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AN ULTRA-PRODUCT METHOD VIA LEFT REVERSIBLE SEMIGROUPS TO STUDY BRUCK'S GENERALIZED CONJECTURE

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Abstract. It has been asked by Lau several times whether a Banach space with weak fixed point property has weak fixed point property for left reversible semigroups. This problem is known as Bruck generalized conjecture (BGC). The aim of this note is to propose a new approach to tackle the BGC. Our approach uses the order structure of the semigroup for the first time in literature to construct an ultra-product structure. Then, we use this ultra-product structure to give an affirmative answer to BGC for the case of nearly uniformly convex (NUC) Banach spaces. One should note that alternatives proofs are available in the case of NUC Banach spaces, but what we hope for is that the originality of our method could pave the way for studying the BGC in its utmost generality.
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1. Introduction

Let K be a subset of a Banach space E. A self mapping T on K is said to be *non-expansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. We say that E has the *weak fixed point property (weak fpp)* if for every weakly compact convex non-empty subset K of E, any non-expansive self mapping on K has a fixed point.

Let S be a *semi-topological semigroup*, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous. S is called *left reversible* if any two closed right ideals of S have non-void intersection.

An *action* of S on a subset K of a topological space E is a mapping $(s, x) \mapsto s(x)$ from $S \times K$ into K such that $(st)(x) = s(t(x))$ for $s, t \in S, x \in K$. The action is *separately continuous* if it is continuous in each variable when the other is kept fixed. We say that S has a *common fixed point* in K if there exists a point x in K such that $sx = x$ for all $s \in S$. When E is a normed space, the action of S on K is *non-expansive* if $\|s(x) - s(y)\| \leq \|x - y\|$ for all $s \in S$ and $x, y \in K$. There are also other types of action for a semi-topological semigroup (see [1] and [4]).

We say that a Banach space E has the *weak fpp for left reversible semigroups* if for every weakly compact convex non-empty subset K of E, any non-expansively separately continuous action of a semi-topological semigroup S on K has a fixed point.

One of the celebrated results in fixed point theory is due to Bruck [2]. He has shown that if a Banach space E has weak fpp, then it has weak fpp for abelian semigroups. Now, we call the following statement *Bruck's Generalized Conjecture (BGC)*:

(BGC) If a Banach space E has weak fpp, then it has weak fpp for any left reversible semi-topological semigroup S.

The above statement has been brought up as an open problem several times by Lau, for example see [5]. In [7], it has been shown that BGC is true for the preduals of von Neumann algebras. It turns out that to study BGC one should use the order structure of the given semigroup in certain ways. For the first time, we have done this by using a new method for a special class of Banach spaces. What is important for us here is the method of the proof not the theorem itself since alternative proofs are available for our theorem. But, our proof uses the natural order structure of the semigroup for the first time and construct an ultra-product of the Banach space based on the order of the semigroup. We hope that our approach can be altered to tackle Bruck's generalized conjecture in its full statement.

2. Weak fixed point property of Bruck type

A Banach space E is called *nearly uniformly convex (NUC),* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every sequence (x_n) in the closed unit ball of E with $sep(x_n) := \inf\{\| x_n - x_m\| : n \neq m\} > \epsilon$ the distance $dist(0, co\{x_n\})$ is strictly less than $1 - \delta$ where $co\{x_n\}$ denotes the convex hull of the sequence.

When S is a left reversible semigroup, we make it to a directed set by declaring: $\alpha \geq \beta$ if and only if $\alpha S \subseteq \overline{\beta S}$. Thus, we can use S as an index set for nets and speak about limit and limit-supremum with respect to this directed set. Also, note that when S acts on a weakly compact subset K of Banach space E, each α acts non-expansively on K and sometimes we use the notation T_α instead of α; even, we may use $\alpha.\beta$ to denote the composition $T_\alpha \circ T_\beta$.

Let $l_\infty(E) = \{x = (x_\alpha) : x_\alpha \in E; \quad \|x\| = \sup\|x_\alpha\| < \infty\}$, and

$$\mathcal{N} = \{x = (x_\alpha) : x_\alpha \in E; \quad \lim_\alpha \|x_\alpha\| = 0\}.$$

Put $\widetilde{E} = l_\infty(E)/\mathcal{N}$ and endow it with the quotient norm $\|[(x_\alpha)]\| = \limsup_\alpha \|x_\alpha\|$. One can embed E and its subsets into $\widetilde{E}$ by using constant classes. For example, for $x \in K$ let $\dot{x} = [(x)]$ denotes the equivalence class containing the constant net $(..., x, x, x, ...)$. So, $\dot{K} = \{\dot{x} : x \in K\}$ is a subset of $\widetilde{E}$. Also, we define

$$\widetilde{K} = \{[(k_\alpha)] \in \widetilde{E} : k_\alpha \in K \quad \text{for each } \alpha\}$$

and note that, $\dot{K} \subseteq \widetilde{K}$. The process of embedding preserves the properties of being closed, bounded and convex for subsets of E. If each T_α is non-expansive on K, then the mega mapping $\widetilde{T} : \widetilde{K} \longrightarrow \widetilde{K}$ defined by $\widetilde{T}[(x_\alpha)] = [(T_\alpha x_\alpha)]$ is also non-expansive.

Another piece of notation, when (x_k) is a sequence in E, then $f_k = [(\alpha.x_k)_\alpha]$ is a sequence in $\widetilde{E}$.

Definition 2.1. (a) Let K be a non-empty subset of a Banach space E and $(x_\alpha)_{\alpha\in S}$ be a bounded net in E. Noting that $k \in K, \alpha \in S$ define

$$r(K,(x_\alpha)) = \inf\{\limsup_\alpha \| x_\alpha - k \| : k \in K\}$$

$$AC(K,(x_\alpha)) = \{k \in K : \limsup_\alpha \| x_\alpha - k \| = r(K,(x_\alpha))\}$$

The set $AC(K,(x_\alpha))$ (the number $r(K,(x_\alpha))$) will be called the *asymptotic center (asymptotic radius)* of $(x_\alpha)_{\alpha\in S}$ in K. These are the generalizations of Chebyshev center and radius and are due to Edelstein[3]. The asymptotic center is always non-empty for weakly compact set K.

(b) When viewed in $\widetilde{E}$, these notions are seen as:

$$r(\dot{K},[(x_\alpha)]) = \inf\{\| [(x_\alpha)] - \dot{k} \| \; k \in K\},$$

$$AC(\dot{K},[(x_\alpha)]) = \{\dot{k} \in \dot{K} : \| [(x_\alpha)] - \dot{k} \| = r(\dot{K},[(x_\alpha)])\}.$$

The proof of the following theorem uses some ideas from [8, section 5]. As we mentioned earlier, it is the method of the proof which is important for us not the theorem itself!

Theorem 2.2. *Let K be a bounded closed convex non-empty subset of a Banach space E, and let S be a left reversible semigroup acting non-expansively and separately continuous on K. Suppose there exists a $\lambda \in [0,1)$ such that for each net (x_α) in K and each net (y_α) in $AC(K,(x_\alpha))$ we have $r(K,(y_\alpha)) \le \lambda r(K,(x_\alpha))$. Then, S has a common fixed point in K.*

Proof. Let $x_0 \in K$. Put $r_0 = r(K,(\alpha.x_0))$ and $A_1 = AC(K,(\alpha.x_0))$. Let $x_1 \in A_1$. Since the action is non-expansive we see that $\alpha.x_1 \in A_1$ and $[(\alpha.x_1)] \in \widetilde{A_1}$. Put $r_1 = r(K,(\alpha.x_1))$ and $A_2 = AC(K,(\alpha.x_1))$. By induction, we get a sequence x_k such that $x_k \in A_k$, $[(\alpha.x_k)] \in \widetilde{A_k}$, $r_k = r(K,(\alpha.x_k))$, $A_{k+1} = AC(K,(\alpha.x_k))$ and $r_k \le \lambda^k r_0$. As in the proof of [8, Theorem 5.3], there exists a $u \in K$ such that the sequence $f_k = [(\alpha.x_k)]$ converges to $\dot{u}$ in $\widetilde{K}$. Note that the use of $\alpha \in S$ instead of $n \in \mathbb{N}$ in those theorems for the part we need, cause no problem since the convergence occurs in k. So, the limit of $[(\alpha.x_k)]$ is $[(u)]$ regardless of the indexing αs. Let $\beta \in S$ be arbitrary and consider the elements $[(\beta.\alpha.x_k)]$ and the constant element $[(\beta.u)]$ which are indexed by αs. Since the action is non-expansive we get the $\|[(\beta.\alpha.x_k)] - [(\beta.u)]\| \le \|[(\alpha.x_k)] - [(u)]\|$. Hence, the sequence $[(\beta.\alpha.x_k)]$ converges to $[(\beta.u)]$ in $\widetilde{K}$. But, $\beta.\alpha \in S$, so by the first part of the current proof, $[(\beta.\alpha.x_k)]$ is a another sequence like $[(\alpha.x_k)]$ with the same limit. That is, by the same discussion, $[(\beta.\alpha.x_k)]$ must have the limit $[(\beta.u)] = [(u)]$. Hence, $(\beta.u - u)_\alpha \in \mathcal{N}$. Therefore, $\beta \cdot u = u$, which is the desired result. □

Now, we are ready to prove an important theorem in light of Theorem 2.2. We give two proofs of it. The first one is new and is based on Theorem 2.2 while the second proof is a classic one based on Lim's fixed point theorem [6].

Corollary 2.3. *Let K be a weakly compact convex non-empty subset of a nearly uniformly convex Banach space E, and let S be a left reversible semigroup acting non-expansively and separately continuous on K. Then, S has a common fixed point in K.*

Proof 1. The set K is closed, bounded and convex. Also, nearly uniformly convex Banach spaces satisfy the inequality in Theorem 2.2. So, the result follows.

Proof 2. Every nearly uniformly convex Banach space has normal structure. So, an application of Lim's fixed point theorem [6] will do the job. □

Though the above corollary is an extension of Kirk's classic theorem and is a special case of Lim's fixed point theorem, one should not underestimate the method of its proof. Above all, it is a common practice in mathematics to prove an old problem from a new perspective and it turns out the new view has many ramifications.

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