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ALGORITHMS FOR APPROXIMATING SOLUTIONS OF EQUILIBRIUM PROBLEMS AND FIXED POINTS OF NONEXPANSIVE-TYPE SEMIGROUP

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Abstract. In this paper, a Krasnoselskii-type and a Halpern-type algorithms for approximating a fixed point of a totally quasi-ϕ-asymptotically nonexpansive multi-valued semigroup and a solution of a generalized mixed equilibrium problem is studied. Strong convergence of the sequences generated by these algorithms is proved in real Banach spaces. Finally, the theorems proved are significant improvement on several important recent results.

Key Words and Phrases: Halpern-type algorithm, Krasnoselskii-type algorithm, generalized mixed equilibrium problems, totally quasi-ϕ-asymptotically nonexpansive multi-valued maps, equally continuous maps.
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1. Introduction

Throughout this paper, we assume X is a real Banach space with dual space X^*, K is a nonempty, closed and convex subset of X, $\to$ and $\rightharpoonup$ will respectively denote strong and weak convergence.

A map $J : X \to 2^{X^*}$ defined by $Ju := \{u^* \in X^* : \langle u, u^*\rangle = ||u||||u^*||, ||u|| = ||u^*||\}$ is called a *normalized duality map* on X, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between elements of X and X^*.

Let $G : K \to 2^X$ be any map. A point $u \in K$ is called a *fixed point* of G if $u \in Gu$. We denote the set of fixed points of G by $F(G)$.

Define the functional $\phi : X \times X \to \mathbb{R}$, introduced by Alber [1], by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2 \ \forall \ u, v \in X.$$

From the definition of ϕ it is obvious that

$$(\|u\| - \|v\|)^2 \le \phi(u, v) \le (\|u\| + \|v\|)^2 \ \forall \ u, v \in X. \tag{1.1}$$

Definition 1.1. (see e.g., Zhang *et al.* [22]) Let $\mathcal{G} := \{G(t) : t \geq 0\}$ be a one parameter family of multi-valued maps from K to 2^K. For each $t \geq 0$, let

$$F(G(t)) := \{x \in K : x \in G(t)x\}.$$

- $\mathcal{G}$ is said to be a *multi-valued quasi-ϕ-nonexpansive semigroup* if

  $$\mathcal{F} := \bigcap_{t \geq 0} F(G(t))$$

  is nonempty and the following conditions are satistfied:

  (i) $G(0)u = u$ for all $u \in K$;

  (ii) $G(s+t) = G(s)G(t)$ for all $s, t \geq 0$;

  (iii) for each $u \in K$, the mapping $t \to G(t)u$ is continuous;

  (iv) $\phi(p, \eta(t)) \leq \phi(p, u)\ \forall\ t \geq 0,\ p \in \mathcal{F}, u \in K, \eta(t) \in G(t)u.$

- $\mathcal{G}$ is said to be a $(\{\beta_n\})$-*multi-valued quasi-ϕ-asymptotically nonexpansive semigroup* if $\mathcal{F} := \bigcap_{t \geq 0} F(G(t))$ is nonempty and there exists a sequence $\{\beta_n\} \subset [1, \infty), \beta_n \downarrow 1$ such that the conditions $(i) - (iii)$ and the following condition (v) are satisfied:

  (v) $\phi(p, \eta_n(t)) \leq \beta_n \phi(p, u)\ \forall\ , n \geq 1,\ t \geq 0,\ p \in \mathcal{F}, u \in K, \eta_n(t) \in G^n(t)u.$

- $\mathcal{G}$ is said to be a $(\{\gamma_n\}, \{\delta_n\}, \rho)$-*multi-valued totally-quasi-ϕ-asymptotically nonexpansive semigroup* if $\mathcal{F} := \bigcap_{t \geq 0} F(G(t))$ is nonempty and there exist nonnegative sequences $\{\gamma_n\}, \{\delta_n\}$ with $\gamma_n \to 0, \delta_n \to 0\ (n \to \infty)$ and a strictly increasing continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$ such that the conditions $(i) - (iii)$ and the following condition (iv) are satisfied:
  (vi) $\phi(p, \eta_n(t)) \leq \phi(p, u) + \gamma_n \rho(\phi(p, u)) + \delta_n,\ \forall\ n \geq 1,\ t \geq 0,\ p \in \mathcal{F},$
  $u \in K, \eta_n(t) \in G^n(t)u.$

- A multi-valued $(\{\gamma_n\}, \{\delta_n\}, \rho)$-*totally-quasi-$\phi$-asymptotically nonexpansive semigroup* $\mathcal{G}$ is said to uniformly Lipschitzian if there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that

  $$\|\eta_n(t) - w_n(t)\| \leq L(t)\|u - v\|,\ \forall\ u, u \in K,\ \eta_n(t) \in G^n(t)u,$$

  $$w_n(t) \in G^n(t)v,\ \forall\ n \geq 1,\ t \geq 0.$$

Definition 1.2. (see e.g., Feng *et al.* [11] for similar definition for self maps) A semigroup $\mathcal{G} := \{G(t) : t \geq 0\}$ define from K to 2^K is said to be *equally continuous* if for $u_n, y_n \in K$ we have that

$$\lim_{n \to \infty} \|u_n - y_n\| = 0 \Rightarrow \lim_{n \to \infty} \|\eta_n(t) - w_n(t)\| = 0\ \forall\ \eta_n(t) \in G^n(t)u_n, w_n(t) \in G^n(t)y_n.$$

Remark 1. It is easy to see that the class of uniformly L-Lipschitz multi-valued semigroups is a subclass of the class of equally continuous multi-valued semigroups.

A motivation for the study of the class of totally quasi-ϕ-asymptotically nonexpansive maps is to unify various definitions of classes of maps associated with the class of relatively nonexpansive maps which are extensions, of nonexpansive maps with nonempty fixed point sets and to prove general convergence theorems applicable to all these classes.

Let $\psi : K \to \mathbb{R}$ be a real-valued function, $A : K \to X^*$ be a nonlinear map and $f : K \times K \to \mathbb{R}$ be a bifunction. The *generalized mixed equilibrium problem* is to find $u^* \in K$ such that

$$f(u^*, v) + \psi(v) - \psi(u^*) + \langle v - u^*, Au^* \rangle \geq 0, \ \forall\, v \in K.$$

We denote the set of solutions of the generalized mixed equilibrium problem by **GMEP**(f, A, ψ).

The class of generalized mixed equilibrium problem includes, as special cases, the class of *mixed equilibrium problem* ($A \equiv 0$, see e.g., Ceng and Yao [5] and the references contained therein); the class of *generalized equilibrium problem* ($\zeta \equiv 0$, see e.g., Takahashi and Takahashi [12]); the class of *equilibrium problem* ($A \equiv 0,\ \zeta \equiv 0$, see e.g., Fan [10], Blum and Oettli [3] and the references contained in them); the class of *variational inequality problem* ($h \equiv 0,\ \zeta \equiv 0$, see e.g., Stampacchia [15]) and the class of *convex minimization problem* ($A \equiv 0,\ h \equiv 0$).

It is well known that these classes of problems on their own right, are exremely important and applicable in several area of sciences and social sciences such as in physics, economics, finance, transportation, network and structural analysis, ecology, image reconstruction, and elasticity, (see, e.g., Blum and Otelli [3], Dafermos and Nagurney [9], Su [16], Barbagallo [2], Moudafi [14]). Therefore, the $GMEP(f, A, \psi)$ is a unifying model for several problems arising in physics, engineering, optimization, finance, economics, and so on.

Several strong and weak convergence theorems for nonexpansive semigroups and asymptotically nonexpansive semigroups have been established in various Banach spaces via iterative methods by numerous authors (see e.g., Suzuki [17], Xu [19], Chang *et al.* ([7], [6]), Thong [18], Buong [4], Yao [20] and the references contained in them).

Recently, Zhang *et al.* [22] studied the class of totally quasi-ϕ-asymptotically nonexpansive semigroups and considered in a uniformly convex and uniformly smooth real Banach space X, the following *Halpern-type* algorithm;

$$\begin{cases} u_1 \in X, \ \textit{chosen arbitrary}, \ K_1 = K, \\ y_{n,t} = J^{-1}\Big(\alpha_n J u_1 + (1-\alpha_n)\big(\beta_n J u_n + (1-\beta_n) J G^n(t) u_n\big)\Big), \ t \geq 0, \\ K_{n+1} = \{z \in K_n : \sup_{t \geq 0} \phi(z, y_{n,t}) \leq \alpha_n \phi(z, u_1) + (1-\alpha_n)\phi(z, u_n) + \theta_n\}, \\ u_{n+1} = \Pi_{K_{n+1}} u_1, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\mathcal{G} := \{G(t) : t \geq 0\}$ is a closed, uniformly L-Lipschitzian and $\{\delta_n, \gamma_n, \rho\}$-totally quasi-$\phi$-asymptotically nonexpansive multi-valued semigroup, $\mathcal{F} := \bigcap\limits_{t \geq 0} F(G(t))$ and

$\theta_n = \gamma_n \sup_{p\in\mathcal{F}} \rho(\phi(p,u_n)) + \delta_n$. The authors prove that the sequence $\{u_n\}$ generated by the above iterative scheme converges strongly to $\Pi_{\mathcal{F}}u_1$, under the following conditions:

$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0$; $(C2)$ $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$; $(C3)$ $\mathcal{F}$ is a nonempty and bounded subset of K, where $\alpha_n \in [0,1]$ and $\beta_n \in (0,1)$.

This results of Zhang *et al.* [22] is an important generalizations and improvements of several recent results.

In this paper, it is our purpose to study the following *Halpern-type* and *Krasnoselskii-type* algorithms:

$$\begin{cases} u_0 \in X, \ arbitrary, \ K_1 = K, \quad u_1 = \Pi_{K_1}u_0, \\ y_{n,t} = J^{-1}\Big(\sigma_n Ju_0 + (1-\sigma_n)J\eta_n(t)\Big), \qquad (\eta_n(t) \in G^n(t)u_n), t \geq 0, \\ v_{n,t} = \Lambda_{r_n}y_{n,t}, \ t \geq 0, \\ K_{n+1} = \{z \in K_n : \sup_{t\geq 0} \phi(z, v_{n,t}) \leq \sigma_n\phi(z,u_0) + (1-\sigma_n)\phi(z,u_n) + \omega_n\}, \\ u_{n+1} = \Pi_{K_{n+1}}u_0, \quad n \geq 1, \end{cases} \tag{1.3}$$

$$\begin{cases} u_0 \in X, \ chosen\ arbitrary, \ K_1 = K, \quad u_1 = \Pi_{K_1}u_0, \\ y_{n,t} = J^{-1}\Big(\sigma Ju_n + (1-\sigma)J\eta_n(t)\Big), \qquad (\eta_n(t) \in G^n(t)u_n), t \geq 0, \\ v_{n,t} = \Lambda_{r_n}y_{n,t}, \ t \geq 0, \\ K_{n+1} = \{z \in K_n : \sup_{t\geq 0} \phi(z, v_{n,t}) \leq \phi(z,u_n) + \omega_n\}, \\ u_{n+1} = \Pi_{K_{n+1}}u_0, \quad n \geq 1, \end{cases} \tag{1.4}$$

and to prove, in uniformly convex and smooth real Banach spaces that the sequences generated by these algorithms converge strongly to an element in $\Omega := F(\mathcal{G}) \cap GMEP(h,A,\zeta)$, where $\mathcal{G} := \{G(t) : t \geq 0\}$ is an **equally continuous** and **multi-valued** $\{\delta_n,\gamma_n,\rho\}$-totally quasi-$\phi$-asymptotically nonexpansive semigroup; $A : K \to X^*$ is a continuous and monotone map; $h : K \times K \to \mathbb{R}$ is a bifunction satisfying appropriate conditions and $\zeta : K \to \mathbb{R}$ is a lower semi-continuous and convex function. Our theorems are significant improvements and generalizations of numerous results for this class of nonlinear problems (in particular, the results of Zhang *et al.* [22], Chang *et al.* ([7], [6]), Xu [19], Buong [4], Suzuki [17], Yao *et al.* [20], Thong [18] and the results of a host of other authors (see concluding remarks).

2. Preliminaries

A Banach space X is said to have the *Kadec-Klee property* if for any sequence $\{u_n\}$ in X such that $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$, then $u_n \to u$. Every uniformly convex real Banach space has the Kadec-Klee property.

We now present some lemmas that will be used in the sequel.

Lemma 2.1 (Kamimura and Takahashi, [13]). *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n,y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.2 (see Alber [1]). *Let K be a nonempty closed and convex subset of a reflexive strictly convex and smooth Banach space X. Then,*

$$\phi\Big(u, \Pi_K v\Big) + \phi\Big(\Pi_K v, v\Big) \leq \phi(u, v), \forall\ u \in K,\ v \in X. \tag{2.1}$$

Recall that a multi-valued mapping $G : K \to 2^K$ is said to be closed if for any sequences $\{u_n\}$ and $\{w_n\}$ in K with $w_n \in G(u_n)$, if $u_n \to u$ and $w_n \to y$, then $y \in Gu$.

Let K be a nonempty closed and convex subset of a Banach space X. For solving the generalized mixed equilibrium problem (1.2), we assume that a bifunction $h : K \times K \to \mathbb{R}$ satisfies the following conditions:

$(B1)$ $h(u, u) = 0,\ \forall u \in X$,
$(B2)$ h is monotone, that is, $h(u, v) + h(v, u) \leq 0,\ \forall u, v \in X$,
$(B3)$ for all $u, y, z \in X$, $\limsup_{t\downarrow 0} h(tz + (1-t)u, v) \leq h(u, v)$,
$(B4)$ for all $u \in K, h(u, .) : K \to \mathbb{R}$ is convex and lower semi-continuous.

Lemma 2.3 (see Zhang [21]). *Let X be a smooth, strictly convex and reflexive Banach space, and K be a nonempty, closed and convex subset of X. Let $A : K \to X^*$ be a continuous and monotone mapping, $\zeta : K \to \mathbb{R}$ be a lower semi-continuous and convex function, and $h : K \times K \to \mathbb{R}$ be a bifunction satisfying the conditions $(B1) - (B4)$. Let $r > 0$ be any given number and $u \in X$ be any given point. Then, the followings hold:*

(1) *There exists $z \in K$ such that*

$$h(z, v) + \zeta(v) - \zeta(z) + \langle v - z, Az\rangle + \frac{1}{r}\Big\langle v - z, Jz - Ju\Big\rangle \geq 0, \quad \forall\ v \in K$$

(2) *If we define a mapping $\Lambda_r : K \to K$ by*

$$\Lambda_r(u) = \Big\{z \in K : h(z, v) + \zeta(v) - \zeta(z) + \langle v - z, Az\rangle + \frac{1}{r}\Big\langle v - z, Jz - Ju\Big\rangle \geq 0,\ \forall v \in K\Big\},\ u \in K,$$

the map Λ_r has the following properties:
(a) *Λ_r is single-valued;*
(b) *$F(\Lambda_r) = GMEP(h, A, \zeta) = \hat{F}(\Lambda_r)$;*
(c) *$GMEP(h, A, \zeta)$ a is closed convex set of K;*
(d) *$\phi(q, \Lambda_r u) + \phi(\Lambda_r u, u) \leq \phi(q, u)\ \forall q \in F(\Lambda_r), u \in X$.*

3. Main results

We now prove the following strong convergence theorem using a **Halpern-type algorithm**.

Theorem 3.1. *Let X be a uniformly convex and smooth real Banach space with dual space X^* and let K be a nonempty, closed and convex subset of X. Let*

$$\mathcal{G} := \{G(t) : t \geq 0\}$$

be an equally continuous and $(\{\gamma_n\}, \{\delta_n\}, \rho)$-totally quasi-$\phi$-asymptotically nonexpansive multi-valued semigroup, $h : K \times K \to \mathbb{R}$ be a bifunction satisfying $(B1) - (B4)$,

$A : K \to X^*$ be a continuous monotone map, and $\zeta : K \to \mathbb{R}$ be a convex and lower-semi continuous function. Suppose $\Omega := F(\mathcal{G}) \cap GMEP(h, A, \zeta)$ is a nonempty subset of K. Then, the sequence $\{u_n\}$ generated by the algorithm (1.3) converges strongly to $\Pi_\Omega u_0$, where $\{\sigma_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \sigma_n = 0$ and $\omega_n = \gamma_n \rho\big[\phi(p, u_n)\big] + \delta_n$, $p \in \Omega$.

Proof. The proof is presented in a number of steps.

Step 1: K_n is closed and convex for all $n \geq 1$.
We proceed by induction. Clearly, $K_1 = K$ is closed and convex. Assume K_n is closed and convex for some $n \geq 1$. It therefore follows that

$$\begin{aligned} K_{n+1} &= \{z \in K_n : \sup_{t\geq 0} \phi(z, v_{n,t}) \leq \sigma_n \phi(z, u_0) + (1 - \sigma_n)\phi(z, u_n) + \omega_n\} \\ &= \bigcap_{t\geq 0} \{z \in K : \phi(z, v_{n,t}) \leq \sigma_n \phi(z, u_0) + (1 - \sigma_n)\phi(z, u_n) + \omega_n\} \cap K_n \\ &= \bigcap_{t\geq 0} \{z \in K : 2\sigma_n \langle z, Ju_0\rangle + 2(1 - \sigma_n)\langle z, Ju_n\rangle - 2\langle z, Jv_{n,t}\rangle \\ &\leq \sigma_n \|u_0\|^2 + (1 - \sigma_n)\|u_n\|^2 \\ &- \|v_{n,t}\|^2 + \omega_n\} \cap K_n, \end{aligned}$$

is closed and convex. Hence, Step 1 is established.

Step 2: $\Omega \subset K_n$ for all $n \geq 1$.

Clearly, $\Omega \subset K_1$. Suppose $\Omega \subset K_n$ for some $n \geq 1$. Let $q \in \Omega$. Then, by applying Lemma 2.3 (d), property of ϕ, and totally quasi-ϕ-asymptotically nonexpansiveness of $\mathcal{G}$ we have that

$$\begin{aligned} \phi(q, v_{n,t}) = \phi(q, \Lambda_{r_n} y_{n,t}) &\leq \phi(q, y_{n,t}) \\ &= \phi\Big(q, J^{-1}\big(\sigma_n Ju_0 + (1 - \sigma_n)J\eta_n(t)\big)\Big) \\ &\leq \sigma_n \phi(q, u_0) + (1 - \sigma_n)\phi(q, \eta_n(t)) \\ &\leq \sigma_n \phi(q, u_0) + (1 - \sigma_n)\big[\phi(q, u_n) + \gamma_n \rho\big(\phi(q, u_n)\big) + \delta_n\big] \\ &\leq \sigma_n \phi(q, u_0) + (1 - \sigma_n)\phi(q, u_n) + \omega_n. \end{aligned} \tag{3.1}$$

Therefore, $\sup_{t\geq 0} \phi(q, v_{n,t}) \leq \sigma_n \phi(q, u_0) + (1 - \sigma_n)\phi(q, u_n) + \omega_n$, which implies that $q \in K_{n+1}$. Hence, by induction $\Omega \subset K_n$ for all $n \geq 1$.

Step 3: $\{\phi(u_n, u_0)\}$ is convergent and $\lim_{n\to\infty} \omega_n = 0$.
Since $u_n = \Pi_{K_n} u_0$ and $K_{n+1} \subset K_n$ for all $n \geq 1$, we have that

$$\phi(u_n, u_0) \leq \phi(u_{n+1}, u_0),$$

which implies that $\{\phi(u_n, u_0)\}$ is nondecreasing. Furthermore, by Lemma 2.2 and the fact that $\Omega \subset K_n$, we have that

$$\phi(u_n, u_0) = \phi\Big(\Pi_{K_n} u_0, u_0\Big) \leq \phi(q, u_0) - \phi(q, u_n) \leq \phi(q, u_0),$$

for all $n \geq 1$ and $q \in \Omega$. Thus, $\{\phi(u_n, u_0)\}$ is bounded. Hence, $\{\phi(u_n, u_0)\}$ is convergent. Consequently, $\lim_{n\to\infty} \omega_n = 0$.

Step 4: $u_n \to u^*$, $v_{n,t} \to u^*$ and $y_{n,t} \to u^*$ uniformly in $t \geq 0$ as $n \to \infty$, for some $u^* \in K$.
Since $\{\phi(u_n, u_0)\}$ is bounded, using inequality (1.1) we obtain that $\{u_n\}$ is also bounded. Let $m > n$. Then, using Lemma 2.2, we obtain that

$$\phi(u_m, u_n) = \phi(u_m, \Pi_{K_n} u_0) \leq \phi(u_m, u_0) - \phi(\Pi_{K_n} u_0, u_0) = \phi(u_m, u_0) - \phi(u_n, u_0),$$

which implies that $\phi(u_m, u_n) \to 0$ as $n, m \to \infty$. Therefore, by Lemma 2.1, we obtain that $\|u_m - u_n\| \to 0$ as $n, m \to \infty$. Hence, $u_n \to u^* \in K$ as $n \to \infty$. Since $u_{n+1} \in K_{n+1}$, we have that

$$\sup_{t\geq 0} \phi(u_{n+1}, v_{n,t}) \leq \sigma_n \phi(u_{n+1}, u_0) + (1 - \sigma_n)\phi(u_{n+1}, u_n) + \omega_n \to 0$$

as $n \to \infty$. This implies that for each $t \geq 0$, $\phi(u_{n+1}, v_{n,t}) \to 0$ as $n \to \infty$. Again, by Lemma 2.1 we obtain that $v_{n,t} \to u^*$ uniformly in $t \geq 0$. From inequality (3.1) we have that $\phi(q, y_{n,t}) \leq \sigma_n \phi(q, u_0) + (1 - \sigma_n)\phi(q, u_n) + \omega_n$ for any $q \in \Omega$. Combining this with the fact that $v_{n,t} = \Lambda_{r_n} y_{n,t}$ and Lemma 2.3 (d), we have that:

$$\begin{aligned}
\phi(v_{n,t}, y_{n,t}) = \phi(\Lambda_{r_n} y_{n,t}, y_{n,t}) &\leq \phi(q, y_{n,t}) - \phi(q, \Lambda_{r_n} y_{n,t}) \\
&\leq \sigma_n \phi(q, u_0) + (1 - \sigma_n)\phi(q, u_n) - \phi(q, \Lambda_{r_n} y_{n,t}) + \omega_n \\
&= \sigma_n \phi(q, u_0) + (1 - \sigma_n)\phi(q, u_n) - \phi(q, v_{n,t}) + \omega_n,
\end{aligned}$$

for any $q \in \Omega$. This implies that $\lim_{n\to\infty} \phi(v_{n,t}, y_{n,t}) = 0$. Hence, by Lemma 2.1 we have that $y_{n,t} \to u^*$ uniformly in $t \geq 0$.

Step 5: $u^* \in \Omega$.
From Step 4 and continuity of J, we have that for each $t \geq 0$, $\|Jy_{n,t} - Ju^*\| \to 0$ as $n \to \infty$. Thus, for each $t \geq 0$,

$$\begin{aligned}
\|Jy_{n,t} - Ju^*\| &= \big\|\sigma_n Ju_0 + (1 - \sigma_n)J\eta_n(t) - Ju^*\big\| \\
&= \big\|(1 - \sigma_n)(J\eta_n(t) - Ju^*) - \sigma_n(Ju^* - Ju_0)\big\| \\
&\geq (1 - \sigma_n)\|J\eta_n(t) - Ju^*\| - \sigma_n\big\|Ju^* - Ju_0\big\|,
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|J\eta_n(t) - Ju^*\| = 0$ uniformly in $t \geq 0$. By norm-to-weak continuity of J^{-1}, we have that $\eta_n(t) \rightharpoonup u^*$ as $n \to \infty$. Furthermore,

$$\big|\|\eta_n(t)\| - \|u^*\|\big| = \big|\|J\eta_n(t)\| - \|Ju^*\|\big| \leq \|J\eta_n(t) - Ju^*\| \to 0.$$

Thus, $\lim_{n\to\infty} \|\eta_n(t)\| = \|u^*\|$. Hence, by the Kadec-Klee property of X, we have that

$$\lim_{n\to\infty} \eta_n(t) = u^* \qquad \textit{uniformly in } t \geq\ 0. \tag{3.2}$$

By continuity of $\{G(t) : t \geq 0\}$, we have that

$$\lim_{n\to\infty} s_n(t) = s, \quad \textit{uniformly in } t \geq 0,\ s_n(t) \in G(t)\eta_n(t), s \in G(t)(u^*). \tag{3.3}$$

Now, consider the sequence $\{z_n(t)\}$ generated by

$$z_2(t) \in G(t)(\eta_1(t)) \subset G^2(t)(u_1), z_3(t) \in G(t)(\eta_2(t)) \subset G^3(t)(u_2), ...,$$
$$z_n(t) \in G(t)(\eta_{n-1}(t)) \subset G^n(t)(u_{n-1}), z_{n+1} \in G(t)(\eta_n(t)) \subset G^{n+1}(t)(u_n), ...$$

Using the equal continuity of $\{G(t) : t \geq 0\}$ and (3.2), we obtain that

$$\begin{aligned} \big\|z_{n+1}(t) - \eta_n(t)\big\| \quad &\leq \quad \big\|z_{n+1}(t) - \eta_{n+1}(t)\big\| + \big\|\eta_{n+1}(t) - u_{n+1}\big\| + \|u_{n+1} - u_n\| \\ &\quad + \big\|u_n - \eta_n(t)\big\| \to 0, \quad n \to \infty. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} z_{n+1}(t) = u^*$ uniformly in $t \geq 0$. Since $z_{n+1}(t) \in G(t)\eta_n(t)$, by (3.3), $\lim\limits_{n\to\infty} z_{n+1}(t) = z$ uniformly in $t \geq 0$, $z \in G(t)(u^*)$. Hence, we have that $z = u^*$. Thus, $u^* \in F(G(t))$. By arbitrariness of t, we have that $u^* \in F(\mathcal{G})$.

We now show that $u^* \in \text{GMEP}(h, A, \zeta)$. Define a function $\Psi : K \times K \to \mathbb{R}$ by

$$\Psi(u, z) = h(u, z) + \zeta(z) - \zeta(u) + \langle z - u, Au\rangle \quad \forall u, z \in K.$$

Then, as shown in Zhang [21], Ψ satisfies $(B1) - (B4)$.
Now, the equation $v_{n,t} = \Lambda_{r_n} y_{n,t}$ implies that

$$\Psi(v_{n,t}, z) + \frac{1}{r_n}\Big\langle z - v_{n,t}, Jv_{n,t} - Jy_{n,t}\Big\rangle \geq 0 \quad \forall z \in K. \tag{3.4}$$

By applying $(B2)$, we have that,

$$\frac{1}{r_n}\Big\langle z - v_{n,t}, Jv_{n,t} - Jy_{n,t}\Big\rangle \geq -\Psi(v_{n,t}, z) \geq \Psi(z, v_{n,t}) \quad \forall z \in K, \tag{3.5}$$

which implies that

$$\begin{aligned} \Psi(z, v_{n,t}) \quad &\leq \quad \frac{1}{r_n}\Big\langle z - v_{n,t}, Jv_{n,t} - Jy_{n,t}\Big\rangle \\ &\leq \quad \frac{1}{r_n}\Big\|z - v_{n,t}\Big\|\Big\|Jv_{n,t} - Jy_{n,t}\Big\| \\ &\leq \quad \frac{1}{r_n}\Big(\|z\| + M\Big)\Big\|Jv_{n,t} - Jy_{n,t}\Big\|, \end{aligned}$$

for all $z \in K$, and some $M > 0$. This implies that $\liminf_{n\to\infty} \Psi(z, v_{n,t}) \leq 0$ for all $z \in K$. From $(B4)$, we obtain that

$$\Psi(z, u^*) \leq \liminf_{n\to\infty} \Psi(z, v_{n,t}) \leq 0 \quad \forall\ z \in K.$$

Let $\lambda \in (0, 1)$ and $z \in K$. Then, $z_\lambda = \lambda z + (1 - \lambda)u^* \in K$. Therefore, $\Psi(z_\lambda, u^*) \leq 0$. From conditions $(B1)$ and $(B4)$ we have that

$$0 = \Psi(z_\lambda, z_\lambda) \leq \lambda\Psi(z_\lambda, z) + (1 - \lambda)\Psi(z_\lambda, u^*) \leq \lambda\Psi(z_\lambda, z) \Rightarrow \Psi(z_\lambda, z) \geq 0 \quad \forall\ z \in K.$$

By $(B3)$, we have that $\Psi(u^*, z) \geq \limsup_{\lambda\downarrow 0} \Psi(z_\lambda, z) \geq 0 \quad \forall\ z \in K$. Therefore, $u^* \in GMEP(h, \zeta, A)$.

Step 6: $u^* = \Pi_\Omega u_0$. Let $d = \Pi_\Omega u_0$. Since $u^* \in \Omega$, we have that

$$\phi(d, u_0) \leq \phi(u^*, u_0). \tag{3.6}$$

Also, since $u_n = \Pi_{K_n} u_0$ and $d \in \Omega \subset K_n$, we have that $\phi(u_n, u_0) \leq \phi(d, u_0)$. Since $u_n \to u^*$, we have that

$$\phi(u^*, u_0) \leq \phi(d, u_0). \tag{3.7}$$

From inequalities (3.6) and (3.7) we obtain that $\phi(u^*, u_0) = \phi(d, u_0)$.
Thus, $u^* = d = \Pi_\Omega u_0$. This completes the proof. □

A prototype for the control parameter in Theorem 3.1 is the canonical choice, $\sigma_n = \dfrac{1}{n}$.

The following corollary concerning the approximation of fixed point of fixed point of totally quasi-ϕ-asymptotically nonexpansive multi-valued semigroup, is easily deducible from Theorem 3.1.

Corollary 3.2. *Let X be a uniformly convex and smooth real Banach space with dual space X^* and let K be a nonempty, closed and convex subset of X. Let*

$$\mathcal{G} := \{G(t) : t \geq 0\}$$

be an equally continuous and $(\{\gamma_n\}, \{\delta_n\}, \rho)$-totally quasi-$\phi$-asymptotically nonexpansive multi-valued semigroup. Suppose $\Omega := F(\mathcal{G})$ is a nonempty subset of K. Then, the sequence $\{u_n\}$ generated by the algorithm

$$\begin{cases} u_0 \in X, \ arbitrary, \ K_1 = K, \quad u_1 = \Pi_{K_1} u_0, \\ y_{n,t} = J^{-1}\Big(\sigma_n J u_0 + (1-\sigma_n) J\eta_n(t)\Big), \qquad (\eta_n(t) \in G^n(t) u_n), t \geq 0, \\ K_{n+1} = \{z \in K_n : \sup_{t\geq 0} \phi(z, y_{n,t}) \leq \sigma_n \phi(z, u_0) + (1-\sigma_n)\phi(z, u_n) + \omega_n\}, \\ u_{n+1} = \Pi_{K_{n+1}} u_0, \quad n \geq 1, \end{cases}$$

converges strongly to $\Pi_\Omega u_0$, where $\{\sigma_n\} \subset (0,1)$ with

$$\lim_{n\to\infty} \sigma_n = 0 \ \text{and} \ \omega_n = \gamma_n \rho\big[\phi(p, u_n)\big] + \delta_n, \ p \in \Omega.$$

We now prove the following strong convergence theorem using a **Krasnoselskii-type** algorithm.

Theorem 3.3. *Let X be a uniformly convex and smooth real Banach space with dual space X^* and let K be a nonempty, closed and convex subset of X. Let*

$$\mathcal{G} := \{G(t) : t \geq 0\}$$

be an equally continuous and $(\{\gamma_n\}, \{\delta_n\}, \rho)$-totally quasi-$\phi$-asymptotically nonexpansive multi-valued semigroup, $h : K \times K \to \mathbb{R}$ be a bifunction satisfying $(B1) - (B4)$, $A : K \to X^*$ be a continuous monotone map, and $\zeta : K \to \mathbb{R}$ be a convex and lower-semi continuous function. Suppose $\Omega := F(\mathcal{G}) \cap GMEP(h, A, \zeta)$ is a nonempty subset of K. Then, the sequence $\{u_n\}$ generated by the algorithm (1.4) converges strongly to $\Pi_\Omega u_0$, where $\sigma \in (0,1)$ and $\omega_n = \gamma_n \rho\big[\phi(p, u_n)\big] + \delta_n$, $p \in \Omega$.

Proof. As in the proof of Theorem 3.1, the proof of this theorem is presented in six steps.

Step 1: K_n is closed and convex for all $n \geq 1$.

This follows easily by induction, just as in the proof of Theorem 3.1.

Step 2: $\Omega \subset K_n$ for all $n \geq 1$. Clearly, $\Omega \subset K_1$. Suppose $\Omega \subset K_n$ for some $n \geq 1$. Let $q \in \Omega$. Then, by applying Lemma 2.3 (d), property of ϕ, and totally quasi-ϕ-asymptotically nonexpansiveness of $\mathcal{G}$, as in Step 2 of the proof of Theorem 3.1, we have that for any $t \geq 0$,

$$\begin{aligned}
\phi(q, v_{n,t}) = \phi\big(q, \Lambda_{r_n} y_{n,t}\big) &\leq \phi\big(q, y_{n,t}\big) \\
&= \phi\Big(q, J^{-1}\big(\sigma J u_n + (1-\sigma) J\eta_n(t)\big)\Big) \\
&\leq \sigma\phi(q, u_n) + (1-\sigma)\phi(q, \eta_n(t)) \\
&\leq \sigma\phi(q, u_n) + (1-\sigma)\big[\phi(q, u_n) + \gamma_n \rho\big(\phi(q, u_n)\big) + \delta_n\big] \\
&\leq \phi(q, u_n) + \gamma_n \rho_n\big(\phi(q, u_n)\big) + \delta_n = \phi(q, u_n) + \omega_n,
\end{aligned}$$

which implies that $\sup_{t\geq 0} \phi(q, v_{n,t}) \leq \phi(q, u_n) + \omega_n$. Thus, $q \in K_{n+1}$. Hence, by induction $\Omega \subset K_n$ for all $n \geq 1$.

Step 3: $\{\phi(u_n, u_0)\}$ is convergent and $\lim\limits_{n\to\infty} \omega_n = 0$.
This follows just as in Step 3 of the proof of Theorem 3.1.

Step 4: $u_n \to u^*, v_{n,t} \to u^*$, and $y_{n,t} \to u^*$ uniformly in $t \geq 0$ as $n \to \infty$, for some $u^* \in K$.
Following the same pattern of argument as in Step 4 of the proof of Theorem 3.1, we obtain that $u_n \to u^* \in K$ as $n \to \infty$. Since $u_{n+1} \in K_{n+1}$, we have that

$$\sup_{t\geq 0} \phi(u_{n+1}, v_{n,t}) \leq \phi(u_{n+1}, u_n) + \omega_n \to 0$$

as $n \to \infty$. This implies that for each $t \geq 0$, $\phi(u_{n+1}, v_{n,t}) \to 0$ as $n \to \infty$. Thus, by Lemma 2.1 we obtain that $v_{n,t} \to u^*$ uniformly in $t \geq 0$. From inequality (3.8) we have that $\phi(q, y_{n,t}) \leq \phi(q, u_n) + \omega_n$ for any $q \in \Omega$. Combining this with the fact that $v_{n,t} = \Lambda_{r_n} y_{n,t}$ and Lemma 2.3 (d), we have that

$$\begin{aligned}
\phi(v_{n,t}, y_{n,t}) = \phi(\Lambda_{r_n} y_{n,t}, y_{n,t}) &\leq \phi(u^*, y_{n,t}) - \phi(u^*, \Lambda_{r_n} y_{n,t}) \\
&\leq \phi(u^*, u_n) - \phi(u^*, \Lambda_{r_n} y_{n,t}) + \omega_n \\
&= \phi(u^*, u_n) - \phi(u^*, v_{n,t}) + \omega_n,
\end{aligned}$$

which implies that $\lim\limits_{n\to\infty} \phi(v_{n,t}, y_{n,t}) = 0$.
Hence, by Lemma 2.1 we have that $y_{n,t} \to u^*$ uniformly in $t \geq 0$.

Step 5: $u^* \in \Omega$.
We first show that $u^* \in F(\mathcal{G})$. From Step 4 and continuity of J, we have that

$$\|Jy_{n,t} - Ju^*\| \to 0,\ \|Ju_n - Ju^*\| \to 0$$

uniformly in $t \geq 0$ as $n \to \infty$. Therefore,

$$\begin{aligned}
\|Jy_{n,t} - Ju^*\| &= \big\|\sigma J u_n + (1-\sigma) J\eta_n(t) - Ju^*\big\| \\
&= \big\|(1-\sigma)\big(J\eta_n(t) - Ju^*\big) - \sigma(Ju^* - Ju_n)\big\| \\
&\geq (1-\sigma)\|J\eta_n(t) - Ju^*\| - \sigma\big\|Ju^* - Ju_n\big\|,
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|J\eta_n(t) - Ju^*\| = 0$ uniformly in $t \geq 0$. The rest of the justification of this steps follows the same pattern as in the justification of Step 5 in the proof of Theorem 3.1.

Step 6: $u^* = \Pi_\Omega u_0$.
This is the same as Step 6 of the proof of Theorem 3.1. Hence, the proof is established. □

Concluding remarks

Remark 2. It is well known that the Krasnoselskii sequence whenever it conveges, it converge as fast as a geometrical progression. In this paper, in addition to proving strong convergence of a Halpern-type algorithm *that approximates a common point in the fixed point set of totally quasi-ϕ-asymptotically nonexpansive multi-valued semi-groups and solution set of a generalized mixed equilibrium problem*, we also established strong convergence of a Krasnoselskii-type algorithm that serves the same purpose (Theorem 3.3).

Remark 3. Corollary 3.2 improves the results of Zhang *et al.* [22] in the following ways:

- The algorithm in Corollary 3.2 involves only one control parameter $\{\sigma_n\} \subset (0,1)$ satisfying condition $(C1)$, whereas the algorithm of Zhang *et al.* [22] contains two control parameters $\{\beta_n\} \subset (0,1)$ and $\{\sigma_n\} \subset [0,1]$ satisfying conditions $(C1)$ and $(C2)$. Consequently, the algorithm in Corollary 3.2 is more efficient than algorithm (1.2) studied in the paper of Zhang *et al.* [22].
- The Banach spaces considered in Corollary 3.2 are uniformly convex and smooth real Banach spaces. Such spaces include uniformly smooth and uniformly convex real Banach spaces considered in Zhang *et al.* [22].
- The requirement that the semigroup $\mathcal{G} = \{G(t) : t \geq 0\}$ is uniformly L-Lipschitz continuous in Zhang *et al.* [22] is weakened to the condition that $\mathcal{G} = \{G(t) : t \geq 0\}$ is equally continuous in Corollary 3.2.

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