \beta(K)\Big],$$

  for a.e. $t \in \mathbb{R}_+$, where $D(\theta) = \{u(\theta) : u \in D\}$.

(H3) $h : \mathbb{R}_+ \times E \times \mathcal{BM}_g \to E$ is continuous and satisfies the following conditions:
  - (i) there exists a constant $q_2 \in [0,q)$ such that for each $C > 0$, there is a nonnegative function $v \in L^{1/q_1}(\mathbb{R}_+)$ such that $|h(t,x,y)| \le v(t)$ for a.e. $t \in \mathbb{R}_+$ and for all $(x,y) \in E \times \mathcal{BM}_g$.
  - (ii) there exists a sequence $\{H_n > 0 : n \in \mathbb{N}\}$ such that $\forall t \in [0,n]$ we have

$$|h(t,x,y) - h(t,x',y')| \le H_n\left(|x-x'| + \|y-y'\|_{\mathcal{BM}_g}\right).$$

5. Main result

Theorem 5.1. *Suppose that (H1)-(H3) are hold and*

$$M_n := \frac{nqM\Gamma(3)}{\Gamma(1+2q)}\int_0^n k(s)ds + \frac{n^q M H_n}{\Gamma(1+q)}(m+1) < 1, \quad \forall n \in \mathbb{N}.$$

Then the mild solution set, $\mathcal{S}$, of problem (1.2) is an R_δ-set.

In order to prove the Theorem 5.1 we first consider the operators $\mathcal{B}, \mathcal{Q} : F \to F$ defined by

$$\mathcal{B}y(t) = \int_0^t (t-s)^{q-1}\,\widehat{C}_q(t-s)h\left(\Phi(s,y)\right)ds,$$
$$\mathcal{Q}y(t) = \int_0^t (t-s)^{q-1}\,\widehat{S}_q(t-s)f\left(\Phi(s,y)\right)ds,$$

for all $t \in \mathbb{R}_+$.

Lemma 5.2. *For each $n \in \mathbb{N}$ we have*

$$p_n\left(\mathcal{B}(x) - \mathcal{B}(y)\right) \le k_n p_n(x-y),$$

for all $x, y \in F$, where $k_n = \dfrac{n^q M H_n}{\Gamma(1+q)}(m+1)$.

Proof. Let $x, y \in F$. For every $t \in [0,n]$, we have

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| \le \left|\int_0^t (t-s)^{q-1}\,\widehat{C}_q(t-s)\left[h\left(\Phi(s,x)\right) - h\left(\Phi(s,y)\right)\right]ds\right|.$$

Hence, by applying the assumption (H3) and Lemma 4.6 we get

$$\begin{aligned}|\mathcal{B}x(t) - \mathcal{B}y(t)| &\le \frac{qMH_n}{\Gamma(1+q)}(m+1)p_n(x-y)\int_0^t (t-s)^{q-1}ds\\ &= \frac{n^q M H_n}{\Gamma(1+q)}(m+1)p_n(x-y)\\ &\le k_n p_n(x-y).\end{aligned}$$

This implies that $p_n\left(\mathcal{B}(x) - \mathcal{B}(y)\right) \le k_n p_n(x-y)$. The proof of Lemma is complete. □

Lemma 5.3. *The operator $\mathcal{Q}$ is continuous and $\mathcal{Q}(F)$ is equicontinuous. Futhermore, $\mathcal{B}(F)$ is also equicontinuous.*

Proof. The proof of this lemma consists several steps.
Step 1. $\mathcal{Q}$ is continuous. Indeed, assume that (x_k) be a sequence in F converging to $x \in F$. Put

$$G = \{x_k : k \in \mathbb{N}\} \cup \{x\}.$$

There is a nonnegative function $r_G \in L^{1/q_1}\left(\mathbb{R}_+\right)$ such that

$$|f\left(\Phi(s,y)\right)| \le r_G(s)$$

for all $y \in F$ and for a.e. $s \in [0,n]$ by using (H2 - (iii)). This implies

$$|f\left(\Phi(s,x_k)\right) - f\left(\Phi(s,x)\right)| \le 2r_G(s),$$

for all $k \in \mathbb{N}$ and for a.e. $s \in [0,n]$. On the other hand, it follows from the continuity of ξ and the assumption (H2-(ii)) that $f\left(\Phi(s,x_k)\right) - f\left(\Phi(s,x)\right)$ converges to 0, for almost every where $s \in [0,n]$. Hence, by the Lebesgue dominated convergence theorem,

$$\lim_{k\to\infty} \int_0^n \left|f\left(\Phi(s,x_k)\right) - f\left(\Phi(s,x)\right)\right|^{1/q_1} ds = 0.$$

From Lemma 4.6 and Hölder's inequality we get

$$\begin{aligned}
|\mathcal{Q}x_k(t) - \mathcal{Q}x(t)| &= \left|\int_0^t (t-s)^{q-1}\,\widehat{S}_q(t-s)\left[f\left(\Phi(s,x_k)\right) - f\left(\Phi(s,x)\right)\right] ds\right| \\
&\le \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\left[f\left(\Phi(s,x_k)\right) - f\left(\Phi(s,x)\right)\right] ds \\
&\le \frac{qM}{\Gamma(1+q)} \left(\int_0^t (t-s)^{(q-1)/(1-q_1)} ds\right)^{1-q_1} \times \\
&\quad \left(\int_0^t \left|f\left(\Phi(s,x_k)\right) - f\left(\Phi(s,x)\right)\right|^{1/q_1}(s) ds\right)^{q_1} \\
&\le \frac{qMn^{q-q_1}}{\Gamma(1+q)} \left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} \left\|f\left(\Phi(\cdot,x_k)\right) - f\left(\Phi(\cdot,x)\right)\right\|_{L^{\frac{1}{q_1}}(0,n)},
\end{aligned}$$

for all $t \in [0,n]$. Here we used the following estimate

$$\left(\int_0^t (t-s)^{(q-1)/(1-q_1)} ds\right)^{1-q_1} = t^{q-q_1}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} \le n^{q-q_1}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}.$$

Hence $p_n\left(\mathcal{Q}x_k - \mathcal{Q}x\right)$ converges to 0 when $k \to \infty$. Therefore, $\mathcal{Q}$ is continuous.

Step 2. $\mathcal{Q}(F)$ is equicontinuous. In fact, there is a nonnegative function $r_Q \in L^{1/q_1}(\mathbb{R}_+)$ such that

$$\left|f\left(\Phi(t,x)\right)\right| \le r_Q(t)$$

for all $x \in F$ and for a.e. $t \in [0,n]$ by using again (H2 - (iii)).

⋄ For $x \in F$ and $0 \le t_1 < t_2 \le n$, we have

$$\begin{aligned}
|\mathcal{Q}x(t_2) - \mathcal{Q}x(t_1)| &= \left|\int_0^{t_2} (t_2-s)^{q-1}\,\widehat{S}_q(t_2-s) f\left(\Phi(s,x)\right) ds \right. \\
&\quad \left. - \int_0^{t_1} (t_1-s)^{q-1}\,\widehat{S}_q(t_1-s) f\left(\Phi(s,x)\right) ds\right| \\
&\le I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left|\int_{t_1}^{t_2} (t_2-s)^{q-1}\,\widehat{S}_q(t_2-s) f\left(\Phi(s,x)\right) ds\right|, \\
I_2 &= \left|\int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] \widehat{S}_q(t_2-s) f\left(\Phi(s,x)\right) ds\right|, \\
I_3 &= \left|\int_0^{t_1} (t_1-s)^{q-1}\left[\widehat{S}_q(t_2-s) - \widehat{S}_q(t_1-s)\right] f\left(\Phi(s,x)\right) ds\right|.
\end{aligned}$$

Estimate I_1. By using Hölder's inequality we have

$$I_1 \leq \frac{qMn^q\Gamma(3)}{\Gamma(1+2q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\|r_Q\|_{L^{\frac{1}{q_1}}(0,n)}(t_2-t_1)^{q-q_1}.$$

Estimate I_2. It follows from Hölder's inequality and the following inequality

$$(a^\alpha - b^\alpha)^\gamma \leq a^{\alpha\gamma} - b^{\alpha\gamma}, \text{ for all } 0 \leq a < b, \alpha < 0, \gamma > 1.$$

In order to prove this inequality we note that the the function $j(t) = (t^\alpha - 1)^\gamma - t^{\alpha\gamma} + 1$ is increasing on $(0, 1]$ by

$$j'(t) = \alpha\gamma\left[(t^\alpha - 1)^{\gamma-1}t^{\alpha-1} - t^{\alpha\gamma-1}\right] \geq 0, \quad \forall t \in (0, 1].$$

Hence $j(t) \leq j(1) = 0$ which implies that $(t^\alpha - 1)^\gamma \leq t^{\alpha\gamma} - 1$. Let $t = a/b$ we obtain the desired inequality.

By using the above inequality, Lemma 4.6 and Hölder's inequality we get

$$\begin{aligned}
I_2 &\leq \frac{qMn^q\Gamma(3)}{\Gamma(1+2q)}\|r_Q\|_{L^{\frac{1}{q_1}}(0,n)}\left(\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\\
&\leq \frac{qMn^q\Gamma(3)}{\Gamma(1+2q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\|r_Q\|_{L^{\frac{1}{q_1}}(0,n)}\left(t_1^{\frac{q-q_1}{1-q_1}}-t_2^{\frac{q-q_1}{1-q_1}}+(t_2-t_1)^{\frac{q-q_1}{1-q_1}}\right)^{1-q_1}\\
&\leq \frac{qMn^q\Gamma(3)}{\Gamma(1+2q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\|r_Q\|_{L^{\frac{1}{q_1}}(0,n)}(t_2-t_1)^{q-q_1}.
\end{aligned}$$

Estimate I_3. Without loss of generality to assume that $t_1 > 0$. For $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
I_3 &\leq \int_0^{t_1-\varepsilon}(t_1-s)^{q-1}\left|\widehat{S}_q(t_2-s)f(\Phi(s,x)) - \widehat{S}_q(t_1-s)f(\Phi(s,x))\right|ds\\
&\quad + \int_{t_1-\varepsilon}^{t_1}(t_1-s)^{q-1}\left|\widehat{S}_q(t_2-s)f(\Phi(s,x)) - \widehat{S}_q(t_1-s)f(\Phi(s,x))\right|ds\\
&\leq \left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\|r_Q\|_{L^{\frac{1}{q_1}}[0,n]}\left[2\frac{qMn^q\Gamma(3)\varepsilon^{1-q_1}}{\Gamma(1+2q)}\right.\\
&\quad \left.+\left(t_1^{\frac{q-q_1}{1-q_1}} - \varepsilon^{\frac{q-q_1}{1-q_1}}\right)^{1-q_1}\sup_{s\in[0,t_1-\varepsilon]}\|\widehat{S}_q(t_2-s)-\widehat{S}_q(t_1-s)\|_{\mathcal{L}(E)}\right].
\end{aligned}$$

Since $S(t)$ is equicontinuous, it follows that $\widehat{S}_q(t)$ $(t > 0)$ is equicontinuous in t. Hence I_3 converges to zero independently of $x \in F$ as $t_2 - t_1 \to 0$ and $\varepsilon \to 0$. Combining the above estimates we can conclude that $\mathcal{Q}(F)$ is equicontinuous.
By proving similarly, $\mathcal{B}(F)$ is equicontinuous. The proof of this lemma is complete. □

Lemma 5.4. *The operator $\mathcal{B} + \mathcal{Q}$ is Palais-Smale.*

Proof. Let $(x_m)_m \subset F$ be a sequence satisfying the condition

$$\lim_{m\to\infty}(I - \mathcal{B} - \mathcal{Q})(x_m) = 0.$$

We will prove that $V := \{x_m : m \in \mathbb{N}\}$ is relatively compact in F.

Thanks to the equicontinuousness of $\mathcal{B}(F), \mathcal{Q}(F)$, we have V is equicontinuous. It's suffice to prove the relatively compactness in E of $V(t) = \{x(t) : x \in V\}$ for all $t \in \mathbb{R}$. If $t \in (-\infty, 0]$ then $V(t) = 0$. Thus $V(t)$ is relatively compact.

Set $F_n := \left\{ y|_{(-\infty,n]} : y \in F \right\}$ is a Banach space endowed with the norm $\{p_n\}$ defined by

$$p_n(y) = \sup_{s\in[0,n]} |y\,(s)|\,.$$

Put $W = \{x|_{(-\infty,n]} : x \in V\}$. With $x \in F$, set $u = x|_{(-\infty,n]} \in F_n$. We have $x_t = u_t$ for all $t \in [0, n]$. Thus we may observe $\mathcal{B}, \mathcal{Q} : F_n \to F_n$. By (H2)(iii) and (H3)(i), $\mathcal{B}(W)$and $\mathcal{Q}(W)$ are bounded in F_n. Notice that $\mathcal{B} : F_n \to F_n$ is k_n contraction, so we have

$$\beta(\mathcal{B}(W)) \le k_n \beta(W).$$

⋄ For each $t \in [0, n]$ we put

$$W(t) = \{x(t) : x \in W\},$$
$$W_t = \{x_t : x \in W\}.$$

By Lemma 4.6, Lemma 2.13 and (H2)(iv), for all $t \in [0, n], x \in F_n$ we have

$$\begin{aligned}\beta(\{\mathcal{Q}x(t) : x \in W\}) &\le \frac{qM\Gamma(3)}{\Gamma(1 + 2q)} \int_0^t (t - s)^{2q-1} \beta(f\,(s, x(s) + \widehat{\varphi}(s), x_s + \widehat{\varphi}_s))ds \\ &\le \frac{qM\Gamma(3)}{\Gamma(1 + 2q)} \int_0^t (t - s)^{2q-1} k(s) \\ &\times \Big[\sup_{\theta\in(-\infty,0]} \beta(W_s(\theta)) + \beta(W(s))\Big] ds.\end{aligned}$$

By $x \in F_n$, we have $x(\theta) = 0$ for all $\theta \in (-\infty, 0]$. Thus, we have

$$\sup_{\theta\in(-\infty,0]} \beta(W_s(\theta)) = \sup_{s\in[0,n]} \beta(W(s)),$$

for all $s \in [0, n]$.

Thus,

$$\beta(\{\mathcal{Q}x(t) : x \in W\}) \le \sup_{s\in[0,n]} \beta(W(s)) \frac{nqM\Gamma(3)}{\Gamma(1 + 2q)} \int_0^t k(s)ds.$$

By Lemma 2.12, we have

$$\beta(\mathcal{Q}(W)) = \sup_{s\in[0,n]} \beta(\{\mathcal{Q}x(t) : x \in W\}) \le \beta(W) \frac{nqM\Gamma(3)}{\Gamma(1 + 2q)} \int_0^t k(s)ds.$$

We have

$$W \subset \Big(W - \mathcal{B}(W) - \mathcal{Q}(W)\Big) + \mathcal{B}(W) + \mathcal{Q}(W).$$

By Pallais-Smale condition, $W - \mathcal{B}(W) - \mathcal{Q}(W)$ is relatively compact. Thus

$$\beta(W) \le \beta(\mathcal{B}(W)) + \beta(\mathcal{Q}(W)) \le M_n\beta(W).$$

By $M_n < 1$, $\beta(W) = 0$. This implies that $V(t) = W(t)$ is compact for all $t \in [0, n]$. Combining the above results, V is relatively compact in F. The proof of this lemma is complete. □

Finally, we shall prove the Theorem 5.1.

Proof of Theorem 5.1. It's suffice to prove that $\mathrm{Fix}(\mathcal{B}+\mathcal{Q})$ is R_δ. It's clear that $\mathcal{Q}x(0)=0$ for all $x\in F$. Let $\varepsilon\in(0,n]$. Assume that $x,y\in F$ and satisfy

$$x|_{I_\varepsilon}=y|_{I_\varepsilon},\ I_\varepsilon=[0,n]\cap[-\varepsilon,\varepsilon]=[0,\varepsilon].$$

By definition of x and y we have $x(t)=y(t)$ and $x_t=y_t$ for all $t\in I_\varepsilon$. This implies

$$\mathcal{B}(x)|_{I_\varepsilon}=\mathcal{B}(y)|_{I_\varepsilon},\ \mathcal{Q}(x)|_{I_\varepsilon}=\mathcal{Q}(y)|_{I_\varepsilon}.$$

So it follows from the Lemma 5.2, Lemma 5.3, Lemma 5.4 and Theorem 3.4 that $\mathrm{Fix}(\mathcal{B}+\mathcal{Q})$ is R_δ. □

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References

[1] J. Andres, G. Gabor, L. Górniewicz, *Boundary value problems on infinite intervals*, Trans. Amer. Math. Soc., **351**(1999), 4861-4903.

[2] J. Andres, L. Górniewicz, *Topology Fixed Point Priciples for Boundary Value Problems*, Topol. Fixed Point Theory Appl, Kluwer, Dordrecht, 2003.

[3] P.N. Aronszajn, *Le correspondant topologique de l'unicité dans la theorie des equations différentielles,* Annals of Mathematics, **43**(1942), 730-78.

[4] E.G. Bajlekova, *Fractional Evolution Equations in Banach Spaces*, Ph.D. Thesis, Eindhoven University of Technology, 2001.

[5] D. Baleanu, O.G. Mustafa, *On the global existence of solutions to a class of fractional differential equations*, Comput. Math. Appl., **59**(5)(2010), 835-1841.

[6] D. Baleanu, O.G. Mustafa, R.P. Agarwal, *An existence Result for a superlinear fractional differential equation*, Appl. Math. Lett., **23**(9)(2010), 1129-1132.

[7] J. Banas, K. Goebel, *Measure of Noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Math, **60**, Dekker, New York, 1990.

[8] K. Borsuk, *Theory of Retracts*, vol. 44, Monografie Matematyczne PWN, Warsaw, 1967.

[9] F.E. Browder, C.P. Gupta, *Topological degree and nonlinear mappings of analytic type in Banach space,* J. Math. Anal. Appl, **26**(2010), 390-402.

[10] D. Bugajewska, *On the structure of solution sets of differential equations in Banach spaces*, Math. Slovaca, **50**(2000), 463-471.

[11] M. Cichón, I. Kubiaczyk, *Some remarks on the structure of the solution set for differential inclusions in Banach spaces*, J. Math. Anal. Appl., **233**(1990), 597-606.

[12] F.S. De Blasi, J. Myjak, *On the solution sets for differential inclusion*, Bull. Pol. Acad. Sci. Math., **12**(1985), 17-23.

[13] S.D. Eidelman, A.N. Kochubei, *Cauchy problem for fractional diffusion equations*, J. Differential Equations, **199**(2004), 211-255.

[14] M.M. El-Borai, *Semigroups and some nonlinear fractional differential equations*, Applied Math. Comput., **149**(2004), 823-831.

[15] M.M. El-Borai, *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos Solitons & Fractals, **149**(2004), 823-831.

[16] X. Fu, K. Ezzinbi, *Existence of solutions for neutral differential evolution equations with nonlocal conditions*, Nonlinear Anal., **54**(2003), 215-227.

[17] G. Gabor, *On the acyclicity of fixed point sets of multivalued maps*, Topological Methods in Nonlinear Analysis, **14**(1999), 327-343.

[18] L. Górniewicz, *Topological structure of solution sets: current results*, Archivum Mathematicum, **36**(2000), 343-382.

[19] L. Górniewicz, *Solving equations by topological methods*, Opuscula Mathematica, **25**(2005), 195-225.
[20] H.P. Heinz, *On the behaviour of measure of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal., **7**(1983), 1351-1371.
[21] L.H. Hoa, K. Schmitt, *Fixed point theorems of Krasnosel'skii type in locally convex space and applications to integral equation*, Results in Mathematics, **25**(1994), 291-313.
[22] L.H. Hoa, K. Schmitt, *Periodic solutions of functional-differential equations of retarded and neutral types in Banach spaces,* Boundary Value Problems for Functional-Differential Equations, World Scientific, New Jersey, (1995), 177-185.
[23] O.K. Jardat, A. Al-Omari, S. Momani, *Existence of the mild solution for fractional semilinear initial value problems,* Nonlinear Anal., **69**(9)(2008), 3153-3159.
[24] S. Ji, G. Li, M. Wang, *Controllability of impulsive differential systems with nonlocal conditions,* Appl. Math. Comput., **217**(2011), 6981-6989.
[25] M.A. Krasnoselskii, *Two remarks on the method of successive approximations,* Uspekhi Mat. Nauk., **10**(1955), 123-127.
[26] W. Kryszewski, *Topological structure of solution sets of differential inclusions: the constrained case*, Abstract and Applied Analysis, (2003), 325-351.
[27] Z. Kubacek, *On the structure of fixed point sets of some compact maps in the Fréchet space*, Mathematica Bohemica, **118**(1993), 343-358.
[28] F. Mainardi, P. Paradisi, R. Gorenflo, *Probability distributions generated by fractional diffusion equations*, in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science, Kluwer, Dordrecht, 2000.
[29] J.J. Nieto, *Aronszajn's theorem for some nonlinear Dirichlet problems with unbounded nonlinearities*, Proc. Edinburgh Mathematical Society, **31**(1988), 345-351.
[30] S. Park, *Generalizations of the Krasnosels'kii fixed point theorem*, Nonlinear Anal., **67**(2007), 3401-3410.
[31] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.
[32] J.P.C. Santos, M.M. Arjunan, C. Cuevas, *Existence results for fractional neutral integro-differential equations with state dependent delay,* Comput. Math. Appl., **62**(2011), 1275-1283.
[33] J.P.C. Santos, V. Vijayakumar, R. Murugesu, *Existence of mild solutions for nonlocal Cauchy problem for fractional neutral integro-differential equation with unbounded delay*, Commun. Math. Anal., **14**(2013), 59-71.
[34] S. Szufla, *Sets of fixed points nonlinear mappings in function spaces*, Funkcial. Ekvac., **22**(1979), 121-126.
[35] C.C. Travis, G.F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Sci. Hung., **32**(1978), 75-96.
[36] J.R. Wang, Z. Fan, Y. Zhou, *Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces,* J. Optim. Theory Appl., **154**(2012), 292-302.
[37] J. Wang, Y. Zhou, M. Fekan, *Abstract Cauchy problem for fractional differential equations*, Nonlinear Dynam., **74**(2013), 685-700.
[38] B. Yan, *Boundary value problems on the half-line with impulses and infinite delay,* J. Math. Anal. Appl., **259**(2001), 94-114.
[39] X. Zheng, X. Yang, *The structure of weak Pareto solution sets in piecewise linear multiobjective optimization in normed spaces*, Science in China, Series A: Mathematics, **51**(2008), 1243-1256.
[40] Y. Zhou, F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl., **59**(2010), 1063-1077.

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