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SATURATED VERSIONS OF SOME FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS

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Abstract. In this paper, we will give extended versions of two standard fixed point principles: one for Hardy-Rogers type operators and the other one for Ćirić type operators in complete metric space. Our results generalize similar theorems given in [9].

Key Words and Phrases: Fixed point, complete metric space, Hardy-Rogers type operators, Ćirić type operators, well-posed property, Ostrowski property, quasi-contraction.
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1. Introduction

The most important metric fixed point theorem is the well-known Contraction Principle proved by St. Banach in 1922 in normed spaces and by R. Caccioppoli in 1930 in complete metric spaces. Very recently, in [9] I.A. Rus proved a saturated version of the Banach-Caccioppoli Contraction Principle, together with an extended version of it.

The purpose of this paper is to present extended versions of some fixed point theorems for generalized contractions. Hardy-Rogers contractions and Ćirić generalized contractions are considered. Our results generalize similar theorems given in I.A. Rus [9]. For the case of non-self operators see [2].

For a better understanding of the main part of the paper, we introduce some important definitions. If X is a nonempty set and $f : X \to X$ is an operator, then we denote by $F_f := \{x \in X : x = f(x)\}$ the fixed point set for f.

Definition 1.1 ([7])**.** If X is a nonempty set and $f : X \to X$ is an operator such that, $F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$. then f is called a Bessaga oprator.

Definition 1.2 ([7])**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is a (strict) Picard mapping if there exists $x^* \in X$ such that $F_f = \{x^*\}$ and $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* (uniformly) for all $x \in X$.

Definition 1.3. Let (X, d) be a metric space, $f : X \to X$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function which is continuous at 0 and $\psi(0) = 0$. If the following assumptions are satisfies:

(i) $F_f = \{x^*\}$
(ii) $f^n(x) \to x^*$ as $n \to \infty$, $\forall x \in X$.
(iii) $d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall x \in X$, where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$,

then f is called a ψ-Picard operator.

Definition 1.4. Let (X, d) be a metric space, f be a self-mapping in (X, d). Then the fixed point equation $x = f(x)$ is said to be well-posed if:

(i) $F_f = \{x^*\}$
(ii) $y_n \in X$, $n \in \mathbb{N}$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$.

Moreover, if f satisfies (i) and

(iii) $y_n \in X$, $n \in \mathbb{N}$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$,

then we say that f has the Ostrowski property.

Definition 1.5 ([5])**.** Let X be a nonempty set and $f : X \to X$ be an operator such that

$$\bigcap_{n \in \mathbb{N}} f^n(X) = \{x^*\},$$

then f is a Janos operator.

In 2016, Rus [9] presented a new variant of the contraction principle, a variant with generous conclusions. That variant is the following:

Theorem 1.6 (Saturated Principle of Contraction (SPC), [9])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) *There exists $x^* \in X$ such that, $F_{f^n} = \{x^*\}, \forall n \in \mathbb{N}$.*
(ii) *For all $x \in X$, $f^n(x) \to x^*$ as $n \to \infty$.*
(iii) *$d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall x \in X$, where $\psi(t) = ct$, $c > 0$.*
(iv) *If $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in X such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.*
(v) *If $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in X such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.*
(vi) *If $Y \subset X$ is a closed subset such that $f(Y) \subset Y$, then $x^* \in Y$. Moreover, if in addition Y is bounded, then*

$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}.$$

Another result of the above type is the following.

Theorem 1.7 (Saturated Principle of Quasicontraction (SPQC),[9])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We suppose that there exists a fixed point x^* of f and $0 < l < 1$ such that:*

$$d(f(x), x^*) \leq l d(x, x^*), \quad \forall x \in X.$$

Then we have $(i) - (vi)$ in Theorem 1.6.

In connection with the above results, I.A. Rus proposes in [9] the concept of relevant metrical conditions.

Definition 1.8. Let (X, d) be a complete metric space and $f : X \to X$ be an operator. A metric condition on f is relevant if all of the conclusion of the saturated principle of contraction (SPC) take place.

For example, in [9] it is proved that Kannan's condition on f is relevant from the SPC point of view.

Theorem 1.9 ([9])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be such that there exists $0 < l < 1$, with*

$$d(f(x), f(y)) \leq l[d(x, f(x)) + d(y, f(y))], \forall x, y \in X.$$

Then we have the conclusions in SPC, with
(iii) $d(x, x^*) \leq \frac{1}{1-2l} d(x, f(x)), \forall x \in X.$

In this work we will give some examples of relevant metrical condition. More precisely, we will extend the above mentioned results to the case of Hardy-Rogers and Ćirić metrical conditions on a self operator $f : X \to X$.

2. Main results

First, we give the definition of Hardy-Rogers type operators as follows.

Definition 2.1. A mapping $f : X \to X$ is said to be a Hardy-Rogers type operators if and only if for every $x, y \in X$ there exist non-negative numbers α, β, γ such that

$$\alpha + 2\beta + 2\gamma = \lambda < 1$$

and

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))] + \gamma[d(x, f(y)) + d(y, f(x))], \quad (2.1)$$

hold for every $x, y \in X$.

Now, we prove the saturated principle contraction for Hardy-Rogers type operators.

Theorem 2.2. *Let (X, d) be f-orbitally complete metric space with $f : X \to X$ be Hardy-Rogers type operators where $\alpha + 2\beta + 2\gamma = \lambda < 1$ for $\alpha, \beta, \gamma \in \mathbb{R}^+$. Then we have the conclusions in SPC with*
(iii) $d(x, x^*) \leq \frac{1}{1-\eta} d(x, f(x)), \forall x \in X$ *where* $\eta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$.

Proof. $(i) - (ii)$ Let $x \in X$ be arbitrary and define a sequence (x_n) by

$$x_0 = x, x_1 = f(x_0), \ldots x_n = f(x_{n-1}) = f^n(x_0) \ldots .$$

Since f is a Hardy-Rogers type operators and by (2.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, f(x_{n-1})) + d(x_n, f(x_n))] + \gamma[d(x_{n-1}, f(x_n)) \\ &\quad + d(x_n, f(x_{n-1}))] \\ &= (\alpha + \beta) d(x_{n-1}, x_n) + \beta d(x_n, f(x_n)) + \gamma d(x_{n-1}, f(x_n)) \\ &\leq (\alpha + \beta) d(x_{n-1}, x_n) + \beta d(x_n, f(x_n)) + \gamma[d(x_{n-1}, x_n) + d(x_n, f(x_n))] \\ &= (\alpha + \beta + \gamma) d(x_{n-1}, x_n) + (\beta + \gamma) d(x_n, x_{n+1}). \end{aligned}$$

Then we have

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \right) d(x_{n-1}, x_n) = \eta d(x_{n-1}, x_n) \ \forall x, y \in X, \tag{2.2}$$

and $\eta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$ be given. Since $\alpha + 2\beta + 2\gamma < 1, \forall \alpha, \beta, \gamma \in \mathbb{R}^+ \Rightarrow \eta = \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma} \right) < 1$. Repeating this argument n-times, we obtain

$$d(x_n, x_{n+1}) \leq \eta d(x_{n-1}, x_n) \leq \cdots \leq \eta^n d(x, f(x)).$$

Consider

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \eta^n d(x, f(x)) + \eta^{n+1} d(x, f(x)) + \cdots + \eta^{n+p-1} d(x, f(x)) \\ &= \eta^n \left(\frac{1 - \eta^p}{1 - \eta} \right) d(x, f(x)) \\ &\leq \frac{\eta^n}{1 - \eta} d(x, f(x)), \text{ as } p \to \infty, \forall n \in \mathbb{N}. \end{aligned}$$

Since $\eta < 1$, then $\eta^n \to 0$ as $n \to \infty$. So It's Cauchy sequence. Because X is f-orbitally complete, then there is a point x^* in X such that $f^n(x) \to x^*$ as $n \to \infty, \forall x \in X$.

Now, we shall show that $f(x^*) = x^*$ i.e., $x^* \in F_f$. Let $n = 1$ in (2.2), we have

$$d(f(x), f^2(x)) = d(x_1, x_2) \leq \eta d(x, f(x)) \ \forall x \in X. \tag{2.3}$$

Consider

$$\begin{aligned} d(f(x^*), x_{n+1}) &= d(f(x^*), f^n(x)) \\ &\leq \alpha d(x^*, f^{n-1}(x)) + \beta[d(x^*, f(x^*)) + d(f^{n-1}(x), f^n(x))] \\ &\quad + \gamma[d(x^*, f^n(x)) + d(f^{n-1}(x), f(x^*))] \\ &\leq \alpha d(x^*, x_n) + \beta[d(x^*, x_{n+1}) + d(x_{n+1}, f(x^*))] + \beta d(x_n, x_{n+1}) \\ &\quad + \gamma d(x^*, x_{n+1}) + \gamma[d(x_n, x_{n+1}) + d(x_{n+1}, f(x^*))] \\ &\leq \alpha d(x^*, x_n) + \beta d(x^*, x_{n+1}) + (\beta + \gamma) d(x_n, x_{n+1}) \\ &\quad + \gamma d(x^*, x_{n+1})) + (\beta + \gamma) d(x_{n+1}, f(x^*)). \end{aligned}$$

Consequently

$$\begin{aligned} d(f(x^*), x_{n+1}) &\leq \frac{\alpha}{1-\beta-\gamma} d(x^*, x_n) + \frac{\beta}{1-\beta-\gamma} d(x^*, x_{n+1}) + \frac{\beta+\gamma}{1-\beta-\gamma} d(x_n, x_{n+1}) \\ &+ \frac{\gamma}{1-\beta-\gamma} d(x^*, x_{n+1}) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

So we proved that f has at least one fixed point x^* in X i.e., $x^* \in F_f$ and $x^* \in F_{f^n}$, because $(f^n(x))_{n\in\mathbb{N}}$ is a successive sequence of x.
To show a uniqueness of x^*, let $y^* \neq x^* \in F_f$. Then by (2.1) it follows

$$0 < d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \alpha d(x^*, y^*),$$

implies that $\alpha = 1$ contradicts with $\alpha < 1 - 2\beta - 2\gamma < 1, \forall \alpha, \beta, \gamma \in \mathbb{R}^+$.

(iii) From (2.3). Consider

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^{n-1}(x), f^n(x)) + d(f^n(x), x^*) \\ &\leq d(x, f(x)) + \eta d(x, f(x)) + \cdots + \eta^{n-1} d(x, f(x)) + d(f^n(x), x^*) \\ &= (1 + \eta + \cdots + \eta^{n-1}) d(f^n(x), x^*) \\ &= \left(\frac{1-\eta^n}{1-\eta}\right) d(x, f(x)) \\ &\leq \frac{1}{1-\eta} d(x, f(x)) \text{ as } n \to \infty. \end{aligned}$$

Therefore we have $d(x, x^*) \leq \frac{1}{1-\eta} d(x, f(x)), \forall x \in X$ where $\eta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$, so (iii) is proved.

$(iv) - (vi)$ By (2.1). Consider

$$\begin{aligned} & d(f(x), f(x^*))) \\ &\leq \alpha d(x, x^*) + \beta[d(x, f(x)) + d(x^*, f(x^*))] + \gamma[d(x, f(x^*)) + d(x^*, f(x))] \\ &= \alpha d(x, x^*) + \beta d(x, f(x)) + \gamma d(x, f(x^*)) + \gamma d(x^*, f(x)) \\ &\leq (\alpha + \gamma) d(x, x^*) + \beta[d(x, x^*) + d(x^*, f(x))] + \gamma d(x^*, f(x)) \\ &= (\alpha + \beta + \gamma) d(x, x^*) + (\beta + \gamma) d(f(x^*), f(x)),\ \forall x \in X. \end{aligned}$$

Thus $d(f(x), f(x^*)) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) d(x, x^*)$, for every $x \in X$.
This implies that f is a l-contraction where $l = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} = \eta$. It follows from SPQC (see [9]), Theorem 1.7. We have $(iv) - (vi)$.

Theorem 2.3 (SPC for Hardy-Rogers type operator with respect to a strongly equivalent metric)**.** *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that*

(a) *(X, ρ) is complete metric space.*
(b) *$f : X \to X$ is a Hardy-Rogers type operator with respect to the metric ρ.*
(c) *There exists $c_1, c_2 > 0$ such that*

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y), \quad \forall x, y \in X.$$

Then we have:

(i) $F_{f^n} = \{x^*\}, \forall n \in \mathbb{N}^*$;
(ii) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty, \forall x \in X$;
(iii) $d(x, x^*) \leq \frac{c_2}{c_1}\left(\frac{1}{1-\eta}\right) d(x, f(x)), \forall x \in X$ *where* $\eta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$;
(iv) *The fixed point problem for* f *is well-posed with respect to the metric* d;
(v) *The operator* f *has the Ostrowski property with respect to the metric* d;
(vi) *If* $Y \subset X$ *is a nonempty bounded and closed subset in* (X, d) *with* $f(Y) \subset Y$, *then* $x^* \in Y$ *and*
$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (i) it is a set-theoretical one, we obtain it from previous theorem.

(ii) For all $n \in \mathbb{N}^*$, consider
$$d(f^n(x), x^*) \leq \frac{1}{c_1}\rho(f^n(x), x^*) \to 0 \text{ as } n \to \infty, \ \forall x \in X.$$
This follows that $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$.

(iii) We know from Theorem2.2 that $\rho(x, x^*) \leq \frac{1}{1-\eta}\rho(x, f(x)), \forall x \in X$ where $\eta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$. By property (c) we have that
$$d(x, x^*) \leq \frac{1}{c_1}\rho(x, x^*) \leq \frac{1}{c_1}\left(\frac{1}{1-\eta}\right)\rho(x, f(x)) \leq \frac{c_2}{c_1}\left(\frac{1}{1-\eta}\right) d(x, f(x)).$$

(iv) Suppose that $y_n \in X$ and $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. From (iii), replacing x by $y_n \in X$, then
$$d(y_n, x^*) \leq \frac{c_2}{c_1}\left(\frac{1}{1-\eta}\right) d(y_n, f(y_n))$$
Taking $n \to 0$, we obtain
$$d(y_n, x^*) \to 0 \text{ as } n \to \infty.$$

(v) Since (X, ρ) is a complete metric space by (a), it follows form Theorem 2.2 that f has the Ostrowski property with respect to ρ. We wiil show that f has the Ostrowski property with respect to d. Indeed, if for each $(y_n)_{n \in \mathbb{N}} \subset X$ such that $\rho(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$, then there exists $x \in X$ such that $\rho(y_n, x^*) \to 0$ as $n \to \infty$.
Suppose that $(y_n)_{n \in \mathbb{N}} \subset X$ with $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. From (c) we have
$$\rho(y_{n+1}, f(y_n)) \leq c_2 d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty.$$
By our hypothesis, $\rho(y_n, x^*) \to 0$ as $n \to \infty$. Using (c) again we obtain that
$$d(y_n, x^*) \leq \frac{1}{c_1}\rho(y_n, x^*) \to 0 \text{ as } n \to \infty.$$

(vi) Suppose that $\emptyset \neq Y \subset X$ is a bounded and closed subset in (X, d) with $f(Y) \subset Y$. Let $x \in Y$. Then $f^n(x) \in Y$, $\forall n \in \mathbb{N}$. From (ii) we have $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$, and since Y is closed, this implies that $x^* \in Y$.

For the second part, since $Y \subset X$, from (a) and Theorem 2.2(ii) and (vi), we have that $f^n(x) \xrightarrow{\rho} x^*$ as $n \to \infty$. Then there exists subsequence $f^k(x) \subset Y$, such that

$f^k(x) \xrightarrow{\rho} x^*$, $\forall k \in \mathbb{N}$ as $k \to \infty$, and $x^* \in Y$. Thus Y is a closed subset in (X, ρ). Since $Y \subset X$ is bounded subset in (X, d) and (c) we obtain

$$\rho(x, y) \leq c_2 d(x, y) \leq c_2 M, \quad \forall x, y \in Y, \exists M \in \mathbb{R}^+.$$

So we have Y is bounded closed subset of X in (X, ρ) and $f(Y) \subset Y$. Then

$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\},$$

by Theorem 2.2. The proof is complete. □

Theorem 2.4 (see [9], [10] Maia's Theorem p. 40)**.** *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose*

(a) $d(x, y) \leq \rho(x, y)$, $\forall x, y \in X$.
(b) (X, d) *is a complete metric space.*
(c) f *is a Hardy-Rogers type operator with respect to* ρ.
(d) f *is continuous with respect to* d

Then we have:

(i) $F_{f^n} = \{x^*\}, \forall n \in \mathbb{N}^*$.
(ii) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$, $\forall x \in X$.
(iii) $f^n(x) \xrightarrow{\rho} x^*$ *as* $n \to \infty$, $\forall x \in X$.
(iv) $\rho(x, x^*) \leq \frac{1}{1-l}\rho(x, f(x))$, $\forall x \in X$, *where* $l = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$, $\alpha + 2\beta + 2\gamma < 1$.
(v) *The fixed point problem for* f *is well-posed with respect to the metric* ρ.
(vi) *The operator* f *has the Ostrowski property with respect to the metric* ρ.
(vii) *If* $Y \subset X$ *is a nonempty bounded and closed subset in* (X, ρ) *with* $f(Y) \subset Y$, *then* $x^* \in Y$ *and*

$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. $(i) - (ii)$. Let $x \in X$ and $(f^n(x))_{n \in \mathbb{N}}$ be the corresponding sequence of successive approximations. From (c), we know that $\rho(f(x), f^2(x)) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)\rho(x, f(x))$, for all $x \in X$ where $\frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$. It follows that this sequence is a Cauchy sequence in (X, ρ). From (a), we also get that it is a Cauchy sequence in (X, d) too. By (b), there exists $x^* \in X$ such that $f^n(x) \xrightarrow{d} x^*$ for $n \in \mathbb{N}$. From (d) we obtain that $x^* \in F_f$ and we have $x^* \in F_{f^n}$, by hypothesis. Let $x^* \neq y^* \in X$ and y^* be another fixed point. Then

$$\begin{aligned} 0 \leq \rho(f(x^*), f(y^*)) &\leq \alpha\rho(x^*, y^*) + \beta[\rho(x^*, f(x^*)) + \rho(y^*, f(y^*))] \\ &\quad + \gamma[\rho(x^*, f(y^*) + \rho(y^*, f(x^*))] \\ &= (\alpha + 2\gamma)\rho(x^*, y^*) \ \Rightarrow \alpha + 2\gamma = 1, \end{aligned}$$

This contradict to $\alpha + 2\beta + 2\gamma < 1$, $\forall \alpha, \beta \in \mathbb{R}^+$. Thus, we have that $F_f = F_{f^n} = \{x^*\}$.

(iii) Taking $y = x^*$ in (c) we have

$$\rho(f(x), x^*) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)\rho(x, x^*), \text{ where } \alpha + 2\beta + 2\gamma < 1. \tag{2.4}$$

Consider

$$\rho(f^n(x), x^*) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n \rho(x, x^*) \to 0 \text{ as } n \to \infty, \text{ since } \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1.$$

Suppose that y^* is an another fixed point and $x^* \neq y^*$, then

$$0 \leq \rho(x^*, y^*) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) \rho(x^*, y^*).$$

This implies that $\frac{\alpha+\beta+\gamma}{1-\beta-\gamma} = 1$ which is a contradiction, since $\frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$. Hence $f^n(x) \xrightarrow{\rho} x^*$ as $n \to \infty$.

$(iv) - (vii)$ Using (2.4), we will have that

$$\rho(x, x^*) \leq \frac{1}{1-l}\rho(x, f(x)), \forall x \in X, \text{ where } l = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1.$$

Notice that f is a l-qusicontraction where $l = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$. So the proof follows from SPQC (see [9]), Theorem 1.7. □

Next, we will consider the SPC for some Ćirić type operators.

Theorem 2.5. *Let (X, d) be f-orbitally complete metric space with $f : X \to X$ be Ćirić type operator such that*

$$d(f(x), f(y)) \leq q \cdot \max\{d(x, y); d(x, f(x)); d(y, f(y)); \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\}, \tag{2.5}$$

$\forall x, y \in X$, *where* $0 < q < 1$. *Then we have the conclusions (i)-(iv) in SPC with*

(iii) $d(x, x^*) \leq \frac{1}{1-q} d(x, f(x)), \forall x \in X$. *Additionally, if* $q \in (0, \frac{1}{2})$, *then:*

(v) the fixed point problem for f has Ostrowski's property;

(vi) f is a Janos operator.

Proof. $(i) - (ii)$ Let $x \in X$ be arbitrary and define a sequence

$$x_0 = x, x_1 = f(x_0), ..., x_n = f(x_{n-1}) = f^n(x_0), ...$$

From (2.5) and the definition of the sequence, it follows that

$$\begin{aligned}
&d(x_n, x_{n+1}) \\
&= d(f(x_{n-1}), f(x_n)) \\
&\leq q \max\{d(x_{n-1}, x_n); d(x_{n-1}, x_n); d(x_n, x_{n+1}); \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\} \\
&= q \max\{d(x_{n-1}, x_n); d(x_n, x_{n+1}); \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} \\
&\leq q \max\{d(x_{n-1}, x_n); d(x_n, x_{n+1}); \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} \\
&\leq q d(x_{n-1}, x_n) \quad \forall x, y \in X.
\end{aligned}$$

Repeating this process for n-times, we obtain that

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n) \leq ... \leq q^n d(x_0, x_1) = q^n d(x, f(x)).$$

Consider

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &\leq q^n d(x, f(x)) + q^{n+1} d(x, f(x)) + ... + q^{n+p-1} d(x, f(x)) \\ &= q^n (1 + q + ... + q^{p-1}) d(x, f(x)) \\ &= q^n \left(\frac{1 - q^p}{1 - q} \right) d(x, f(x)), \ \forall p \in \mathbb{N}. \end{aligned}$$

Since $0 < q < 1$ then $q^n \to 0$ as $n \to \infty$. So $(f^n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence. From X is f-orbitally complete, so there is a point x^* in X such that $f^n(x) \to x^*$ as $n \to \infty, \forall x \in X$. Now (ii) is proved, let us prove that $x^* \in F_f$. Consider

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, f^n(x)) + d(f^n(x), f(x^*)) \\ &\leq d(x^*, f^n(x)) + q \max\{d(f^{n-1}(x), x^*); d(f^{n-1}(x), f^n(x)); d(x^*, f(x^*)); \\ &\quad \frac{1}{2}[d(f^{n-1}(x), f(x^*)) + d(x^*, f^n(x))]\} \\ &= 0 + q d(x^*, f(x^*)) \quad \text{by taking } n \to \infty, \quad \text{it is a contradiction.} \end{aligned}$$

Hence $x^* = f(x^*)$ that is $x^* \in F_f$. From (ii) we have that $F_f = F_{f^n} = x^*$ for each $n \in \mathbb{N}$.

(iii) Consider

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^{n-1}(x), f^n(x)) + d(f^n(x), x^*) \\ &\leq d(x, f(x)) + q d(x, f(x)) + \cdots + q^{n-1} d(x, f(x)) + d(f^n(x), x^*) \\ &= (1 + q + \cdots + q^{n-1}) d(x, f(x)) + d(f^n(x), x^*) \\ &= \left(\frac{1 - q^n}{1 - q} \right) d(x, f(x)) + d(f^n(x), x^*). \end{aligned}$$

So we have $d(x, x^*) \leq \frac{1}{1-q} d(x, f(x)), \forall x \in X$.

(iv) Suppose that $y_n \in X$ and $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. From (iii), replacing x by $y_n \in X$, then

$$d(y_n, x^*) \leq \frac{1}{1 - q} d(y_n, f(y_n)).$$

Taking $n \to 0$ and by our assumption we obtain that

$$d(y_n, x^*) \to 0 \text{ as } n \to \infty.$$

(v) Suppose that $y_n \in X$ and $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. We evaluate

$$\begin{aligned}
&d(y_{n+1}, x^*)\\
&\le d(y_{n+1}, f(y_n)) + d(f(y_n), f(x^*))\\
&\le d(y_{n+1}, f(y_n)) + q\max\{d(y_n, x^*), d(y_n, f(y_n)), \frac{1}{2}[d(y_n, x^*) + d(x^*, f(y_n))]\}\\
&\le d(y_{n+1}, f(y_n)) + q\max\{d(y_n, x^*), d(y_n, x^*) + d(x^*, f(y_n))\\
&\quad, \frac{1}{2}[d(y_n, x^*) + d(x^*, f(y_n))]\}\\
&= d(y_{n+1}, f(y_n)) + q\max\{d(y_n, x^*), d(y_n, x^*) + d(x^*, f(y_n))\}\\
&= d(y_{n+1}, f(y_n)) + q(d(y_n, x^*) + d(x^*, f(y_n)))\\
&\le d(y_{n+1}, f(y_n)) + qd(y_n, x^*) + q(d(x^*, y_{n+1}) + d(y_{n+1}, f(y_n))).
\end{aligned}$$

Thus

$$d(y_{n+1}, x^*) \le \frac{1+q}{1-q} d(y_{n+1}, f(y_n)) + \frac{q}{1-q} d(y_n, x^*), \text{ for every } n \in \mathbb{N}.$$

Continuing this procedure we obtain that

$$d(y_{n+1}, x^*) \le \frac{1+q}{1-q} \sum_{k=0}^{n} d(y_{k+1}, f(y_k)) \left(\frac{q}{1-q}\right)^{n-k} + \left(\frac{q}{1-q}\right)^{n+1} d(y_0, x^*),$$

for every $n \in \mathbb{N}$.

By Cauchy Lemma the first sequence tends to 0 as $n \to \infty$, while the second one goes to zero, since $q \in (0, \frac{1}{2})$.

(vi) Suppose that $\emptyset \ne Y \subset X$ is a closed subset in (X, d) with $f(Y) \subset Y$. Let $x \in Y$. Then $f^n(x) \in Y$, $\forall n \in \mathbb{N}$. From (ii) we have $f^n(x) \to x^*$ as $n \to \infty$, and since Y is closed, this implies that $x^* \in Y$. For the second part of (vi), From (2.5) we have

$$d(f(x), x^*) \le \frac{q}{1-q} d(x, x^*) \tag{2.6}$$

and $\frac{q}{1-q} < 1$ by our hypothesis that $q \in (0, \frac{1}{2})$. So $\delta(f(Y), \{x^*\}) \le \delta(Y, \{x^*\})$ where δ is the diameter functional with respect to d. Consider

$$d(f(y), x^*) \le ld(y, x^*) \le l \sup_{z \in Y} d(z, x^*) \le l\delta(Y, \{x^*\})$$

for all $y \in Y$ and $l = \frac{q}{1-q}$. Then $\sup_{y \in Y} d(f(y), x^*) \le l\delta(Y, \{x^*\})$. Hence

$$\delta(f(Y), \{x^*\}) \le l\delta(Y, \{x^*\}).$$

Furthermore, we have $\delta(f^n(Y), \{x^*\}) \le l^n \delta(Y, \{x^*\}) \to 0$ as $n \to \infty$.

So $\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}$ □

Theorem 2.6 (SPC for Ćirić type operator with respect to a strongly equivalent metric). *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that*

(a) *(X, ρ) is f-orbitally complete metric space.*

(b) $f : X \to X$ *is Ćirić type operators with respect to the metric* ρ *and* $q \in (0, \frac{1}{2})$ *for gaurantee that* f *has Ostrowski property and* f *is a Janos operator.*

(c) *There exists* $c_1, c_2 > 0$ *such that*

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y),\ \forall x, y \in X.$$

Then we have $(i) - (vi)$ *in Theorem 2.5, except*
(iii) $d(x, x^*) \leq \psi(d(x, f(x))),\ \forall x \in X$, *where* $\psi = \frac{c_2}{c_1(1-q)}$, $0 < q < 1$ *and* $q \in (0, \frac{1}{2})$ *for* (v) *and* (vi).

Proof. $(i) - (vi)$ The proof follows from Theorem 2.3, by applying Theorem 2.5. □

3. An open question

The most general extension of the Contraction Principle was given by Lj. Ćirić. We give here his result.

Definition 3.1 ([6])**.** Let f be a mapping of a metric space X into it self. For each $x \in X$, let

$$O(x, n) = \{x, fx, \ldots, f^n x\},\ n = 1, 2, \ldots$$
$$O(x, \infty) = \{x, fx, \ldots\}.$$

A space X is said to be f-orbitally complete if and only if every Cauchy sequence which is contained in (O, ∞) for some $x \in X$ converges in X.

Definition 3.2 ([3])**.** A mapping $f : X \to X$ of a metric space X into itself is said to be quasi-contraction if and only if there exists a number $q \in (0, 1)$, such that

$$d(f(x), f(y)) \leq q \cdot \max\{d(x, y); d(x, f(x)); d(y, f(y)); d(x, f(y)); d(y, f(x))\}$$

holds for every $x, y \in X$.

Theorem 3.3 ([3])**.** *Let* $f : X \to X$ *be a quasi-contraction on a metric space* X *and let* X *be* f*-orbitally complete. Then*

(a) f *has a unique fixed point* x^* *in X,*
(b) $f^n(x) \to x^*$, *and*
(c) $d(f^n(x), x^*) \leq \frac{q^n}{1-q} d(x, fx)$ *for every* $x \in X$.

The open question is to give an extended version of the above principle and to study if the quasi-contraction condition is a relevant one, in the sense of Definition 1.8.

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