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ON BEST PROXIMITY PAIRS WITH APPLICATION TO DIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider the following system of differential equations,

$$y' = f(x,y),\ y(x_0) = y_1 \text{ and } z' = g(x,z),\ z(x_0) = z_1,$$

where $f,\ g$ are bounded L^1 functions defined on a rectangle in $\mathbb{R}^2$. We give sufficient conditions for the existence of two functions ϕ and ψ, on an interval I containing x_0, such that

$$|y_1 + \int_{x_0}^{x} f(t,\phi(t))dt - \phi(x)| \leq |y_1 - z_1|,$$

$$|z_1 + \int_{x_0}^{x} g(t,\psi(t))dt - \psi(x)| \leq |y_1 - z_1|$$

for all $x \in I$. To establish the same, we introduce a notation of c-cyclic contractive mapping and prove the existence of best proximity pairs for such a mapping.
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1. Introduction

Let (A,B) be a pair of subsets of a metric space and T be a mapping from A into B. Fixed point theorems analyze conditions for the existence of a solution for the fixed point equation $d(x,Tx) = 0$. If the fixed point equation does not posses any solution then it is natural to explore the optimal solution for the real valued function $x \to d(x,Tx)$ on a suitable domain space A. Ky Fan in 1969 [6] established a fundamental result for the existence of a point $x_0 \in A$, under certain conditions, such that

$$\|x_0 - Tx_0\| = \text{dist}(Tx_0, A).$$

The most optimal solution for the problem of minimizing the real valued function $x \to d(x,Tx)$ is the one which attains the value at

$$\text{dist}(A,B) := \{d(u,v) : u \in A,\ v \in B\}.$$

A pair $(x,Tx) \in A \times B$ is said to be a best proximity pair for the map T if $d(x,Tx) = \text{dist}(A,B)$. Such a point x, if it exists, is said to be best proximity point of T. In view of this stand point, many researchers [2, 3, 5, 7, 10, 12, 14] expound the condition that asserts the existence of a best proximity point for a similar kind of mappings.

Suppose $T : A \cup B \to A \cup B$ satisfying $TA \subseteq B$ and $TB \subseteq A$. In the application point of view, it is natural interest to explore the necessary conditions, either on A and B or on T, for the sequence(s) $\{T^{2n}x\}$ (and/or $\{T^{2n+1}x\}$), for any $x \in A \cup B$, converge to best proximity points. In this manuscript we introduce a class mappings called c-contractive mappings and prove the existence of best proximity pairs for such a mapping. Our results include some known existence theorems from the existing literature.

As an application, we consider the following system of differential equations:

$$y' = f(x, y),\ y(x_0) = y_1 \text{ and } z' = g(x, z),\ z(x_0) = z_1,$$

where $f,\ g$ are functions on a suitable rectangle in $\mathbb{R}$ containing (x_0, y_1) and (x_0, z_1). If $f = g$, then we have $y_1 = z_1$. In this case, the both IVPs considers and Picard's existence and uniqueness theorem for the existence of the solution of the IVP, if f and g are Lipschitz functions. Also the celebrated Piano's theorem ensures the existence of solution for a differential equation, if the functions f and g is continuous (for more details reader can refer [1, 4, 8]). In any case, there exists a function ϕ on a neighborhood I of x_0 such that

$$\int_{x_0}^{x} f(t, \phi(t))dt - \phi(x) = y_1,$$

It is to be noted that, if $|y_1 - z_1| > 0$, then it is natural to explore the optimal solution for the real valued function

$$\theta(x) = \left| y_1 - \int_{x_0}^{x} f(t, \phi(t))dt - \phi(x) \right|.$$

In this stand point we consider f and g are two bounded functions on L_1, the space of absolute integrable functions on the rectangle. We prove that there exists an interval I containing x_0 and two continuous functions ϕ and ψ on I such that

$$\left| y_1 + \int_{x_0}^{x} f(t, \phi(t))dt - \phi(x) \right| \leq |y_1 - z_1| \text{ for all } x \in I,$$

$$\left| z_1 + \int_{x_0}^{x} g(t, \psi(t))dt - \psi(x) \right| \leq |y_1 - z_1| \text{ for all } x \in I.$$

It is to be noted that if $y_1 = z_1$ then (ϕ, ψ) turns out to be a solution the system of differential equations (y', z').

2. Existence of best proximity pairs

Let A and B be two subsets of a metric space (X, d). Let T is a mapping on $A \cup B$ satisfying $TA \subseteq B$ and $TB \subseteq A$. If T is a contractive mapping, that is $d(Tx, Ty) < d(x, y)$ for all $x,\ y$ in $A \cup B$ and further if A and B are compact then it is easy to notice that $A \cap B \neq \emptyset$ and T has a unique fixed point in $A \cap B$. Also if $d(Tx, Ty) < d(x, y)$, for all $x \in A$ and $y \in B$ and $A,\ B$ are compact, then $A \cap B \neq \emptyset$ and T has a unique fixed point in $A \cap B$. In [9], the authors defined cyclic contractive mappings and proved the existence of a best proximity pair for such a mapping in the setting a strictly convex Banach space. Also it is to be noted that the existing best proximity results mostly depend on the concept of strictly convex space. Motivated by this fact, we present the following definition.

Definition 2.1. Let A and B be subsets of metric space (X, d). A mapping $T : A \cup B \to A \cup B$ with $TA \subseteq B$ and $TB \subseteq A$ is said to be *c-contractive mapping* if T is continuous and

$$d(Tx, Ty) < d(x, y), \text{ whenever } d(x, y) > \operatorname{dist}(A, B) \text{ for } x \in A,\ y \in B.$$

It is observed form the continuity property of the c-contractive mapping on $A \cup B$ that, $d(Tx, Ty) = \operatorname{dist}(A, B)$, for every $x \in A$ and $y \in B$ with $d(x, y) = \operatorname{dist}(A, B)$. Hence, if T is c-contractive mapping on $A \cup B$ then $TA_0 \subseteq B_0$ and $TB_0 \subseteq A_0$, where

$$A_0 = \{x \in A : d(x, y) = \operatorname{dist}(A, B), \text{ for some } y \in B\} \text{ and}$$
$$B_0 := \{y \in B : d(x, y) = \operatorname{dist}(A, B) \text{ for some } x \in A\}.$$

Therefore, we have $d(Tx, Ty) \leq d(x, y)$ for all $x \in A, y \in B$.

Definition 2.2. [13] A nonempty subset A of a metric space (X, d) is said to be *approximatively compact* if for any y in X and any sequence $\{x_n\}$ in A with $d(x_n, y)$ converges to $\operatorname{dist}(y, A)$ then $\{x_n\}$ has a convergent subsequence.

It is evident from the above definition that every boundedly compact subset of a metric space is approximatively compact. Also it was proved in [9] that, if B is compact and A is approximatively compact subsets of a metric space, then $A_0,\ B_0$ are nonempty compact subsets of X.

Definition 2.3. [11] A pair (A, B) of nonempty subsets of a metric space (X, d) is said to have *projectional property* if for $(x, y) \in A \times B$ with $d(x, y) = \operatorname{dist}(A, B)$ and for any sequences $\{x_n\}$ in A, $\{y_n\}$ in B satisfying

$$d(x_n, y) \to \operatorname{dist}(A, B),\ d(x, y_n) \to \operatorname{dist}(A, B) \text{ as } n \to \infty,$$

then $x_n \to x$ and $y_n \to y$.

It is to be observed that if (A, B) satisfies projectional property, then for any $x \in A$ there exists atmost one $y \in B$ such that $d(x, y) = \operatorname{dist}(A, B)$

Lemma 2.4. *Let A be an approximatively compact and B is compact be subsets of a metric space (X, d). Suppose that (A, B) satisfies projectional property. If T is a cyclic contractive map, then T is c-contractive on $A_0 \cup B_0$.*

Proof. Suppose $\{u_n\}$ converges to u in A_0 (or in B_0), then there exists a unique v in B_0 (or in A_0) such that $d(u, v) = \operatorname{dist}(A, B)$. Now

$$\begin{aligned} d(Tu_n, Tv) &\leq d(u_n, v) \\ &\leq d(u_n, u) + d(u, v) \to \operatorname{dist}(A, B). \end{aligned}$$

Hence $d(Tu_n, (Tu)') \to \operatorname{dist}(A, B)$. In a similar fashion one can prove that $d((Tu_n)', Tu) \to \operatorname{dist}(A, B)$. Since (A, B) has projectional property, $Tu_n \to Tu$. Hence T is c-contractive. $\square$

Let $X = \mathbb{R}^2$ with $\|\cdot\|_\infty$. If

$$A := \{(0,x) : 0 \leq x \leq 1\} \text{ and}$$
$$B := \{(0,x) : 1 \leq x \leq 1\}$$

then every cyclic map on $A \cup B$ is cyclic contractive. Also it is to be noted that every point on $A \cup B$ is a best proximity point for T.

Now we prove the existence of a best proximity pair for such a map in the setting of a metric space.

Theorem 2.5. *Let A be an approximatively compact and B be a compact subsets of a metric space (X,d). Suppose T is a c-contractive map on $A \cup B$. Then there exists a best proximity pair of T. Further for any sub-sequence of x_n, where $x_n = T(x_{n-1})$ for any $x_0 \in A$, converges to x in A with (x, Tx) is a best proximity pair.*

Proof. Define $f : A_0 \to \mathbb{R}^+$ as

$$f(u) = d(u, Tu), \quad \text{for all} \quad u \in A_0.$$

As f is continuous and A_0 is compact, there exists $v \in A_0$ such that

$$d(v, Tv) = \inf\{d(u, Tu) : u \in A\}.$$

Now we claim that $d(v, Tv) = \text{dist}(A, B)$. Suppose not. Then we have

$$d(Tv, T^2v) < d(v, Tv).$$

Now $T^2v \in A_0$ and so

$$\begin{aligned} d(v, Tv) &= \inf_{u \in A_0} d(u, Tu) \\ &\leq d(T^2v, T^3v) \\ &\leq d(Tv, T^2v) \\ &< d(v, Tv), \end{aligned}$$

a contradiction. Hence (v, Tv) is a best proximity point of T in A. Now choose an $x_0 \in A_0$ and define $x_n = T(x_{n-1})$ for all $n \in \mathbb{N}$. As $A_0 \cup B_0$ is compact, the sequence $\{x_n\}$ convergent subsequence say $\{x_{n_k}\}$ that converges to x. Suppose both

$$\{n_k : d \in \mathbb{N}\} \cap \{2n : n \in \mathbb{N}\} \text{ and } \{n_k : d \in \mathbb{N}\} \cap \{2n+1 : n \in \mathbb{N}\}$$

are infinite sets, then $A \cup B \neq \emptyset$. Hence $\text{dist}(A, B) = 0$ and in this case x turn outs to be a fixed point of T and this completes the proof. Hence without loss generality assume that n_k is even for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} x_{n_k} = x$. Then $x \in A_0$. Also,if $d(Tx_{n_k}, Tx) = \text{dist}(A, B)$ for all n_k except finitely many, then we have

$$d(x, Tx) = \text{dist}(A, B),$$

which completes the proof. So we assume that $d(Tx_{n_k}, Tx) > \text{dist}(A, B)$ for all n_k. This gives us that $d(T(x_{n_k}), T(x_{n_k+1})) > \text{dist}(A, B)$ for all k. Suppose

$$\alpha = \text{dist}(x, Tx) := \inf\{d(T(x_{n_k}), T(x_{n_k+1})\} < \text{dist}(A, B).$$

Then now

$$\text{dist}(x, Tx) > \text{dist}(T^{n_1}x, T^{n_1+1}x),$$

which is a contradiction. Hence $d(x,Tx) = \text{dist}(A,B)$. This completes the proof. $\square$

Now we state an alternate statement of the above, which we use in the sequel.

Theorem 2.6. *Let A and B be two closed closed convex subsets of a normed linear space with dist$(A,B) > 0$ and T is a c-contractive map on $A\cup B$. Suppose TA lies in compact subest of X and TB lies in a compact subset of X. Then T has a best proximity pair. Further for any $x_0 \in A$, define $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. Then $\{x_{2n}\}$ has a convergent subsequence that converges to x and $\{x_{2n+1}\}$ has a convergent subsequence that converges to y such that (x,Tx) and (y,Ty) best proximity pairs of T.*

As an immediate consequence of Theorem 2.5, we have the following:

Corollary 2.7. *Let A and B be nonempty subsets of metric space X with either A or B be compact and let T be a cyclic map. If T is a cyclic contractive map on $A\cup B$, then their exists $x\in A$ and $y \in B$ such that $d(x,Tx) = dist(A,B) = d(x,y)$.*

Following example shows that the assumption of compactness on the sets A and B can not be dropped in Theorem 2.5.

Example 2.8. Let $X = \ell_p, 1 \le p \le \infty$ and

$$A := \left\{(1+1)e_1, \left(1+\frac{1}{3}\right)e_3, \dots\right\},$$
$$B := \left\{\left(1+\frac{1}{2}\right)e_2, \left(1+\frac{1}{4}\right)e_4, \dots\right\},$$

where e_n the sequence consisting of 1 at n^{th} place and rest of them are 0's. Define T on $A\cup B$ by

$$T\left(\left(1+\frac{1}{n}\right)e_n\right) = \left(1+\frac{1}{n+1}\right)e_{n+1}.$$

It is clear that T is a cyclic map for odd n and even m.

$$\begin{aligned}\left\|T\left(\left(1+\frac{1}{n}\right)e_n\right) - T\left(\left(1+\frac{1}{m}\right)e_m\right)\right\| &= \left\|\left(1+\frac{1}{n+1}\right)e_{n+1} - \left(1+\frac{1}{m+1}\right)e_{m+1}\right\| \\ &= \left(\left(1+\frac{1}{n+1}\right)^p + \left(1+\frac{1}{m+1}\right)^p\right)^{\frac{1}{p}} \\ &< \left(\left(1+\frac{1}{n}\right)^p + \left(1+\frac{1}{m}\right)^p\right)^{\frac{1}{p}} \\ &= \left\|\left(1+\frac{1}{n}\right)e_n - \left(1+\frac{1}{m}\right)e_m\right\|\end{aligned}$$

Also it is to be observed that, either if X is strictly convex Banach space or if (A,B) has the projectional property, then we have a unique best proximity pair of T. Further for any sub-sequence of x_n, where $x_n = T(x_{n-1})$ for some $x_0 \in A$, converges to the best proximity point x of T in A (see [9]). With this observation and Example 2.8 we conjuncture the following:

Conjecture 2.9. *Let A and B be two convex compact subsets of a normed linear space X and T be a c-contractive map on $A \cup B$. Suppose*

$$dist(A,B) < \delta(A,B) := \sup\{d(x,y) : x \in A, y \in B\}.$$

Then there exists a unique best proximity pair of T. Further for any sub-sequence of x_n, where $x_n = T(x_{n-1})$ for any $x_0 \in A$, converges to the best proximity point x of T in A.

3. Application to system of differential equations

Let x_0 and y_0 in $\mathbb{R}$ and $a,\ b > 0$ and let $S := [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. Consider a pair of differential equations with initial conditions:

$$y' = f(x,y) \text{ with initial condition } y(x_0) = y_1 \text{ and}$$
$$z' = f(x,z) \text{ with initial condition } y(x_0) = z_1.$$

where $f,\ g$ are integrable functions from S to $\mathbb{R}$ and $x, z, z_1 \in [y_0 - b, y_0 + b]$.

Now we discuss the existence of the best proximity solution for the system, if it does not posses any solution. We say that Φ and Ψ is a best proximity solution for the system, if it satisfying the following:

(1) Φ and Ψ are continuous functions on a neighborhood I of x_0

(2) $\left| y_1 + \int_{x_0}^{x} f(t, \Phi(t))dt - \Phi(x) \right| \le |y_1 - z_1|$, for all $x \in I$

(3) $\left| z_1 + \int_{x_0}^{x} g(t, \Phi(t))dt - \Phi(x) \right| \le |y_1 - z_1|$, for all $x \in I$ and

It is easy to see that if $y_0 = z_1$ then a best proximity solution turns out to be a common solution for the system.

The following theorem ensures the existence of a best proximity solution.

Theorem 3.1. *Let $x_0, y_0, y_1, z_1, a, b, S$ be as stated above. Suppose f and g are bounded L_1 functions on S satisfying satisfies*

$$|f(x,t_1) - g(x,t_2)| \le |(t_1 - t_2) - (y_1 - z_1)|,$$

for all $x \in [x_0 - a, x_0 + a]$ and $t_1,\ t_2 \in [y_0 - b, y_0 + b]$ with $|t_1 - t_2| \ge |y_1 - z_1|$. Then there exists $\beta > 0$ and continuous functions ϕ and ψ on $[x_0 - \beta, x + \beta]$ such that ϕ and ψ is a best proximity solution for the system.

Proof. Let

$$\beta = \frac{1}{2} \min\left\{1, a, \frac{b - |y_1 - y_0|}{M}, \frac{b - |z_1 - y_0|}{M}\right\},$$

where M be a common bounded of f and g. Then it is easy to $\beta > 0$. Let

$$I = [x_0 - \beta, x_0 + \beta]$$

and $X = \mathcal{C}[I]$ be the space of all real valued continuous functions on I with supremum norm and set

$$A : \{h \in X : h(x_0) = y_1 \text{ and } |h(x) - y_1| < b,\ \forall\ x\ \in\ I\},$$
$$B : \{h \in X : h(x_0) = z_1 \text{ and } |h(x) - z_1| < b,\ \forall\ x\ \in\ I\}.$$

Then it is easy see that A and B are closed convex subsets of X with

$$\mathrm{dist}(A,B) = |y_1 - z_1|.$$

Now let us define a map T on $A \cup B$ by

$$Th(x) = \begin{cases} z_1 + \displaystyle\int_{x_0}^{x} g(u,h(u))du & \text{if } f \in A \\ y_1 + \displaystyle\int_{x_0}^{x} f(u,h(u))du & \text{if } f \in B \end{cases}$$

It is evident from the above that $Th \in X$ for all $h \in X$. Also for $h \in A$, We have

$$Th(x_0) = z_1$$

and

$$\begin{aligned} |Th(x) - y_0| &= |z_1 + \int_{x_0}^{x} g(u,h(u))du - y_0| \\ &\leq |z_1 - y_0| + |\int_{x_0}^{x} g(u,h(u))du| \\ &\leq |z_1 - y_0| + M\beta \\ &< b, \end{aligned}$$

therefore $Th \in B$. In a similar way one can show that $Th \in A$, if for $h \in B$. Also it is to be noted that, as the integral operator is continuous, T is continuous. Now for any $h \in A$ and $k \in B$ with $\|h - k\| > |y_1 - z_1|$.

$$\begin{aligned} |Th(x) - Tk(x)| &= |z_1 + \int_{x_0}^{x} g(u,h(u))du - y_1 + \int_{x_0}^{x} f(u,k(u))du| \\ &\leq |z_1 - y_1| + \int_{x_0}^{x} |g(u,h(u)) - f(u,k(u))|du \\ &< |z_1 - y_1| + \int_{x_0}^{x} \big(\|h-k\| - |y_1 - z_1|\big)du \\ &\leq |z_1 - y_1| + \beta\big(\|h-k\| - |y_1 - z_1|\big) \\ &\leq \beta\|h-k\| + (1-\beta)|z_1 - y_1| \\ &< \beta\|h-k\| + (1-\beta)\|h-k\|. \end{aligned}$$

Therefore $|Th(x) - Tk(x)| < \|h-k\| + (1-\beta)\|h-k\|$, so

$$\|Tf - Tg\| \leq \|h-k\| + (1-\beta)\|h-k\|.$$

As $\|h-k\| > |y_1 - z_1|$, we have

$$\|Tf - Tg\| < \|h-k\|.$$

This show that T is a c-contractive map on $A \cup B$.

Now for $h \in A$, we have $|Th(x)| \leq z_1 + M\beta$. Therefore $\{Th : h \in A\}$ is a uniformly bounded subset of X. Also for $x_1 < x_2 \in I$ and for $h \in A$, we have

$$\begin{aligned} |Tf(x_1) - Tf(x_2)| &= \left| z_1 + \int_{x_0}^{x_1} g(u, h(u))du - z_1 + \int_{x_0}^{x_2} g(u, h(u))du \right| \\ &= \left| \int_{x_1}^{x_1} g(u, h(u))du \right| \\ &\leq \int_{x_1}^{x_1} |g(u, h(u))|du \\ &= M(x_2 - x_1). \end{aligned}$$

Therefore $\{Th : h \in A\}$ is a equi-continuous family. Therefore by Ascoli theorem, $T(A)$ lies in a compact subset of X. In a similar way one can show that $T(B)$ lies in a compact subset of X. Hence by Theorem 2.6, there exists continuous functions ϕ and ψ on I such that ϕ and ψ is a best proximity solution for the system. For $h \in A$, define $h_n = Th_{n-1}$. Then by Theorem 2.5, we have a subsequence of h_{2n} that converges to ϕ and a subsequence of h_{2n+1} that converges to ψ. □

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