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WARDOWSKI-FENG-LIU TYPE FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Abstract. In this work, we present some Wardowski-Feng-Liu type fixed point theorems for multivalued mappings in complete (ordered) metric spaces. The obtained results generalize and improve several existing theorems in the literature. The given notions and outcome are illustrated by an appropriate example. An application to existence of solutions for Fredholm-type integral inclusions is presented.
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1. Introduction

First, we recall the following concept and some related results.

Let $F : (0, \infty) \to \mathbb{R}$ be a function. We will consider the following conditions:

(F1) F is strictly increasing;

(F2) for each sequence $\{t_n\}$ of positive numbers,
$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} F(t_n) = -\infty;$$

(F3) there exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$;

(F4) $F(\inf A) = \inf F(A)$ for all $A \subseteq (0, 1)$ with $\inf A > 0$.

We denote the sets of all functions F satisfying (F1)-(F3), resp. (F1)-(F4) by $\mathfrak{F}$, resp. $\mathfrak{F}_*$. It is clear that $\mathfrak{F}_* \subset \mathfrak{F}$ and some examples of functions belonging to $\mathfrak{F}_*$ are $F_1(t) = \ln t$, $F_2(t) = t + \ln t$, $F_3(t) = -1/\sqrt{t}$, $F_4(t) = \ln(t^2 + t)$, see [15]. If we define $F_5(t) = \ln t$ for $t \le 1$ and $F_5(t) = 2t$ for $t > 1$, then $F_5 \in \mathfrak{F} \setminus \mathfrak{F}_*$.

Note that, if F satisfies (F1), then it satisfies (F4) if and only if it is right-continuous.

Wardowski [15] called a self-mapping T on a metric space (X,d) an F-contraction if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$. It is clear that each (Banach-type) contraction (with parameter $\lambda \in (0,1)$) is an F-contraction (with $F(t) = \ln t$ and $\tau = -\ln \lambda$). However, using other functions $F \in \mathfrak{F}$, more general conditions can be obtained. It was proved in [15] that each F-contraction in a complete metric space has a fixed point. Afterwards, several researchers obtained various fixed point results using the idea of F-contractions, see, e.g., [16, 14, 1].

In what follows, for a metric space (X,d), let $CL(X)$ denote the family of non-empty closed subsets of X. Sgroi and Vetro in [13], called a multivalued mapping $T : X \to CL(X)$ an F-contraction if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x,y \in X$ with $y \in Tx$ there exists $z \in Ty$ for which

$$\tau + F(d(y,z)) \leq F\left(M(x,y)\right) \tag{1.1}$$

if $d(y,z) > 0$, where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\right\}. \tag{1.2}$$

They proved some fixed point results for mappings satisfying condition (1.1) (recall that a point $x \in X$ is called a fixed point of a mapping $T\colon X \to CL(X)$ if $x \in Tx$).

Several fixed point results for multivalued F-contractions were also obtained in the papers [3, 4, 7, 8, 2].

On the other hand, the following result was obtained by Feng and Liu (recall that a function $f\colon X \to \mathbb{R}$ is called lower semi-continuous if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ for all sequences $\{x_n\}$ in X with $\lim_{n\to\infty} x_n = x \in X$).

Theorem 1.1. [6] *Let (X,d) be a metric space, $T\colon X \to CL(X)$ and let the function $f\colon X \to \mathbb{R}$, $f(x) = d(x,Tx)$ be lower semi-continuous. If there exist $b,c \in (0,1)$ with $b < c$ such that for any $x \in X$ there is $y \in Tx$ satisfying*

$$c\,d(x,y) \leq f(x) \quad \text{and} \quad f(y) \leq b\,d(x,y),$$

then T has a fixed point.

In this paper, we obtain fixed point results for multivalued mappings satisfying conditions of Wardowski-Feng-Liu type. An example is given to illustrate their use. An application to Fredholm-type integral inclusions is given at the end.

2. Main results

Let $T : X \to CL(X)$ be a multivalued map, $F \in \mathfrak{F}$ and $\eta : (0,\infty) \to (0,\infty)$. For $x \in X$ with $d(x,Tx) > 0$, define a set $F_\eta^x \subseteq X$ as

$$F_\eta^x = \{\, y \in Tx : F(d(x,y)) \leq F(\max\{d(x,Tx), d(y,Ty)\}) + \eta(M(x,y))\,\},$$

where $M(x,y)$ is defined by (1.2).

Theorem 2.1. *Let (X,d) be a complete metric space, $T\colon X \to CL(X)$ and $F \in \mathfrak{F}_*$. Assume that the following conditions hold:*

(i) *the mapping $x \mapsto d(x,Tx)$ is lower semi-continuous;*
(ii) *there exist functions $\theta, \eta\colon (0,\infty) \to (0,\infty)$ such that*

$$\theta(t) > \eta(t), \quad \liminf_{s\to t^+} \theta(s) > \liminf_{s\to t^+} \eta(s) \text{ for all } t \geq 0;$$

(iii) *for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\eta^x$ satisfying*

$$\theta(M(x,y)) + F(d(y,Ty)) \leq F(d(x,y)).$$

Then T has a fixed point.

Proof. Suppose that T has no fixed points. Then for all $x \in X$, we have $d(x,Tx) > 0$. Since $Tx \in CL(X)$ for every $x \in X$, the set F_η^x is nonempty for any $x \in X$. If $x_0 \in X$ is any initial point, then there exists $x_1 \in F_\eta^{x_0}$ such that

$$\theta(M(x_0,x_1)) + F(d(x_1,Tx_1)) \leq F(d(x_0,x_1))$$

and for $x_1 \in X$, there exists $x_2 \in F_\eta^{x_1}$ satisfying

$$\theta(M(x_1,x_2)) + F(d(x_2,Tx_2)) \leq F(d(x_1,x_2)).$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in F_\eta^{x_n}$ and

$$\theta(M(x_n,x_{n+1})) + F(d(x_{n+1},Tx_{n+1})) \leq F(d(x_n,x_{n+1})),$$

where

$$\begin{aligned} M(x_n,x_{n+1}) = \max\Big\{ & d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), \\ & \frac{1}{2}[d(x_n,Tx_{n+1}) + d(x_{n+1},Tx_n)]\Big\} \\ \leq \max\{ & d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{1}{2}d(x_n,x_{n+2})\} \\ = \max\{ & d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}. \end{aligned}$$

Therefore

$$\theta(\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}) + F(d(x_{n+1},Tx_{n+1})) \leq F(d(x_n,x_{n+1})). \tag{2.1}$$

We will verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in F_\eta^{x_n}$, then by the definition of $F_\eta^{x_n}$, we have

$$\begin{aligned} F(d(x_n,x_{n+1})) \leq\ & F(\max\{d(x_n,Tx_n), d(x_{n+1},Tx_{n+1})\}) \\ & + \eta(\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}). \end{aligned} \tag{2.2}$$

Put $\sigma_n = d(x_n,x_{n+1})$ for $n \in \mathbb{N}$, then $\sigma_n > 0$. From (2.1) and (2.2) we have

$$\begin{aligned} F(d(x_{n+1},Tx_{n+1})) \leq\ & F(\max\{d(x_n,Tx_n), d(x_{n+1},Tx_{n+1})\}) + \eta(\max\{\sigma_n,\sigma_{n+1}\}) \\ & - \theta(\max\{\sigma_n,\sigma_{n+1}\}), \end{aligned} \tag{2.3}$$

implying

$$F(\sigma_{n+1}) \leq F(\max\{\sigma_n,\sigma_{n+1}\}) + \eta(\max\{\sigma_n,\sigma_{n+1}\}) - \theta(\max\{\sigma_n,\sigma_{n+1}\}).$$

If $\sigma_n \leq \sigma_{n+1}$, then we have

$$F(\sigma_{n+1}) \leq F(\sigma_{n+1}) + \eta(\sigma_{n+1}) - \theta(\sigma_{n+1}),$$

which is a contradiction since $\theta(t) > \eta(t)$. Therefore,

$$F(\sigma_{n+1}) \leq F(\sigma_n) + \eta(\sigma_n) - \theta(\sigma_n). \tag{2.4}$$

From (2.4), $\{\sigma_n\}$ is decreasing. Therefore, there exists $\delta > 0$ such that $\lim_{n\to\infty} \sigma_n = \delta$. Now let $\delta > 0$. Let $\beta(t) = \liminf_{s\to t^+} \theta(s) - \liminf_{s\to t^+} \eta(s) > 0$. Then using (2.4), the following holds:

$$\begin{aligned} F(\sigma_{n+1}) &\leq F(\sigma_n) - \beta(\sigma_n) \\ &\leq F(\sigma_{n-1}) - \beta(\sigma_n) - \beta(\sigma_{n-1}) \\ &\vdots \\ &\leq F(\sigma_0) - \beta(\sigma_n) - \beta(\sigma_{n-1}) - \ldots - \beta(\sigma_0). \end{aligned} \tag{2.5}$$

Let p_n be the greatest number in $\{0, 1, \ldots, n-1\}$ such that

$$\beta(\sigma_{p_n}) = \min\{\beta(\sigma_0), \beta(\sigma_1), \ldots, \beta(\sigma_n)\}$$

for all $n \in \mathbb{N}$. In this case, $\{p_n\}$ is a nondecreasing sequence. From (2.5) we get

$$F(\sigma_n) \leq F(\sigma_0) - n\beta(\sigma_{p_n}). \tag{2.6}$$

In a similar way, from (2.3) we can obtain

$$F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_0, x_1)) - n\beta(\sigma_{p_n}). \tag{2.7}$$

Now consider the sequence $\{\beta(\sigma_{p_n})\}$. We distinguish two cases.
Case 1. For each $n \in \mathbb{N}$ there is $m > n$ such that $\beta(\sigma_{p_n}) > \beta(\sigma_{p_m})$. Then we obtain a subsequence $\{\sigma_{p_{n_k}}\}$ of $\{\sigma_{p_n}\}$ with $\beta(\sigma_{p_{n_k}}) > \beta(\sigma_{p_{n_{k+1}}})$ for all k. Since $\sigma_{p_{n_k}} \to \delta^+$ we deduce that

$$\liminf_{n\to\infty} \beta(\sigma_{p_{n_k}}) > 0.$$

Hence

$$F(\sigma_{n_k}) \leq F(\sigma_0) - n^k \beta(\sigma_{p_{n_k}}) \text{ for all } k.$$

Consequently, $\lim_{k\to\infty} F(\sigma_{n_k}) = -\infty$ and by (F2), $\lim_{k\to\infty} \sigma_{n_k} = 0$ which contradicts the fact that $\lim_{k\to\infty} \sigma_{n_k} > 0$.
Case 2. There is $n_0 \in \mathbb{N}$ such that $\beta(\sigma_{p_0}) > \beta(\sigma_{p_m})$ for all $m > n_0$. Then $F(\sigma_m) \leq F(\sigma_0) - m\beta(\sigma_{p_{n_0}})$ for all $m > n_0$. Hence, $\lim_{m\to\infty} F(\sigma_m) = -\infty$ and by (F2), $\lim_{m\to\infty} \sigma_m = 0$, which contradicts the fact that $\lim_{m\to\infty} \sigma_m > 0$. Thus $\lim_{m\to\infty} \sigma_m = 0$. From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\sigma_n)^k F(\sigma_n) = -\infty.$$

By (2.6), the following holds for all $n \in \mathbb{N}$:

$$\begin{aligned} (\sigma_n)^k F(\sigma_n) - (\sigma_n)^k F(\sigma_0) &\leq (\sigma_n)^k (F(\sigma_0) - n\beta(\sigma_{p_n})) - (\sigma_n)^k (F(\sigma_0) \\ &= -n(\sigma_n)^k \beta(\sigma_{p_n}) \leq 0. \end{aligned} \tag{2.8}$$

Passing to the limit as $n \to \infty$ in (2.8), we obtain

$$\lim_{n\to\infty} n(\sigma_n)^k \beta(\sigma_{p_n}) = 0.$$

Since $\zeta := \liminf_{n\to\infty} \beta(\sigma_{p_n}) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(\sigma_{p_n}) > \frac{\zeta}{2}$ for all $n \neq n_0$. Thus,

$$n(\sigma_n)^k \frac{\zeta}{2} < n(\sigma_n)^k \beta(\sigma_{p_n}) \tag{2.9}$$

for all $n \geq n_0$.
Letting $n \to \infty$ in (2.9), we have $0 \leq \lim_{n\to\infty} n(\sigma_n)^k \frac{\zeta}{2} < \lim_{n\to\infty} n(\sigma_n)^k \beta(\sigma_{p_n}) = 0$, that is,

$$\lim_{n\to\infty} n(\sigma_n)^k = 0 \tag{2.10}$$

From (2.10), there exits $n_1 \in \mathbb{N}$ such that $n(\sigma_n)^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$,

$$\sigma_n \leq \frac{1}{n^{1/k}}. \tag{2.11}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.11), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \sigma_n + \sigma_{n+1} + \cdots + \sigma_{m-1} \\ &= \sum_{i=n}^{m-1} \sigma_i \leq \sum_{i=n}^{\infty} \sigma_i \leq \sum_{i=n}^{\infty} \frac{1}{n^{1/k}}. \end{aligned}$$

By the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/k}}$, passing to the limit as $n \to \infty$, we get $d(x_n, x_m) \to 0$ and hence $\{x_n\}$ is a Cauchy sequence in (X, d).

Since X is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. On the other hand, from (2.7) and (F2) we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $x \mapsto d(x, Tx)$ is lower semi-continuous, then

$$0 \leq d(z, Tz) \leq d(x_n, Tx_n) \to 0.$$

This is a contradiction. Hence, T has a fixed point.

In a similar way, one can prove

Theorem 2.2. *Let (X, d) be a complete metric space, $T\colon X \to CL(X)$ and $F \in \mathfrak{F}_*$. Assume that the following conditions hold:*

(i) *the mapping $x \mapsto d(x, Tx)$ is lower semi-continuous;*

(ii) *there exist functions $\theta, \eta\colon (0, \infty) \to (0, \infty)$ such that*

$$\theta(t) > \eta(t), \quad \liminf_{s\to t^+} \theta(s) > \liminf_{s\to t^+} \eta(s) \text{ for all } t \geq 0;$$

(iii) *for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in Tx$ satisfying*

$$\begin{aligned} \theta(d(x, y)) + F(d(y, Ty)) &\leq F(d(x, y)) \\ &\leq F(d(x, Tx)) + \eta(d(x, y)). \end{aligned}$$

Then T has a fixed point.

The following example is inspired by [10, Example 2.3].

Example. Let $X = [0, 10]$ be equipped with the standard metric and let $T\colon X \to CL(X)$ be given by

$$Tx = \begin{cases} \{\frac{x}{2}\}, & \text{for } x \neq 6, \\ \{3, 4\} & \text{for } x = 6. \end{cases}$$

Then

$$f(x) = d(x, Tx) = \begin{cases} \frac{x}{2}, & \text{for } x \neq 6, \\ 2, & \text{for } x = 6, \end{cases}$$

is lower semicontinuous, i.e., condition (i) holds true. Take $\theta(t) = \frac{2}{3}$, $\eta(t) = \frac{1}{2}$ and $F(t) = \ln t$ for $t > 0$. Obviously, θ and η satisfy condition (ii).
In order to prove condition (iii) of Theorem 2.2, let $x \in X \setminus \{0\}$, so that $d(x, Tx) > 0$, and consider two cases:

1° If $x \neq 6$, then $F_\eta^x = \{\frac{x}{2}\}$ since for $y = \frac{x}{2}$,

$$F(d(x, y)) = \ln \frac{x}{2} \leq \ln \frac{x}{2} + \frac{1}{2} = F(d(x, Tx)) + \eta(d(x, y)).$$

For these x, y we have

$$\theta(d(x, y)) + F(d(y, Ty)) = \frac{2}{3} + \ln \frac{y}{2} = \ln \left(e^{2/3} \cdot \frac{x}{4} \right) \leq \ln \left(\frac{x}{2} \right) = F(d(x, y)),$$

since $e^{2/3} < 2$.

2° If $x = 6$, one can take $y = 3 \in Tx$. Then $\ln 3 \leq \ln 2 + \frac{1}{2}$ implies that $y \in F_\eta^x$ and $\frac{2}{3} + \ln \frac{3}{2} \leq \ln 3$ implies that (iii) holds true.

All the conditions of Theorem 2.2 are satisfied and T has a fixed point (which is $z = 0$).

Note that it was shown in [10, Example 2.3] that in this situation the conditions of some other known fixed point theorems are not fulfilled.

Our second result is related to multivalued mappings T on the metric space X, where Tx is compact for all $x \in X$. If we take $K(X)$ (the set of all non-empty compact subsets of X) instead of $CL(X)$ in Theorem 2.1, we can remove condition (F4) on F. Further, by taking into account Case 1, we can take $\eta \geq 0$. Therefore, the proof of the following theorem is obvious.

Theorem 2.3. *Let (X, d) be a complete metric space, $T\colon X \to K(X)$ and $F \in \mathfrak{F}$. Assume that the following conditions hold:*

(i) *the mapping $x \mapsto d(x, Tx)$ is lower semi-continuous;*
(ii) *there exist functions $\theta\colon (0, \infty) \to (0, \infty)$ and $\eta : (0, \infty) \to [0, \infty)$ such that*

$$\theta(t) > \eta(t), \quad \liminf_{s \to t^+} \theta(s) > \liminf_{s \to t^+} \eta(s) \text{ for all } t \geq 0;$$

and for any $x \in X$ *with* $d(x, Tx) > 0$, *there exists* $y \in F_\eta^x$ *satisfying*

$$\theta(M(x,y)) + F(d(y,Ty)) \le F(d(x,y)).$$

Then T *has a fixed point.*

Also, we have a version of the above results in spaces equipped with partial order. As usual, $(X, d, \preceq)$ will be called an ordered metric space if:

(i) (X, d) is a metric space,
(ii) $(X, \preceq)$ is a partially ordered set.

Elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. For $x \in X$ with $d(x, Tx) > 0$, define a set $F_{\eta,\preceq}^x \subseteq X$ as

$$F_{\eta,\preceq}^x = \{\, y \in Tx : F(d(x,y)) \le F(\max\{d(x,Tx), d(x,Ty)\} + \eta(M(x,y)),\ x \preceq y\}.$$

Theorem 2.4. *Let* $(X, d, \preceq)$ *be a complete ordered metric space,* $T\colon X \to CL(X)$ *and* $F \in \mathfrak{F}_*$. *Assume that the following conditions hold:*

(i) *the mapping* $x \mapsto d(x, Tx)$ *is lower semi-continuous;*
(ii) *there exist functions* $\theta, \eta\colon (0,\infty) \to (0,\infty)$ *such that*

$$\theta(t) > \eta(t), \quad \liminf_{s\to t^+} \theta(s) > \liminf_{s\to t^+} \eta(s) \text{ for all } t \ge 0;$$

(iii) *for any* $x \in X$ *with* $d(x, Tx) > 0$, *there exists* $y \in F_{\eta,\preceq}^x$, *satisfying*

$$\theta(M(x,y)) + F(d(y,Ty)) \le F(d(x,y)).$$

If the condition

$$\begin{cases} \text{if } \{x_n\} \subset X \text{ is a non-decreasing sequence with } x_n \to z \text{ in } X \\ \text{as } n \to \infty, \text{ then } x_n \preceq z \text{ for all } n \end{cases} \tag{2.12}$$

holds, then T *has a fixed point.*

Proof. Following the lines of proof of Theorem 2.1 and definition of $F_{\eta,\preceq}^x \subseteq X$, we can show that $\{x_n\}$ is a Cauchy sequence in $(X, d, \preceq)$ with $x_{n-1} \preceq x_n$ for $n \in \mathbb{N}$. From the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. By the assumption (2.12), $x_n \preceq z$, for all n. The rest follows in the same way as in the proof of Theorem 2.1.

Theorem 2.5. *Let* $(X, d, \preceq)$ *be a complete ordered metric space,* $T\colon X \to K(X)$ *and* $F \in \mathfrak{F}$. *Assume that the following conditions hold:*

(i) *the mapping* $x \mapsto d(x, Tx)$ *is lower semi-continuous;*
(ii) *there exist functions* $\theta\colon (0,\infty) \to (0,\infty)$ *and* $\eta : (0,\infty) \to [0,\infty)$ *such that*

$$\theta(t) > \eta(t), \quad \liminf_{s\to t^+} \theta(s) > \liminf_{s\to t^+} \eta(s) \text{ for all } t \ge 0;$$

and for any $x \in X$ *with* $d(x, Tx) > 0$, *there exists* $y \in F_{\eta,\preceq}^x$, *satisfying*

$$\theta(M(x,y)) + F(d(y,Ty)) \le F(d(x,y)).$$

Then T *has a fixed point provided* (2.12) *holds.*

3. Application

In this section we are going to apply the obtained results to the problem of existence of solutions for a Fredholm-type integral inclusion. Problems of this kind were treated by several researchers, see, e.g., [11, 12, 9].

Consider the integral inclusion

$$x(t) \in g(t) + \int_a^b K(t, s, x(s))\, ds, \quad t \in [a, b], \tag{3.1}$$

where $g \in X = C[a, b]$ is a given function, $K\colon [a, b] \times [a, b] \times \mathbb{R} \to CL(\mathbb{R})$ is a given set-valued mapping and $x \in X$ is the unknown function. Here, $X = C[a, b]$ is the standard Banach space of continuous real functions with the maximum norm.

Denote by $T\colon X \to CL(X)$ the operator given by

$$Tx(t) = g(t) + \int_a^b K(t, s, x(s))\, ds, \quad t \in [a, b].$$

Obviously, $x \in X$ is a solution of the inclusion (3.1) if and only if x is a fixed point of operator T.

Let $x \in X$ be arbitrary and suppose that the set-valued operator

$$K_x(t, s)\colon [a, b] \times [a, b] \to CL(\mathbb{R}),\ K_x(t, s) := K(t, s, x(s)),\ (t, s) \in [a, b]^2$$

is continuous (w.r.t. Hausdorff-Pompeiu metric on $CL(\mathbb{R})$).
It follows from the Michael's selection theorem that there exists a continuous operator $k_x : [a, b] \times [a, b] \to \mathbb{R}$ such that $k_x(t, s) \in K_x(t, s)$ for each $(t, s) \in [a, b] \times [a, b]$. Hence,

$$g(t) + \int_a^b k_x(t, s)\, ds \in Tx,$$

i.e., $Tx \neq \emptyset$. Since g and K_x are continuous on $[a, b]$, resp. $[a, b]^2$, their ranges are bounded and hence Tx is bounded, i.e., indeed, $T : X \to CL(X)$.

The following existence result is a consequence of Theorem 2.2 (with $F(t) = t + \ln t$).

Theorem 3.1. *Suppose that the following hold:*

(I) *For each $x \in C[a, b]$, the mapping $K_x(t, s) := K(t, s, x(s))$, $(t, s) \in [a, b] \times [a, b]$ is continuous;*

(II) *The mapping $x \mapsto d(x, Tx)$ is lower semi-continuous;*

(III) *There exist functions $\theta, \eta\colon (0, \infty) \to (0, \infty)$ such that*

$$\theta(t) > \eta(t), \quad \liminf_{s \to t^+} \theta(s) > \liminf_{s \to t^+} \eta(s) \text{ for all } t \geq 0;$$

(IV) *For each $x \notin Tx$ there exists $y \in Tx$ such that*

$$d(x, y) + \ln d(x, y) \leq d(x, Tx) + \ln d(x, Tx) + \eta(d(x, y))$$

and

$$\theta(d(x, y)) + d(y, Ty) + \ln d(y, Ty) \leq d(x, y) + \ln d(x, y)$$

hold.

Then the integral inclusion (3.1) *has a solution in* $C[a, b]$.

A similar result can be deduced using the "ordered" version of our results (Theorem 2.4).

Open problem. Can the results of this article be extended for F-Suzuki type contractions in b-metric spaces, as it was done in the recent paper [5]?

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