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A HALPERN TYPE ITERATION WITH MULTIPLE ANCHOR POINTS IN COMPLETE GEODESIC SPACES WITH NEGATIVE CURVATURE

YASUNORI KIMURA* AND KAZUYA SASAKI**

*Department of Information Science, Toho University, Miyama
Funabashi, Chiba 274-8510, Japan
E-mail: yasunori@is.sci.toho-u.ac.jp

**Department of Information Science, Toho University, Miyama
Funabashi, Chiba 274-8510, Japan
E-mail: 6519005s@st.toho-u.jp

Abstract. In this paper, we define a new convex combination on a geodesic space with negative curvature, and show that an iterative sequence generated by using that convex combination converges to a common fixed point of mappings minimizing the specific function of that space.
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1. Introduction

Approximation of fixed points has been studied in various spaces such as Hilbert spaces, Banach spaces, complete CAT(κ) spaces, and others. Halpern type iteration is one of the method to find a fixed point; see [2, 9, 8]. In 2010, Saejung [6] proved a Halpern type approximation theorem using a single mapping and a single anchor point in a complete CAT(0) space. In 2015, Kimura and Wada [4] showed the following theorem using Halpern type iterative scheme with three mappings and three anchor points in a complete CAT(0) space.

Theorem 1.1 (Kimura and Wada [4]). *Let X be a complete* CAT(0) *space and R, S, T nonexpansive mappings from X into itself with $F = F(R) \cap F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\} \subset]0, 1[$ such that*

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty \ \textit{and} \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and $\{\beta_n\}, \{\gamma_n\} \subset]a, b[\subset]0, 1[$ such that

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,\ \lim_{n\to\infty} \beta_n = \beta \in]0, 1[,\ \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$$

and

$$\lim_{n\to\infty}\gamma_n = \gamma \in \,]0,1[.$$

Let $u, v, w, x_1 \in X$ *and define an iterative sequence* $\{x_n\} \subset X$ *by*

$$\begin{cases} r_n = \alpha_n u \oplus (1-\alpha_n)Rx_n, \\ s_n = \alpha_n v \oplus (1-\alpha_n)Sx_n, \\ t_n = \alpha_n w \oplus (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n r_n \oplus (1-\beta_n)(\gamma_n s_n \oplus (1-\gamma_n)t_n) \end{cases}$$

for all $n \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *converges to a point* $p \in F$, *which is a minimizer of the function* $g(x) = \beta d(u,x)^2 + (1-\beta)(\gamma d(v,x)^2 + (1-\gamma)d(w,x)^2)$ *on* F.

In this theorem, the function g can be expressed as

$$g(x) = \lambda d(u,x)^2 + \mu d(v,x)^2 + \nu d(w,x)^2 \quad (\lambda,\mu,\nu > 0,\ \lambda+\mu+\nu = 1).$$

We can consider the function g as a typical function of CAT(0) space, for the following formula is satisfied for any three points x, y, z in a CAT(0) space and $\alpha \in [0,1]$:

$$d(\alpha x \oplus (1-\alpha)y, z)^2 \le \alpha d(x,z)^2 + (1-\alpha)d(y,z)^2.$$

On the other hand, if X is a CAT(-1) space, the following inequality holds for any $x, y, z \in X$ and $\alpha \in [0,1]$:

$$\cosh d(\alpha x \oplus (1-\alpha)y, z) \le \alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z).$$

So the following function can be regarded as a function specific to the CAT(-1) space:

$$h(x) = \lambda \cosh d(u,x) + \mu \cosh d(v,x) + \nu \cosh d(w,x) \quad (\lambda,\mu,\nu > 0,\ \lambda+\mu+\nu = 1).$$

In general, we know that all CAT(-1) spaces are also CAT(0) space. Therefore, the sequence generated by the same method converges to the same point also in a CAT(-1) space. That point is a minimizer of the function g, however, it is not a minimizer of the following function characteristic of the CAT(-1) space:

$$h(x) = \beta \cosh d(u,x) + (1-\beta)(\gamma \cosh d(v,x) + (1-\gamma)\cosh d(w,x)).$$

We consider this problem to be caused by the relationship between convex combination and the geometric structure of CAT(0) space. In fact, the following formula holds for any three points u, v in a CAT(0) space and $\alpha \in [0,1]$:

$$\alpha u \oplus (1-\alpha)v = \operatorname*{argmin}_{x\in X}(\alpha d(u,x)^2 + (1-\alpha)d(v,x)^2).$$

In this paper, we define a new convex combination on a CAT(-1) space in order to resolve that problem, and show that a sequence generated by using that convex combination converges to a minimizer of h in CAT(-1) spaces.

2. Preliminaries

Let (X, d) be a metric space. For $x, y \in X$, a mapping $\gamma\colon [0, l] \to X$ is called a geodesic joining x and y if γ satisfies $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(s) - \gamma(t)) = |s - t|$ for $s, t \in [0, l]$, where $l = d(x, y)$. X is said to be a geodesic space if for any two points $x, y \in X$, there exists a geodesic joining x and y. Further, if a geodesic exists uniquely for any two points $x, y \in X$, then X is called a uniquely geodesic space. In a uniquely geodesic space, an image of geodesic joining x and y is said to be a geodesic segment and is denoted by $[x, y]$.

Let X be a uniquely geodesic space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. The point z is called a convex combination of x and y, and is denoted by $tx \oplus (1-t)y$. For three points $x, y, z \in X$, a geodesic triangle $\triangle(x, y, z) \subset X$ is defined as the union of geodesic segments joining each two points.

For $\kappa \in \mathbb{R}$, let M_κ be a two-dimensional model space with curvature κ. In particular, M_0 is a two-dimensional Euclidean space $\mathbb{R}^2$, M_1 is a two-dimensional unit sphere $\mathbb{S}^2$, and M_{-1} is a two-dimensional hyperbolic space $\mathbb{H}^2$. The diameter of M_κ is denoted by D_κ, that is, $D_\kappa = \infty$ for $\kappa \le 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ otherwise.

Let X be a uniquely geodesic space and let $\kappa \in \mathbb{R}$. For a geodesic triangle $\triangle(x, y, z) \subset X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, a comparison triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z}) \subset M_\kappa$ is defined by $[\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$, where $\bar{x}, \bar{y}, \bar{z}$ are points on M_κ which satisfies $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$. X is called a CAT(κ) space if for any two points $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality $d(p, q) \le d(\bar{p}, \bar{q})$, which is called a CAT$(\kappa)$ inequality, is satisfied for any $\triangle(x, y, z) \subset X$ and its comparison triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z}) \subset M_\kappa$. It is well known that any CAT(κ) space is also a CAT(κ') space whenever $\kappa < \kappa'$.

Let X be a CAT(-1) space. Then the following inequality always holds for any $x, y, z \in X$ and $\alpha \in [0, 1]$:

$$\begin{aligned}&\cosh d(\alpha x \oplus (1-\alpha)y, z) \sinh d(x, y)\\ &\qquad \le \cosh d(x, z) \sinh(\alpha d(x, y)) + \cosh d(y, z) \sinh((1-\alpha) d(x, y)).\end{aligned}$$

This inequality is often called the CN-inequality. Moreover, the following inequality is easily obtained by this inequality:

$$\cosh d(\alpha x \oplus (1-\alpha)y, z) \le \alpha \cosh d(x, z) + (1-\alpha) \cosh d(y, z).$$

Let C be a nonempty set. For $f\colon C \to \mathbb{R}$, the set of all minimizers of f is denoted by $\operatorname{argmin}_{x\in C} f(x)$. In this paper, if $\operatorname{argmin}_{x\in C} f(x)$ is a singleton, then the unique elements p is denoted by $p = \operatorname{argmin}_{x\in C} f(x)$.

Let X be a set and C a nonempty subset of X. For $T\colon C \to X$, the set of all fixed points of T is denoted by $F(T)$.

Let X be a metric space. An asymptotic center of a sequence $\{x_n\} \subset X$ is defined by $\operatorname{argmin}_{x\in X}(\limsup_{n\to\infty} d(x, x_n))$. If the asymptotic center of any subsequences of $\{x_n\}$ is just one point $x \in X$, then $\{x_n\}$ is said to Δ-converge to x, and we denote it by $x_n \xrightarrow{\Delta} x$. A mapping T from X into itself is said to be Δ-demiclosed if for any sequences $\{x_n\} \subset X$ with $x_n \xrightarrow{\Delta} x$, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ implies $x \in F(T)$.

Theorem 2.1 (Kirk and Panyanak [5]). *Let X be a complete* CAT(0) *space, and $\{x_n\}$ a bounded sequence on X. Then there exists a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

Theorem 2.2 (He, Fang, Lopez and Li [3]). *Let X be a complete* CAT(0) *space, and $\{x_n\}$ a sequence on X such that $x_n \overset{\Delta}{\rightharpoonup} x \in X$. Then for any $u \in X$,*

$$d(u, x) \leq \liminf_{n\to\infty} d(u, x_n).$$

Let X be a CAT(-1) space and T a mapping from X into itself with $F(T) \neq \varnothing$. Then T is said to be quasinonexpansive if an inequality $d(Tx, z) \leq d(x, z)$ is satisfied for all $x \in X$ and $z \in F(T)$. We know that the set of all fixed points of quasinonexpansive mapping is closed and convex. Further, T is said to be strongly quasinonexpansive if it is quasinonexpansive and, for any sequence $\{x_n\} \subset X$ and $z \in F(T)$, $\lim_{n\to\infty}(\cosh d(x_n, z) - \cosh d(Tx_n, z)) = 0$ implies that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let X be a complete CAT(-1) space and C a nonempty closed convex subset of X. Then there exists a unique point $p_x \in C$ such that $d(x, p_x) = \inf_{y\in C} d(x, y)$ for each $x \in X$. So we can define the metric projection P_C from X onto C by $P_C x = p_x$ for any $x \in X$.

Now, we introduce some properties of hyperbolic functions.

Lemma 2.3. *For any $a \in\,]{-1}, 1[$,*

$$\sinh(\tanh^{-1} a) = \frac{a}{\sqrt{1 - a^2}},$$

where $\tanh^{-1}\colon]{-1}, 1[\, \to \mathbb{R}$ is an inverse function of the hyperbolic tangent function.

Lemma 2.4. *For any $a, b \in\,]{-1}, 1[$,*

$$\tanh^{-1} a - \tanh^{-1} b = \tanh^{-1} \frac{a - b}{1 - ab}.$$

The following is an important lemma that forms the basis of the proof of the main result.

Lemma 2.5 (Aoyama, Kimura and Kohsaka [1]; Saejung and Yotkaew [7]). *Let $\{a_n\}$ be a sequence of non-negative real numbers and $\{t_n\}$ a sequence of real numbers. Let $\{\beta_n\}$ be a sequence in $]0, 1[$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that*

$$a_{n+1} \leq (1 - \beta_n) a_n + \beta_n t_n$$

for all $n \in \mathbb{N}$. If $\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \geq 0$ implies $\limsup_{i\to\infty} t_{\varphi(i)} \leq 0$ for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that nondecreasing and $\lim_{i\to\infty} \varphi(i) = \infty$, then $a_n \to 0$.

3. Main result

In this section, we prove a Halpern type approximation theorem with multiple anchor points of strongly quasinonexpansive mappings. To prove the main result, we define a new convex combination and introduce some lemmas.

Lemma 3.1. *Let $\{s_n\}, \{t_n\}, \{u_n\}$ be sequences of non-positive real numbers. Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} u_n = 0$ whenever $\lim_{n\to\infty}(s_n + t_n + u_n) = 0$.*

Lemma 3.2. *Let X be a $\mathrm{CAT}(-1)$ space. For $u_1, u_2, u_3 \in X$ and $\beta_1, \beta_2, \beta_3 \in [0,1]$ with $\beta_1 + \beta_2 + \beta_3 = 1$, define a function $g\colon X \to [1, \infty[$ by*

$$g(x) = \beta_1 \cosh d(u_1, x) + \beta_2 \cosh d(u_2, x) + \beta_3 \cosh d(u_3, x)$$

for all $x \in X$. Then for any $x, y \in X$,

$$g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \cosh \frac{d(x,y)}{2} \le \frac{g(x) + g(y)}{2}.$$

Proof. Let x, y be elements of X. It is obvious if $x = y$. Otherwise, we have

$$\begin{aligned} g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \sinh d(x,y) &= \sum_{i=1}^{3} \beta_i \cosh d\left(u_i, \frac{1}{2}x \oplus \frac{1}{2}y\right) \sinh d(x,y) \\ &\le \sum_{i=1}^{3} \beta_i (\cosh d(u_i, x) + \cosh d(u_i, y)) \sinh \frac{d(x,y)}{2} \\ &= (g(x) + g(y)) \sinh \frac{d(x,y)}{2}. \end{aligned}$$

Since $\sinh d(x,y) = 2 \sinh \frac{d(x,y)}{2} \cosh \frac{d(x,y)}{2}$, we obtain the desired result. □

Lemma 3.3. *Let X be a complete $\mathrm{CAT}(-1)$ space and C a nonempty closed convex subset of X. For $u_1, u_2, u_3 \in X$ and $\beta_1, \beta_2, \beta_3 \in [0,1]$ with $\beta_1 + \beta_2 + \beta_3 = 1$, define a function $g\colon X \to [1, \infty[$ by*

$$g(x) = \beta_1 \cosh d(u_1, x) + \beta_2 \cosh d(u_2, x) + \beta_3 \cosh d(u_3, x)$$

for all $x \in X$. Then g has a unique minimizer on C.

Proof. Put $L = \inf_{x \in C} g(x)$ and take a sequence $\{z_n\} \subset C$ with $L \le g(z_n) \le L + 1/n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} g(z_n) = L$.

We show that $\{z_n\}$ is a Cauchy sequence on C. Let $m, n \in \mathbb{N}$ with $m \ge n$. From Lemma 3.2, we get

$$g\left(\frac{1}{2}z_m \oplus \frac{1}{2}z_n\right) \cosh \frac{d(z_m, z_n)}{2} \le \frac{g(z_m) + g(z_n)}{2}.$$

Since $1 \le L \le g\left(\frac{1}{2}z_m \oplus \frac{1}{2}z_n\right)$, we have

$$\cosh \frac{d(z_m, z_n)}{2} \le \frac{g(z_m) + g(z_n)}{2g\left(\frac{1}{2}z_m \oplus \frac{1}{2}z_n\right)} \le \frac{g(z_m) + g(z_n)}{2L} \le \frac{L + 1/n}{L} \to 1 \quad (n \to \infty).$$

Therefore $\{z_n\}$ is a Cauchy sequence on C. From completeness of X and closedness of C, there exists $z \in C$ such that $z_n \to z$ and hence $g(z) = L = \inf_{x \in C} g(x)$. So z is a minimizer of g on C.

Next, we prove its uniqueness. Let $z, z' \in C$ satisfying $g(z) = g(z') = L$. From Lemma 3.2, we have

$$L \cosh \frac{d(z, z')}{2} \le g\left(\frac{1}{2}z \oplus \frac{1}{2}z'\right) \cosh \frac{d(z, z')}{2} \le \frac{g(z) + g(z')}{2} = L.$$

Since $L \geq 1$, we get $\cosh \frac{d(z,z')}{2} \leq 1$ and hence $z = z'$. Therefore we get the conclusion. □

Lemma 3.4. *Let X be a uniquely geodesic space. Then for $u, v \in X$ with $u \neq v$ and $\beta \in [0,1]$,*

$$\sigma u \oplus (1-\sigma)v = \operatorname*{argmin}_{x \in [u,v]} (\beta \cosh d(u,x) + (1-\beta)\cosh d(v,x))$$

if and only if

$$\sigma = \frac{1}{d(u,v)} \tanh^{-1} \frac{\beta \sinh d(u,v)}{1-\beta+\beta\cosh d(u,v)}.$$

Proof. It is obvious if $\beta = 0$ or $\beta = 1$. For $u, v \in X$ with $u \neq v$ and $\beta \in]0,1[$, put $d = d(u,v)$,

$$\begin{aligned} A &= \operatorname*{argmin}_{x \in [u,v]} (\beta \cosh d(u,x) + (1-\beta)\cosh d(v,x)), \\ B &= \operatorname*{argmin}_{0 \leq t \leq 1} (\beta \cosh((1-t)d) + (1-\beta)\cosh td), \text{ and} \\ C &= \operatorname*{argmin}_{0 \leq t \leq d} (\beta \cosh(d-t) + (1-\beta)\cosh t). \end{aligned}$$

Then sets A, B and C consist of one point, respectively. We also have

$$\begin{aligned} &A = \{tu \oplus (1-t)v \mid t \in B\} \text{ and} \\ &B = \operatorname*{argmin}_{0 \leq t \leq 1} (\beta \cosh(d-td) + (1-\beta)\cosh td) = \left\{ \frac{1}{d}t \;\middle|\; t \in C \right\}. \end{aligned}$$

Define a function $f\colon \mathbb{R} \to]1,\infty[$ by $f(t) = \beta\cosh(d-t) + (1-\beta)\cosh t$ for all $t \in \mathbb{R}$, then f is infinitely differentiable and $f'(0) < 0$, $f'(d) > 0$ and $f''(t) > 0$ for all $t \in [0,d]$. So there exists a unique real number $t \in]0,d[$ such that $f'(t) = 0$, that is, there exists a unique minimizer $t \in]0,d[$ of f and it satisfies $f'(t) = 0$. Then we have $f'(t) = 0$ if and only if

$$t = \tanh^{-1} \frac{\beta \sinh d}{1-\beta+\beta\cosh d}.$$

Thus we get

$$C = \left\{ \tanh^{-1} \frac{\beta \sinh d}{1-\beta+\beta\cosh d} \right\} \text{ and } B = \left\{ \frac{1}{d} \tanh^{-1} \frac{\beta \sinh d}{1-\beta+\beta\cosh d} \right\}.$$

So putting

$$\sigma = \frac{1}{d} \tanh^{-1} \frac{\beta \sinh d}{1-\beta+\beta\cosh d},$$

we get $A = \{\sigma u \oplus (1-\sigma)v\}$, that is,

$$\sigma u \oplus (1-\sigma)v = \operatorname*{argmin}_{x \in [u,v]} (\beta \cosh d(u,x) + (1-\beta)\cosh d(v,x)).$$ □

Lemma 3.5. *Let X be a uniquely geodesic space. Then for $u, v \in X$ with $u \neq v$ and $\beta \in [0,1]$,*

$$\begin{aligned}&\operatorname*{argmin}_{x\in[u,v]}(\beta\cosh d(u,x)+(1-\beta)\cosh d(v,x))\\ &=\operatorname*{argmin}_{x\in X}(\beta\cosh d(u,x)+(1-\beta)\cosh d(v,x)).\end{aligned}$$

Proof. Let $u, v \in X$ with $u \neq v$ and $\beta \in [0,1]$, and define a function $f\colon X \to \mathbb{R}$ by $f(x) = \beta\cosh d(u,x) + (1-\beta)\cosh d(v,x)$ for all $x \in X$. Put $z = \operatorname{argmin}_{x\in[u,v]} f(x)$ and let $w \in X$. Further, put

$$t = \frac{d(v,w)}{(d(u,w)+d(v,w))} \text{ and } z' = tu \oplus (1-t)v \in [u,v].$$

Then we get $f(z) \leq f(z')$. Moreover, we obtain

$$d(u,w) = (1-t)(d(u,w)+d(v,w)) \geq (1-t)d(u,v) = d(u,z').$$

Similarly, we also have $d(v,w) \geq d(v,z')$. Therefore we get $f(z') \leq f(w)$ and hence

$$f(z) = \min_{w\in X} f(w).$$ □

Using Lemma 3.4 and Lemma 3.5, we define a new convex combination.

Definition 3.6. Let X be a uniquely geodesic space. For $u, v \in X$ and $\alpha \in [0,1]$, we define a (-1)-convex combination of u and v by

$$\alpha u \overset{-1}{\oplus} (1-\alpha)v \overset{\text{def}}{=} \operatorname*{argmin}_{x\in X}(\alpha\cosh d(u,x)+(1-\alpha)\cosh d(v,x)).$$

From Lemma 3.4 and Lemma 3.5, it can be expressed by $\alpha u \overset{-1}{\oplus} (1-\alpha)v = \sigma u \oplus (1-\sigma)v$, where

$$\sigma = \frac{1}{d(u,v)}\tanh^{-1}\frac{\alpha\sinh d(u,v)}{1-\alpha+\alpha\cosh d(u,v)}$$

whenever $u \neq v$.

Lemma 3.7. *For any $\alpha \in [0,1]$ and $d > 0$,*

$$\frac{1}{d}\tanh^{-1}\frac{\alpha\sinh d}{1-\alpha+\alpha\cosh d} + \frac{1}{d}\tanh^{-1}\frac{(1-\alpha)\sinh d}{\alpha+(1-\alpha)\cosh d} = 1.$$

Proof. It is obvious if $\alpha = 0$ or $\alpha = 1$. We consider the case where $\alpha \in \,]0,1[$. Let $\alpha \in \,]0,1[$ and $d > 0$. From Lemma 2.4, we have

$$\begin{aligned}1-\frac{1}{d}\tanh^{-1}\frac{\alpha\sinh d}{1-\alpha+\alpha\cosh d} &= \frac{1}{d}\left(\tanh^{-1}\frac{\sinh d}{\cosh d} - \tanh^{-1}\frac{\alpha\sinh d}{1-\alpha+\alpha\cosh d}\right)\\ &= \frac{1}{d}\tanh^{-1}\frac{\dfrac{\sinh d}{\cosh d}-\dfrac{\alpha\sinh d}{1-\alpha+\alpha\cosh d}}{1-\dfrac{\sinh d}{\cosh d}\cdot\dfrac{\alpha\sinh d}{1-\alpha+\alpha\cosh d}}\\ &= \frac{1}{d}\tanh^{-1}\frac{(1-\alpha)\sinh d}{\alpha+(1-\alpha)\cosh d}.\end{aligned}$$

So we get the desired result. □

Lemma 3.8. *Let X be a* CAT(-1) *space and $x, y, z \in X$, $\alpha \in [0,1]$. Then*

$$\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \leq \frac{\alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}}.$$

Proof. Let $x, y, z \in X$ and $\alpha \in [0,1]$. It is obvious if $x = y$. Suppose that $x \neq y$ and put

$$\sigma = \frac{1}{d(x,y)} \tanh^{-1} \frac{\alpha \sinh d(x,y)}{1-\alpha+\alpha\cosh d(x,y)}.$$

From Lemma 3.4, Lemma 3.7 and Lemma 2.3, we have

$$\begin{aligned}
&\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \sinh d(x,y) \\
&= \cosh d(\sigma x \oplus (1-\sigma)y, z) \sinh d(x,y) \\
&\leq \cosh d(x,z) \sinh(\sigma d(x,y)) + \cosh d(y,z) \sinh((1-\sigma)d(x,y)) \\
&= \cosh d(x,z) \sinh\left(\tanh^{-1} \frac{\alpha \sinh d(x,y)}{1-\alpha+\alpha\cosh d(x,y)}\right) \\
&\quad + \cosh d(y,z) \sinh\left(\tanh^{-1} \frac{(1-\alpha)\sinh d(x,y)}{\alpha+(1-\alpha)\cosh d(x,y)}\right) \\
&= \cosh d(x,z) \cdot \frac{\alpha \sinh d(x,y)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}} \\
&\quad + \cosh d(y,z) \cdot \frac{(1-\alpha)\sinh d(x,y)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}} \\
&= \frac{\alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}} \cdot \sinh d(x,y)
\end{aligned}$$

and hence

$$\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \leq \frac{\alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}}.$$

Thus we get the desired result. □

Corollary 3.9. *Let X be a* CAT(-1) *space and $x, y, z \in X$, $\alpha \in [0,1]$. Then*

$$\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \leq \alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z).$$

Proof. Since $\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2} \geq 1$, Lemma 3.8 implies the conclusion. □

Lemma 3.10. *Let X be a* CAT(-1) *space. Then for $u, y, z \in X$ and $\alpha \in]0,1[$,*

$$\begin{aligned}
&\cosh d(\alpha u \overset{-1}{\oplus} (1-\alpha)y, z) - 1 \leq (1-\beta)(\cosh d(y,z) - 1) \\
&+ \beta\left(\frac{\left(1-\alpha+\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}\right)\cosh d(u,z)}{\alpha + 2(1-\alpha)\cosh d(u,y)} - 1\right),
\end{aligned}$$

where

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}}.$$

Proof. It is obvious if $u = y$. Otherwise, from Lemma 3.8, we have

$$\begin{aligned}
&\cosh d(\alpha u \overset{-1}{\oplus} (1-\alpha)y, z) - 1 \\
&\le \frac{\alpha \cosh d(u,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}} + (1-\beta)\cosh d(y,z) - 1 \\
&= (1-\beta)(\cosh d(y,z) - 1) + \beta\left(\frac{\alpha \cosh d(u,z)}{\beta\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}} - 1\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\alpha \cosh d(u,z)}{\beta\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}} \\
&= \frac{\alpha \cosh d(u,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2} - (1-\alpha)} \\
&= \frac{\left(1-\alpha + \sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}\right)\cosh d(u,z)}{\alpha + 2(1-\alpha)\cosh d(u,y)},
\end{aligned}$$

we get the conclusion. ☐

Lemma 3.11. *Let $\{\alpha_n\}$ be a sequence on $]0,1[$ such that*

$$\lim_{n\to\infty} \alpha_n = 0,$$

and $\{s_n\}, \{t_n\}$ sequences on $[0,\infty[$ such that

$$\lim_{n\to\infty} s_n = d_1 \in [0,\infty[, \quad \lim_{n\to\infty} t_n = d_2 \in [0,\infty[.$$

Define sequences $\{\sigma_n\}, \{\tau_n\} \subset]0,1[$ by

$$\begin{aligned}
\sigma_n &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cosh s_n + (1-\alpha_n)^2}}, \\
\tau_n &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cosh t_n + (1-\alpha_n)^2}}
\end{aligned}$$

for all $n \in \mathbb{N}$, respectively. Then

$$\lim_{n\to\infty} \frac{\tau_n}{\sigma_n} = \frac{\cosh d_2}{\cosh d_1}.$$

Proof. Put

$$\begin{aligned}
p_n &= \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cosh s_n + (1-\alpha_n)^2}, \\
q_n &= \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cosh t_n + (1-\alpha_n)^2}
\end{aligned}$$

for all $n \in \mathbb{N}$. Then we have

$$\frac{\tau_n}{\sigma_n} = \frac{1 - \dfrac{1-\alpha_n}{\sqrt{\alpha_n{}^2 + 2\alpha_n(1-\alpha_n)\cosh t_n + (1-\alpha_n)^2}}}{1 - \dfrac{1-\alpha_n}{\sqrt{\alpha_n{}^2 + 2\alpha_n(1-\alpha_n)\cosh s_n + (1-\alpha_n)^2}}} = \frac{\dfrac{\alpha_n + 2(1-\alpha_n)\cosh t_n}{q_n(q_n + 1 - \alpha_n)}}{\dfrac{\alpha_n + 2(1-\alpha_n)\cosh s_n}{p_n(p_n + 1 - \alpha_n)}}.$$

Since $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$, we get the conclusion. □

Now, we show the main result.

Theorem 3.12. *Let X be a complete* CAT(-1) *space and R, S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with*

$$F = F(R) \cap F(S) \cap F(T) \neq \varnothing.$$

Let $\{\alpha_n\} \subset]0,1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and $\{\beta_n\}, \{\gamma_n\} \subset]0,1[$ such that

$$\lim_{n\to\infty} \beta_n = \beta \in]0,1[, \ \lim_{n\to\infty} \gamma_n = \gamma \in]0,1[.$$

Let $u, v, w, x_1 \in X$ and define a iterative sequence $\{x_n\} \subset X$ by

$$\begin{cases} r_n = \alpha_n u \overset{-1}{\oplus} (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \overset{-1}{\oplus} (1-\alpha_n) S x_n, \\ t_n = \alpha_n w \overset{-1}{\oplus} (1-\alpha_n) T x_n, \\ x_{n+1} = \beta_n r_n \overset{-1}{\oplus} (1-\beta_n)(\gamma_n s_n \overset{-1}{\oplus} (1-\gamma_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges to a point $p \in F$, which is a minimizer of the function $g(x) = \beta \cosh d(u,x) + (1-\beta)(\gamma \cosh d(v,x) + (1-\gamma)\cosh d(w,x))$ on F.

Proof. Since F is a closed convex subset of X and from Lemma 3.3, the existence and uniqueness of the elements of the set

$$\operatorname*{argmin}_{x\in F} \left(\beta \cosh d(u,x) + (1-\beta)(\gamma \cosh d(v,x) + (1-\gamma)\cosh d(w,x))\right)$$

are guaranteed.

Let

$$p = \operatorname*{argmin}_{x\in F} \left(\beta \cosh d(u,x) + (1-\beta)(\gamma \cosh d(v,x) + (1-\gamma)\cosh d(w,x))\right)$$

and put

$$
\begin{aligned}
a_n &= \cosh d(x_n, p) - 1,\\
b_n^R &= \frac{\left(1-\alpha_n+\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cosh d(u,Rx_n)+(1-\alpha_n)^2}\right)\cosh d(u,p)}{\alpha_n+2(1-\alpha_n)\cosh d(u,Rx_n)} - 1,\\
b_n^S &= \frac{\left(1-\alpha_n+\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cosh d(v,Sx_n)+(1-\alpha_n)^2}\right)\cosh d(v,p)}{\alpha_n+2(1-\alpha_n)\cosh d(v,Sx_n)} - 1,\\
b_n^T &= \frac{\left(1-\alpha_n+\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cosh d(w,Tx_n)+(1-\alpha_n)^2}\right)\cosh d(w,p)}{\alpha_n+2(1-\alpha_n)\cosh d(w,Tx_n)} - 1,\\
\gamma_n^R &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cosh d(u,Rx_n)+(1-\alpha_n)^2}},\\
\gamma_n^S &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cosh d(v,Sx_n)+(1-\alpha_n)^2}},\\
\gamma_n^T &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)\cosh d(w,Tx_n)+(1-\alpha_n)^2}}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Moreover, put

$$\beta_n^R = \beta_n,\ \beta_n^S = (1-\beta_n)\gamma_n,\ \beta_n^T = (1-\beta_n)(1-\gamma_n)$$

for all $n \in \mathbb{N}$ and put

$$\beta^R = \beta,\ \beta^S = (1-\beta)\gamma,\ \beta^T = (1-\beta)(1-\gamma).$$

Then $\{\gamma_n^R\}$, $\{\gamma_n^S\}$ and $\{\gamma_n^T\}$ are sequences on $]0,1[$. From Lemmas 3.8 and 3.10, we have

$$
\begin{aligned}
a_{n+1} &\le \beta_n \cosh d(r_n,p) + (1-\beta_n)\cosh d(\gamma_n s_n \overset{-1}{\oplus} (1-\gamma_n)t_n, p) - 1\\
&\le \beta_n^R \cosh d(r_n,p) + \beta_n^S \cosh d(s_n,p) + \beta_n^T \cosh d(t_n,p) - 1\\
&= \beta_n^R(\cosh d(r_n,p)-1) + \beta_n^S(\cosh d(s_n,p)-1) + \beta_n^T(\cosh d(t_n,p)-1)\\
&\le \beta_n^R\left((1-\gamma_n^R)(\cosh d(Rx_n,p)-1) + \gamma_n^R b_n^R\right)\\
&\quad + \beta_n^S\left((1-\gamma_n^S)(\cosh d(Sx_n,p)-1) + \gamma_n^S b_n^S\right)\\
&\quad + \beta_n^T\left((1-\gamma_n^T)(\cosh d(Tx_n,p)-1) + \gamma_n^T b_n^T\right)\\
&\le \left(\beta_n^R(1-\gamma_n^R) + \beta_n^S(1-\gamma_n^S) + \beta_n^T(1-\gamma_n^T)\right)a_n + \beta_n^R\gamma_n^R b_n^R + \beta_n^S\gamma_n^S b_n^S + \beta_n^T\gamma_n^T b_n^T\\
&= \left(1 - \left(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T\right)\right)a_n\\
&\quad + \left(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T\right)\cdot\frac{\beta_n^R\gamma_n^R b_n^R + \beta_n^S\gamma_n^S b_n^S + \beta_n^T\gamma_n^T b_n^T}{\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T}
\end{aligned}
$$

for all $n \in \mathbb{N}$.

Now we show that the following conditions hold:

(i) $\displaystyle\sum_{n=1}^{\infty}(\beta_n^R\gamma_n^R + \beta_n^S\gamma_n^S + \beta_n^T\gamma_n^T) = \infty$,

(ii) for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ satisfying that φ is nondecreasing and

$$\lim_{i\to\infty} \varphi(i) = \infty,\ \liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \geq 0$$

implies

$$\limsup_{i\to\infty} \frac{\beta^R_{\varphi(i)}\gamma^R_{\varphi(i)}b^R_{\varphi(i)} + \beta^S_{\varphi(i)}\gamma^S_{\varphi(i)}b^S_{\varphi(i)} + \beta^T_{\varphi(i)}\gamma^T_{\varphi(i)}b^T_{\varphi(i)}}{\beta^R_{\varphi(i)}\gamma^R_{\varphi(i)} + \beta^S_{\varphi(i)}\gamma^S_{\varphi(i)} + \beta^T_{\varphi(i)}\gamma^T_{\varphi(i)}} \leq 0.$$

First, we show (i). Since

$$\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cosh d(u, Rx_n) + (1-\alpha_n)^2} \geq 1,$$

we have $\gamma^R_n \geq \alpha_n$. Similarly, we also obtain $\gamma^S_n \geq \alpha_n$ and $\gamma^T_n \geq \alpha_n$. So we get

$$\sum_{n=1}^{\infty}(\beta^R_n\gamma^R_n + \beta^S_n\gamma^S_n + \beta^T_n\gamma^T_n) \geq \sum_{n=1}^{\infty}(\beta^R_n\alpha_n + \beta^S_n\alpha_n + \beta^T_n\alpha_n) = \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Next, we consider (ii). We show boundedness of $\{x_n\}$. By Corollary 3.9, we obtain

$$\begin{aligned}
\cosh d(x_{n+1}, p) &\leq \beta^R_n \cosh d(r_n, p) + \beta^S_n \cosh d(s_n, p) + \beta^T_n \cosh d(t_n, p)\\
&\leq \beta^R_n(\alpha_n \cosh d(u, p) + (1-\alpha_n)\cosh d(Rx_n, p))\\
&\quad + \beta^S_n(\alpha_n \cosh d(v, p) + (1-\alpha_n)\cosh d(Sx_n, p))\\
&\quad + \beta^T_n(\alpha_n \cosh d(w, p) + (1-\alpha_n)\cosh d(Tx_n, p))\\
&\leq \beta^R_n\alpha_n \cosh d(u, p) + \beta^S_n\alpha_n \cosh d(v, p) + \beta^T_n\alpha_n \cosh d(w, p)\\
&\quad + \left(\beta^R_n(1-\alpha_n) + \beta^S_n(1-\alpha_n) + \beta^T_n(1-\alpha_n)\right)\cosh d(x_n, p)\\
&\leq \max\{\cosh d(u, p), \cosh d(v, p), \cosh d(w, p), \cosh d(x_n, p)\}
\end{aligned}$$

for all $n \in \mathbb{N}$. So we have

$$d(x_n, p) \leq \max\{d(u, p), d(v, p), d(w, p), d(x_1, p)\}$$

for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is bounded.
Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function with

$$\lim_{i\to\infty} \varphi(i) = \infty,$$

and put $n_i = \varphi(i)$ for all $i \in \mathbb{N}$. Assume that

$$\liminf_{i\to\infty} (a_{n_i+1} - a_{n_i}) \geq 0,$$

then we get

$$\begin{aligned}
0 &\leq \liminf_{i\to\infty} (a_{n_i+1} - a_{n_i}) \\
&= \liminf_{i\to\infty} (\cosh d(x_{n_i+1}, p) - \cosh d(x_{n_i}, p)) \\
&\leq \liminf_{i\to\infty} \big(\beta^R_{n_i} \cosh d(r_{n_i}, p) + \beta^S_{n_i} \cosh d(s_{n_i}, p) + \beta^T_{n_i} \cosh d(t_{n_i}, p) \\
&\quad - \cosh d(x_{n_i}, p)) \\
&\leq \liminf_{i\to\infty} \big(\beta^R_{n_i}(\alpha_{n_i} \cosh d(u, p) + (1-\alpha_{n_i}) \cosh d(Rx_{n_i}, p)) \\
&\quad + \beta^S_{n_i}(\alpha_{n_i} \cosh d(v, p) + (1-\alpha_{n_i}) \cosh d(Sx_{n_i}, p)) \\
&\quad + \beta^T_{n_i}(\alpha_{n_i} \cosh d(w, p) + (1-\alpha_{n_i}) \cosh d(Tx_{n_i}, p)) - \cosh d(x_{n_i}, p)\big) \\
&= \liminf_{i\to\infty} \big(\beta^R(\cosh d(Rx_{n_i}, p) - \cosh d(x_{n_i}, p)) \\
&\quad + \beta^S(\cosh d(Sx_{n_i}, p) - \cosh d(x_{n_i}, p)) \\
&\quad + \beta^T(\cosh d(Tx_{n_i}, p) - \cosh d(x_{n_i}, p))\big) \\
&\leq \limsup_{i\to\infty} \big(\beta^R(\cosh d(Rx_{n_i}, p) - \cosh d(x_{n_i}, p)) \\
&\quad + \beta^S(\cosh d(Sx_{n_i}, p) - \cosh d(x_{n_i}, p)) \\
&\quad + \beta^T(\cosh d(Tx_{n_i}, p) - \cosh d(x_{n_i}, p))\big) \\
&\leq 0.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\lim_{i\to\infty} (\beta^R(\cosh d(Rx_{n_i}, p) - \cosh d(x_{n_i}, p)) + \beta^S(\cosh d(Sx_{n_i}, p) - \cosh d(x_{n_i}, p)) \\
+ \beta^T(\cosh d(Tx_{n_i}, p) - \cosh d(x_{n_i}, p))) = 0.
\end{aligned}$$

From Lemma 3.1, we have

$$\begin{aligned}
\lim_{i\to\infty} (\cosh d(Rx_{n_i}, p) - \cosh d(x_{n_i}, p)) = 0, \\
\lim_{i\to\infty} (\cosh d(Sx_{n_i}, p) - \cosh d(x_{n_i}, p)) = 0, \\
\lim_{i\to\infty} (\cosh d(Tx_{n_i}, p) - \cosh d(x_{n_i}, p)) = 0.
\end{aligned}$$

Since R, S, T are strongly quasinonexpansive, we obtain

$$\lim_{i\to\infty} d(x_{n_i}, Rx_{n_i}) = \lim_{i\to\infty} d(x_{n_i}, Sx_{n_i}) = \lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0. \tag{1}$$

Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ satisfying

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{\beta^R_{n_i}\gamma^R_{n_i}b^R_{n_i} + \beta^S_{n_i}\gamma^S_{n_i}b^S_{n_i} + \beta^T_{n_i}\gamma^T_{n_i}b^T_{n_i}}{\beta^R_{n_i}\gamma^R_{n_i} + \beta^S_{n_i}\gamma^S_{n_i} + \beta^T_{n_i}\gamma^T_{n_i}} \\
&= \lim_{j\to\infty} \frac{\beta^R_{n_{i_j}}\gamma^R_{n_{i_j}}b^R_{n_{i_j}} + \beta^S_{n_{i_j}}\gamma^S_{n_{i_j}}b^S_{n_{i_j}} + \beta^T_{n_{i_j}}\gamma^T_{n_{i_j}}b^T_{n_{i_j}}}{\beta^R_{n_{i_j}}\gamma^R_{n_{i_j}} + \beta^S_{n_{i_j}}\gamma^S_{n_{i_j}} + \beta^T_{n_{i_j}}\gamma^T_{n_{i_j}}}.
\end{aligned}$$

Moreover, take a subsequence $\{z_r\}$ of $\{x_{n_{i_j}}\}$ with

$$\liminf_{j\to\infty} d(u, x_{n_{i_j}}) = \lim_{r\to\infty} d(u, z_r)$$

and a subsequence $\{z_{r_s}\}$ of $\{z_r\}$ satisfying

$$\liminf_{r\to\infty} d(v, z_r) = \lim_{s\to\infty} d(v, z_{r_s}).$$

Furthermore, take a subsequence $\{z_{r_{s_t}}\}$ of $\{z_{r_s}\}$ such that

$$\liminf_{s\to\infty} d(w, z_{r_s}) = \lim_{t\to\infty} d(w, z_{r_{s_t}}),$$

and a subsequence $\{v_k\}$ of $\{z_{r_{s_t}}\}$ which satisfies $v_k \overset{\Delta}{\rightharpoonup} z \in X$. Then from the formula (1), we have

$$\lim_{k\to\infty} d(v_k, Rv_k) = \lim_{k\to\infty} d(v_k, Sv_k) = \lim_{k\to\infty} d(v_k, Tv_k) = 0$$

and hence $z \in F$. Further, since

$$\begin{aligned}
\lim_{k\to\infty} d(u, v_k) = \liminf_{j\to\infty} d(u, x_{n_{i_j}}) &\le \liminf_{j\to\infty} (d(u, Rx_{n_{i_j}}) + d(Rx_{n_{i_j}}, x_{n_{i_j}})) \\
&= \liminf_{j\to\infty} d(u, Rx_{n_{i_j}}) \\
&\le \liminf_{k\to\infty} d(u, Rv_k) \\
&\le \limsup_{k\to\infty} d(u, Rv_k) \\
&\le \limsup_{k\to\infty} (d(u, v_k) + d(v_k, Rv_k)) \\
&= \lim_{k\to\infty} d(u, v_k),
\end{aligned}$$

we get

$$\lim_{k\to\infty} d(u, v_k) = \lim_{k\to\infty} d(u, Rv_k).$$

Similarly, we also obtain

$$\lim_{k\to\infty} d(v, v_k) = \lim_{k\to\infty} d(v, Sv_k) \text{ and } \lim_{k\to\infty} d(w, v_k) = \lim_{k\to\infty} d(w, Tv_k).$$

By Theorem 2.2, we have the following formulas:

$$\begin{aligned}
\lim_{k\to\infty} d(u, Rv_k) &= \lim_{k\to\infty} d(u, v_k) \ge d(u, z), \\
\lim_{k\to\infty} d(v, Sv_k) &= \lim_{k\to\infty} d(v, v_k) \ge d(v, z), \\
\lim_{k\to\infty} d(w, Tv_k) &= \lim_{k\to\infty} d(w, v_k) \ge d(w, z)
\end{aligned}$$

and hence

$$\begin{aligned}
&\lim_{k\to\infty} \left(\beta^R \cosh d(u, Rv_k) + \beta^S \cosh d(v, Sv_k) + \beta^T \cosh d(w, Tv_k)\right) \\
&\ge \beta^R \cosh d(u, z) + \beta^S \cosh d(v, z) + \beta^T \cosh d(w, z) \\
&\ge \beta^R \cosh d(u, p) + \beta^S \cosh d(v, p) + \beta^T \cosh d(w, p).
\end{aligned}$$

Let

$$d_1 = \lim_{k\to\infty} d(u, Rv_k),\ d_2 = \lim_{k\to\infty} d(v, Sv_k),\ d_3 = \lim_{k\to\infty} d(w, Tv_k)$$

and put $m_k = n_{i_{j_{r_{s_{t_k}}}}}$ for all $k \in \mathbb{N}$. Then from Lemma 3.11, we obtain

$$\lim_{k\to\infty} \frac{\gamma^S_{m_k}}{\gamma^R_{m_k}} = \frac{\cosh d_2}{\cosh d_1}, \quad \lim_{k\to\infty} \frac{\gamma^T_{m_k}}{\gamma^R_{m_k}} = \frac{\cosh d_3}{\cosh d_1}.$$

Put

$$\mu^R = \frac{\beta^R \cosh d_1}{\beta^R \cosh d_1 + \beta^S \cosh d_2 + \beta^T \cosh d_3},$$
$$\mu^S = \frac{\beta^S \cosh d_2}{\beta^R \cosh d_1 + \beta^S \cosh d_2 + \beta^T \cosh d_3},$$
$$\mu^T = \frac{\beta^T \cosh d_3}{\beta^R \cosh d_1 + \beta^S \cosh d_2 + \beta^T \cosh d_3}.$$

Then we get

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{\beta^R_{n_i}\gamma^R_{n_i}b^R_{n_i} + \beta^S_{n_i}\gamma^S_{n_i}b^S_{n_i} + \beta^T_{n_i}\gamma^T_{n_i}b^T_{n_i}}{\beta^R_{n_i}\gamma^R_{n_i} + \beta^S_{n_i}\gamma^S_{n_i} + \beta^T_{n_i}\gamma^T_{n_i}} \\
&= \lim_{k\to\infty} \frac{\beta^R_{m_k}\gamma^R_{m_k}b^R_{m_k} + \beta^S_{m_k}\gamma^S_{m_k}b^S_{m_k} + \beta^T_{m_k}\gamma^T_{m_k}b^T_{m_k}}{\beta^R_{m_k}\gamma^R_{m_k} + \beta^S_{m_k}\gamma^S_{m_k} + \beta^T_{m_k}\gamma^T_{m_k}} \\
&= \lim_{k\to\infty} \frac{\beta^R b^R_{m_k} + \beta^S \cdot \frac{\cosh d_2}{\cosh d_1} \cdot b^S_{m_k} + \beta^T \cdot \frac{\cosh d_3}{\cosh d_1} \cdot b^T_{m_k}}{\beta^R + \beta^S \cdot \frac{\cosh d_2}{\cosh d_1} + \beta^T \cdot \frac{\cosh d_3}{\cosh d_1}} \\
&= \lim_{k\to\infty} \left(\mu^R b^R_{m_k} + \mu^S b^S_{m_k} + \mu^T b^T_{m_k}\right) \\
&= \lim_{k\to\infty} \left(\mu^R \left(\frac{\cosh d(u,p)}{\cosh d(u, Rv_k)} - 1\right) + \mu^S \left(\frac{\cosh d(v,p)}{\cosh d(v, Sv_k)} - 1\right)\right. \\
&\qquad \left. + \mu^T \left(\frac{\cosh d(w,p)}{\cosh d(w, Tv_k)} - 1\right)\right) \\
&= \lim_{k\to\infty} \left(\mu^R \cdot \frac{\cosh d(u,p) - \cosh d(u, Rv_k)}{\cosh d_1} + \mu^S \cdot \frac{\cosh d(v,p) - \cosh d(v, Sv_k)}{\cosh d_2}\right. \\
&\qquad \left. + \mu^T \cdot \frac{\cosh d(w,p) - \cosh d(w, Tv_k)}{\cosh d_3}\right) \\
&= \lim_{k\to\infty} \left(\frac{\beta^R \cosh d(u,p) + \beta^S \cosh d(v,p) + \beta^T \cosh d(w,p)}{\beta^R \cosh d_1 + \beta^S \cosh d_2 + \beta^T \cosh d_3}\right. \\
&\qquad \left. - \frac{\beta^R \cosh d(u, Rv_k) + \beta^S \cosh d(v, Sv_k) + \beta^T \cosh d(w, Tv_k)}{\beta^R \cosh d_1 + \beta^S \cosh d_2 + \beta^T \cosh d_3}\right) \le 0.
\end{aligned}$$

Thus we have (ii). Hence, using Lemma 2.5, we obtain the desired result. □

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