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A GENERALIZATION OF AMINI-HARANDI'S FIXED POINT THEOREM WITH AN APPLICATION TO NONLINEAR MAPPING THEORY

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Abstract. We introduce a new fixed point theorem on complete metric spaces, which generalizes some former results, and we apply this to obtain a surjectivity theorem for Gâteaux differentiable mappings between Banach spaces.
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1. Introduction

In 1922, Banach [8] stated a fixed point theorem, which became one of the most notable results in the history of mathematical analysis, inspiring many other important works. Amongst them, Caristi's result [9] have been generally accepted as a very useful one. It has been also referred as Caristi-Kirk's (or Caristi-Kirk-Browder's) fixed point theorem, and it is essentially equivalent to Ekeland's variational principle [13] and even to completeness of the given metric space [22]. Caristi's result asserts that any self-mapping T of a complete metric space (X, d) such that

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$, has a fixed point, where φ is a nonnegative valued lower semi-continuous function of X.

While there has been many generalizations of Caristi's theorem [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21], here we target especially those two: Downing-Kirk's fixed point theorem [12] and a theorem by Amini-Harandi [5], which we refer to as Amini-Harandi's fixed point theorem.

Throughout this paper, X^Y denotes the set of all functions from Y to X, and $S[X]$ stands for the image of a set X under a mapping S. Downing and Kirk strengthen

Caristi's result considering two complete metric spaces (X, d) and (Y, ρ), a self mapping T of X, a mapping $S : X \to Y$ with closed graph, a constant $c > 0$ and a lower semi-continuous function $\phi : S[X] \to [0, \infty)$ such that

$$\max\{d(x, Tx), c\rho(Sx, STx)\} \le \phi(Sx) - \phi(STx)$$

for all $x \in X$. On the other hand, Amini-Harandi improved the left side of inequality by a self-mapping η of $[0, \infty)$ and the right side by a function $\psi : X \times X \to \mathbb{R}$ with certain properties, such that self-mappings T of X satisfying

$$\eta(d(x, Tx)) \le \psi(Tx, x)$$

for all $x \in X$, would have a fixed point. This is Corollary 2.4 in [5]. For more about background, details on η and ψ, other results obtained and various applications given by Amini-Harandi in the study in subject see [5].

It is worth noting that, beyond generalizing Caristi's theorem, Amini-Harandi's result is strong enough to conclude Downing-Kirk's fixed point theorem, but not in a canonical manner, in the sense that, for given complete metric spaces (X, d) and (Y, ρ), a self mapping T of X, a mapping $S : X \to Y$ with closed graph, a constant $c > 0$ and a lower semi-continuous function $\phi : S[X] \to [0, \infty)$, it is not always possible to find an η and a ψ such that $\eta(d(x, Tx)) = \max\{d(x, Tx), c\rho(Sx, STx)\}$ and $\psi(Tx, x) = \phi(Sx) - \phi(STx)$ for all $x \in Y$. Instead, this generalization depends on defining a new metric. In fact, it is apparent that Downing-Kirk's theorem can be derived as a rather simple conclusion of even Caristi's theorem, since the metric d' defined as $d'(x, y) = \max\{d(x, y), c\rho(Sx, Sy)\}$ on X, makes X complete thanks to closed graph of S and completenesses of (X, d) and (Y, d), and it also makes the function $\varphi : X \to [0, \infty)$ defined as $\varphi(x) = \phi(Sx)$ lower semi-continuous.

We devote this study to introducing a common canonical generalization of Downing-Kirk's and Amini-Harandi's fixed point theorems, which can not be trivially obtained as their corollary. Thereafter, we prove our main result and obtain a surjectivity theorem as an application of it.

2. Main results

In the sequel, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, Ψ_A denotes the set of all mappings $\psi : A \times A \to \mathbb{R}$ such that $\psi(\cdot, a) : A \to \mathbb{R}$ is upper semi-continuous for all $a \in A$, there exists an $\hat{a} \in A$ making $\psi(\hat{a}, \cdot)$ lower semi-continuous and bounded below, $\psi(a, b) + \psi(b, c) \le \psi(a, c)$ for all $a, b, c \in A$ and $\psi(a, a) = 0$ for all $a \in A$, where A is a subspace of a metric space, while Γ denotes the set of all mappings $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that γ is nondecreasing and continuous, $\gamma(\alpha + \beta) \le \gamma(\alpha) + \gamma(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$ and $\gamma^{-1}[\{0\}] = \{0\}$.

Lemma 2.1. *Let (X, d) and (Y, ρ) be complete metric spaces, $T : X \to X$, $S : X \to Y$, $\gamma, \delta \in \Gamma$ and $\psi \in \Psi_A$, where A is a set such that $S[X] \subseteq A \subseteq Y$. Define the relation $\preccurlyeq$ on X with*

$$x \preccurlyeq y \Leftrightarrow \max\{\gamma(d(x, y)), \delta(\rho(Sx, Sy))\} \le \psi(Sx, Sy) \tag{2.1}$$

for all $x, y \in X$. If S has closed graph, then $(X, \preccurlyeq)$ is a partially ordered set and has at least one minimal element.

Proof. It is easy to show that $\preccurlyeq$ is a partial order on X.
Suppose that C is a chain on $(X, \preccurlyeq)$. There exists a totally ordered infinite set $(I, \leq_I)$, such that it can be written $C = \{x_i \in X : i \in I\}$, where $j \leq_I i$ implies $x_i \preccurlyeq x_j$ for all $i, j \in I$. Then

$$0 \leq \gamma(d(x_i, x_j)) \leq \psi(Sx_i, Sx_j) \leq \psi(\hat{a}, Sx_j) - \psi(\hat{a}, Sx_i),$$

which gives $\psi(\hat{a}, Sx_i) \leq \psi(\hat{a}, Sx_j)$, where $\psi(\hat{a}, \cdot)$ is lower semi-cointinuous and bounded below. Hence $\{\psi(\hat{a}, Sx_i)\}_{i \in I}$ is a decreasing net of reals bounded below. Thus we can find an increasing sequence (i_n) on I such that

$$\lim_{n\to\infty} \psi(\hat{a}, Sx_{i_n}) = \inf_{i \in I} \psi(\hat{a}, Sx_i).$$

Let $\varepsilon > 0$. For $\gamma(\varepsilon) > 0$, there exists $n_0 \in \mathbb{N}$ such that $m \geq n \geq n_0$ implies $0 \leq \psi(\hat{a}, Sx_{i_n}) - \psi(\hat{a}, Sx_{i_m}) < \gamma(\varepsilon)$ for all $m, n \in \mathbb{N}$, and

$$\gamma(d(x_{i_m}, x_{i_n})) \leq \psi(Sx_{i_m}, Sx_{i_n}) \leq \psi(\hat{a}, Sx_{i_n}) - \psi(\hat{a}, Sx_{i_m}) < \gamma(\varepsilon).$$

$\gamma(d(x_{i_m}, x_{i_n})) < \gamma(\varepsilon)$ implies $d(x_{i_m}, x_{i_n}) < \varepsilon$, that is (x_{i_n}) is a Cauchy sequence on (X, d), and there is an $x \in X$ such that $(x_{i_n}) \to x$. Then, since γ is continuous we have

$$\gamma(d(x, x_{i_n})) = \limsup_{m,n\to\infty} \gamma(d(x_{i_m}, x_{i_n})) \leq \limsup_{m,n\to\infty} \psi(Sx_{i_m}, Sx_{i_n}). \tag{2.2}$$

On the other hand by (2.1) we similarly have

$$\delta(\rho(Sx_{i_m}, Sx_{i_n})) \leq \psi(Sx_{i_m}, Sx_{i_n}) \leq \psi(\hat{a}, Sx_{i_n}) - \psi(\hat{a}, Sx_{i_m}) < \gamma(\varepsilon),$$

that is $\delta(\rho(Sx_{i_m}, Sx_{i_n})) < \delta(\varepsilon)$, or $\rho(Sx_{i_m}, Sx_{i_n}) < \varepsilon$. So (Sx_{i_n}) is a Cauchy sequence on (Y, ρ) and $(Sx_{i_n}) \to y$ for some $y \in Y$. Since S has closed graph, $(x_{i_n}) \to x$ and $(Sx_{i_n}) \to y$ give $y = Sx$. Also

$$\delta(\rho(Sx, Sx_{i_n})) = \limsup_{m,n\to\infty} \delta(\rho(Sx_{i_m}, Sx_{i_n})) \leq \limsup_{m,n\to\infty} \psi(Sx_{i_m}, Sx_{i_n}). \tag{2.3}$$

Since $\psi(\cdot, x_{i_n})$ is upper semi-continuous we have

$$\limsup_{m,n\to\infty} \psi(Sx_{i_m}, Sx_{i_n}) \leq \psi(Sx, Sx_{i_n}). \tag{2.4}$$

Then $\max\{\gamma(d(x, x_{i_n})), \delta(\rho(Sx, Sx_{i_n}))\} \leq \psi(Sx, Sx_{i_n})$ by (2.2), (2.3) and (2.4), that is $x \preccurlyeq x_{i_n}$ for all $n \in \mathbb{N}$.
Assume that x is not a lower bound for C. So there exists an $x_0 \in C$ such that $x \not\preccurlyeq x_0$. Since C is a chain and $x \preccurlyeq x_{i_n}$, we also have $x_0 \preccurlyeq x_{i_n}$ for all $n \in \mathbb{N}$. Then

$$0 \leq \gamma(d(x_0, x_{i_n})) \leq \psi(\hat{a}, Sx_{i_n}) - \psi(\hat{a}, Sx_0)$$

which gives $\psi(\hat{a}, Sx_0) \leq \psi(\hat{a}, Sx_{i_n})$ for all $n \in \mathbb{N}$ so that

$$\psi(\hat{a}, Sx_0) = \inf_{i \in I} \psi(\hat{a}, Sx_i) = \lim_{n\to\infty} \psi(\hat{a}, Sx_{i_n})$$

and therefore

$$0 \leq \lim_{n\to\infty} \gamma(d(x_0, x_{i_n})) \leq \lim_{n\to\infty} \psi(\hat{a}, Sx_{i_n}) - \psi(\hat{a}, Sx_0) = 0.$$

Then the sequence $\gamma(d(x_0, x_{i_n}))$ converges to $0 = \gamma(0)$ which implies by the properties of γ that $(d(x_0, x_{i_n})) \to 0$, so that $(x_{i_n}) \to x_0$. That is, we have the contradiction $x = x_0$. Hence x must be a lower bound of C and by Zorn's Lemma, $(X, \preccurlyeq)$ has a minimal element. □

In the following, for any mapping $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, Ω_γ denotes the set of all such mappings $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ that, there exists an $\varepsilon_\eta > 0$ such that $\eta(t) \leq \varepsilon_\eta$ implies $\gamma(t) \leq \eta(t)$ for all $t \in \mathbb{R}^+$.

Lemma 2.2. *Let (X, d) and (Y, ρ) be complete metric spaces, $T : X \to X$, $S : X \to Y$, $\gamma, \delta \in \Gamma$, $\eta \in \Omega_\gamma$, $\theta \in \Omega_\delta$ and $\psi \in \Psi_A$, where $S[X] \subseteq A \subseteq Y$. Define a relation $\preceq$ (which is not needed to be a partial order) on X with*

$$x \preceq y \Leftrightarrow \max\{\eta(d(x, y)), \theta(\rho(Sx, Sy))\} \leq \psi(Sx, Sy) \tag{2.5}$$

for all $x, y \in X$. If S has closed graph, then $(X, \preceq)$ has minimal element.

Proof. Let $\hat{a} \in A$ such that $\psi(\hat{a}, \cdot)$ is lower semi-continuous and bounded below and let

$$\psi_0 := \inf_{x \in X} \psi(\hat{a}, Sx).$$

Also let $\varepsilon := \min\{\varepsilon_\eta, \varepsilon_\theta\}$ and $X_0 := \{x \in X : \psi(\hat{a}, x) \leq \psi_0 + \varepsilon\}$. Suppose that (x_n) is a sequence on X_0 and $(x_n) \to x$ on (X, d). Then $\psi(\hat{a}, x_n) \leq \psi_0 + \varepsilon$ for all $n \in \mathbb{N}$ and lower semi-continuity of $\psi(\hat{a}, \cdot) : X \to \mathbb{R}^+$ implies

$$\psi(\hat{a}, x) \leq \liminf_{n \to \infty} \psi(\hat{a}, x_n) \leq \psi_0 + \varepsilon$$

so that $x \in X_0$.
Therefore X_0 is a closed nonempty subset of X such that $\psi_0 \leq \psi(\hat{a}, x) \leq \psi_0 + \varepsilon$ for all $x \in X_0$. Then also X_0 is complete and yet the restriction of S on X_0 has closed graph. If we define a relation $\preccurlyeq$ on X_0 by

$$x \preccurlyeq y \Leftrightarrow \max\{\gamma(d(x, y)), \delta(\rho(Sx, Sy))\} \leq \psi(Sx, Sy)$$

then by Lemma 2.1 $(X_0, \preccurlyeq)$ has a minimal element x_*.
Given an $x \in X$ such that $x \preceq x_*$, that is

$$0 \leq \eta(d(x, x_*)) \leq \psi(\hat{a}, Sx_*) - \psi(\hat{a}, Sx).$$

Then,

$$\psi(\hat{a}, Sx) \leq \psi(\hat{a}, Sx_*) \leq \psi_0 + \varepsilon$$

gives $x \in X_0$. $x, x_* \in X_0$ and $x \preceq x_*$ yield

$$\eta(d(x, x_*)) \leq \psi(Sx, Sx_*) \leq \psi(\hat{a}, Sx_*) - \psi(\hat{a}, Sx) \leq \varepsilon \leq \varepsilon_\eta$$

and

$$\theta(\rho(Sx, Sx_*)) \leq \psi(Sx, Sx_*) \leq \psi(\hat{a}, Sx_*) - \psi(\hat{a}, Sx) \leq \varepsilon \leq \varepsilon_\theta.$$

Since $\eta \in \Omega_\gamma$ and $\theta \in \Omega_\delta$, these implies that

$$\gamma(d(x, x_*)) \leq \eta(d(x, x_*)) \text{ and } \delta(\rho(Sx, Sx_*)) \leq \theta(\rho(Sx, Sx_*)).$$

Hence

$$\begin{aligned}\max\{\gamma(d(x,x_*)),\delta(\rho(Sx,Sx_*))\} &\le \max\{\eta(d(x,x_*)),\theta(\rho(Sx,Sx_*))\}\\ &\le \psi(Sx,Sx_*),\end{aligned}$$

that is $x \preccurlyeq x_*$. Minimality of x_* in $(X_0,\preccurlyeq)$ yields $x = x_*$, which shows that x_* is also minimal in $(X,\preceq)$. □

Theorem 2.3. *Let (X,d) and (Y,ρ) be complete metric spaces, $T : X \to X$, $S : X \to Y$, $\gamma,\delta \in \Gamma$, $\eta \in \Omega_\gamma$, $\theta \in \Omega_\delta$ and $\psi \in \Psi_A$, where $S[X] \subseteq A \subseteq Y$. If S has closed graph and*

$$\max\{\eta(d(Tx,x)),\theta(\rho(STx,Sx))\} \le \psi(STx,Sx).$$

for all $x \in X$, then T has a fixed point.

Proof. We have $Tx \preceq x$ for all $x \in X$, where $\preceq$ is the relation on X defined in (2.5). Then $(X,\preceq)$ has a minimal element x_* by Lemma 2.2 and thus $Tx_* \preceq x_*$ gives $Tx_* = x_*$. □

Note that Theorem 2.3 generalizes Amini-Harandi's fixed point theorem, namely Corollary 2.4 in [5], with $X = Y$, $S = \mathrm{I}_X$, the identity mapping for X, $A = X$, $\delta = \gamma$, $\cup\{\Omega_\gamma : \gamma \in \Gamma\} = \mathcal{A}$ in [5] and $\theta = \eta$. It also generalizes Downing-Kirk's fixed point theorem [12] with $A = S[X]$, $\gamma = \delta = \mathrm{I}_{\mathbb{R}}$, $\eta = \theta = f$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $f(x) = cx$ with a constant $c > 0$, and $\psi(y_1,y_2) = \phi(y_2) - \phi(y_1)$, where $\phi : S[X] \to \mathbb{R}^+$ is lower semi-continuous.

3. Application

Suppose X and Y are locally convex topological vector spaces.

A mapping $S : X \to Y$ is said to be Gâteaux differentiable [12, 14] at a point $x \in X$, if the limit

$$dS_x(y) = \lim_{t\to 0^+} \frac{S(x+ty) - Sx}{t}$$

exists for each $y \in Y$ and $dS_x : X \to Y$ is a linear operator. If $S : X \to Y$ is Gâteaux differentiable, with the derivative dS_x at a point $x \in X$, then the dual of dS_x is denoted with dS'_x, its nullspace is denoted with $N(dS'_x)$ and $N(dS'_x)^\perp$ is the annihilator of $N(dS'_x)$ in Y. We also use the notations

$$D_x(S,\varepsilon) = \{\alpha(Su - Sx) : \alpha \ge 0,\, u \in X,\, \|Su - Sx\| < \varepsilon\}$$

and

$$D_x(S) = \bigcap_{\varepsilon>0} \overline{D_x(S,\varepsilon)}.$$

We first express the following two lemmas from [2], followed by another lemma, which will facilitate the proof of our surjectivity result.

Lemma 3.1. [12] *Let X be a normed space, $x,y,z \in X$, $\alpha \ge 1$, $\beta \in (0,1)$ and*

$$\|\alpha(x-y) - (z-y)\| \le \beta\|z-y\|.$$

Then

$$\|x-y\| \le \frac{1+\beta}{1-\beta}\left(\|y-z\| - \|x-z\|\right).$$

Lemma 3.2. [12] *Let X be a locally convex topological vector space, Y be a Banach space and $S : X \to Y$ be Gâteaux differentiable at the point $x \in X$. Then*

$$N(dS'_x)^{\perp} = \overline{dS_x[X]} \subseteq D_x(S).$$

Lemma 3.3. *Given a complete metric space (X, d) and a Banach space Y. Let $S : X \to Y$ have closed graph, $\gamma, \delta \in \Gamma$ and $\eta \in \Omega_{\gamma}$. Suppose that there exist a $y_0 \in Y$, a constant $\beta \in (0, 1)$, a function $\varepsilon : X \to (0, \infty)$ and sequences (α_n) on $(\mathbb{R}^{+})^{X}$, (U_n) on X^X such that*

(a) *the sets $V_y := \{v \in X : \eta(d(x, v)) \leq \delta(\|Sx - Sv\|),\ Sv = y\}$ are nonempty for all $y \in B(Sx, \varepsilon(x)) \cap S[X]$,*

(b) *$\|(\alpha_n(x))(SU_nx - Sx) - (y_0 - Sx)\| \leq \beta\|y_0 - Sx\|$ for all $n \in \mathbb{N}$,*

(c) *$(SU_nx) \to Sx$,*

(d) *$SU_nx \neq Sx$ for each $n \in \mathbb{N}$.*

Then $y_0 \in S[X]$.

Proof. Assume that $y_0 \notin S[X]$. Let $x \in X$. By (c), there exists an $n_1 \in \mathbb{N}$ such that $\|SU_nx - Sx\| < \varepsilon(x)$ for all $n \geq n_1$. On the other hand, since

$$\|(\alpha_n(x))(SU_nx - Sx) - (y_0 - Sx)\| \leq \beta\|y_0 - Sx\|$$

and $y_0 \notin S[X]$, we have

$$0 \neq (1 - \beta)\|y_0 - Sx\| \leq \alpha_n(x) \cdot \|SU_nx - Sx\|,$$

and $\|SU_nx - Sx\| \to 0$ implies $\alpha_n(x) \to \infty$. Pick $n_2 \in \mathbb{N}$ such that $\alpha_n(x) \geq 1$ for all $n \geq n_2$, and say $n_0 := \max\{n_1, n_2\}$. Then $SU_{n_0}x \neq Sx$ and $\alpha_{n_0}(x) \geq 1$. We have

$$0 < \|SU_{n_0}x - Sx\| \leq \frac{1 + \beta}{1 - \beta}\left(\|Sx - y_0\| - \|SU_{n_0}x - y_0\|\right)$$

by Lemma 3.1. Also, $SU_{n_0}x \in B(Sx, \varepsilon(x)) \cap S[X]$ and $V_{SU_{n_0}x} \neq \varnothing$ by (a).
Let $T : X \to X$ be a choice function for the family of nonempty sets $\{V_{SU_{n_0}x} : x \in X\}$. Then, $\eta(d(x, Tx)) \leq \delta(\|Sx - STx\|)$ and $STx = SU_{n_0}x$, which give

$$\|STx - Sx\| \leq \frac{1 + \beta}{1 - \beta}\left(\|Sx - y_0\| - \|STx - y_0\|\right).$$

We define a mapping $\psi : S[X] \times S[X] \to \mathbb{R}$ such that

$$\psi(a, b) = \delta\left(\frac{1 + \beta}{1 - \beta}\left(\|b - y_0\| - \|a - y_0\|\right)\right)$$

for all $a, b \in S[X]$. Noting that $\delta \in \Gamma$, we observe that $\psi \in \Psi_{S[X]}$. In addition, we have

$$\eta(d(x, Tx)) \leq \delta(\|Sx - STx\|) \leq \psi(STx, Sx),$$

which may be written as

$$\max\{\eta(d(Tx, x)), \theta(\rho(STx, Sx))\} \leq \psi(STx, Sx),$$

where $\theta = \delta \in \Gamma \subseteq \Omega_{\delta}$ and ρ is the metric induced by the norm on Y. Then by Theorem 2.3, T has a fixed point. However, if x_* is a fixed point of T, then $Sx_* = STx_* = SU_{n_0}x$ contradicts with (d). This completes the proof. □

Theorem 3.4. *Let X and Y be Banach spaces and $S : X \to Y$ be a Gâteaux differentiable mapping with a closed graph. Also suppose that given $\delta \in \Gamma$ and $\varepsilon : X \to (0, \infty)$ such that*

(i) *the sets $V_y := \{v \in X : \eta(\|x - v\|)) \leq \delta(\|Sx - Sv\|),\ Sv = y\}$ are nonempty for all $y \in B(Sx, \varepsilon(x)) \cap S[X]$,*

(ii) *$N(dS'_x) = \{0\}$ for each $x \in X$.*

Then S is a surjective mapping.

Proof. We assume the contrary. Then there exists a $y_0 \in Y$ such that $y_0 \neq Sx$ for each $x \in X$. Clearly, (i) is equivalent to the condition (a) of Lemma 3.3.
Let $x \in X$. By $N(dS'_x) = \{0\}$, we have $N(dS'_x)^{\perp} = Y$ and by Lemma 3.2 we also have $D_x(S) = Y$, in particular $y_0 - Sx \in D_x(S)$. Then for every $\varepsilon > 0$, $y_0 - Sx \in \overline{D_x(S, \varepsilon)}$ and for each $n \in \mathbb{N}$, there exists a $(\alpha_n(x))(SU_n(x) - Sx)$ with $\alpha_n(x) \geq 0$, $U_n(x) \in X$, $\|SU_n(x) - Sx\| < \frac{1}{n}$ such that

$$\|((\alpha_n(x))(SU_n(x) - Sx)) - (y_0 - Sx)\| < \frac{\|y_0 - Sx\|}{2n}.$$

This procedure defines the sequences (α_n) on $(\mathbb{R}^+)^X$ and (U_n) on X^X such that the conditions (b), (c) and (d) in Lemma 3.3 are satisfied for $\beta = \frac{1}{2}$. Thus $y_0 \in Y$, which is a contradiction. Hence S is surjective. □

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