Fixed Point Theory, 21(2020), No. 2, 611-630
DOI: 10.24193/fpt-ro.2020.2.43
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SYSTEMS OF UNRELATED GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND UNRELATED HIERARCHICAL FIXED POINT PROBLEMS IN HILBERT SPACE

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Abstract. In this paper, we investigate a hybrid extra-gradient iterative method to approximate the common solution of a system of unrelated generalized mixed equilibrium problems for monotone and Lipschitz continuous mappings and system of unrelated hierarchical fixed point problems for nonexpansive mappings in Hilbert space. We prove a strong convergence theorem for the sequences generated by the proposed iterative algorithm. Further, we give some consequences and applications of our main result. Finally, we discuss a numerical example to demonstrate the applicability of the iterative algorithm.
Key Words and Phrases: System of unrelated generalized mixed equilibrium problems, system of unrelated hierarchical fixed point problems, monotone mapping, Lipschitz continuous mapping, nonexpansive mapping, hybrid extra-gradient iterative method.
2010 Mathematics Subject Classification: 47H05, 47H09, 47J25, 47H10.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. For each $i = 1, 2, ..., N$, let K, K_i be nonempty, closed and convex sets and $\bigcap_{i=1}^{N} K_i \neq \emptyset$. Recall that a mapping $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$. We denote the fixed point set of T by $\text{Fix}(T) := \{x \in T : Tx = x\}$. It is well known that $\text{Fix}(T)$ is closed and convex.

We consider the following new class of hierarchical fixed point problems called the system of unrelated hierarchical fixed point problems (in short, SUHFPP) for nonexpansive mappings $\{T_i : K_i \to K_i\}_{i=1}^{N}$ such that $\bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$ with respect to another nonexpansive mappings $\{S_i : K_i \to K_i\}_{i=1}^N$: Find $x \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ such that

$$\langle x - S_i x, x - y_i \rangle \leq 0,\ \forall y_i \in \mathrm{Fix}(T_i). \tag{1.1}$$

The solution set of SUHFPP(1.1) is denoted by Φ.

For each $i = 1, 2, .., N$, SUHFPP(1.1) is reduced to the hierarchical fixed point problem (in short, HFPP): Find $x \in \mathrm{Fix}(T_i)$ such that

$$\langle x - S_i x, x - y_i \rangle \leq 0,\ \forall y_i \in \mathrm{Fix}(T_i). \tag{1.2}$$

This amounts to saying that $x \in \mathrm{Fix}(T_i)$ satisfies a variational inequality depending on a given criterion S_i, namely: Find $x \in K_i$ such that

$$0 \in (I_i - S_i)x + N_{\mathrm{Fix}(T_i)}(x), \tag{1.3}$$

where $N_{\mathrm{Fix}(T_i)}$ is the normal cone to $\mathrm{Fix}(T_i)$. The solution set of HFPP (1.2) is given by $\Phi_i := \{x \in K_i : x = (P_{\mathrm{Fix}(T_i)} \circ S_i)x\}$. The solution set of HFPP (1.2) is denoted by Φ_i, where $P_{\mathrm{Fix}(T_i)}$ is the metric projection of H onto $\mathrm{Fix}(T_i)$. We easily observe that $\Phi = \bigcap_{i=1}^N \Phi_i$.

The motivation to study SUHFPP(1.1) comes from the fact that it contains, as particular cases, various problems considered in the literature. Below we present some examples of such problems.

If for each $i = 1, 2, .., N$, we set $S_i = I_i$, the identity mapping on K_i, then SUHFPP(1.1) is reduced to the common fixed point problem (in short, CFPP) for a finite family of nonexpansive mappings T_i: Find $x \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ which an extension of convex feasibility problem (in short, CFP). We note that SUHFPP(1.1) covers the following systems of unrelated monotone variational inequalities on fixed point sets, minimization problems over equilibrium constraints, hierarchical minimization problems:

If, for each $i = 1, 2, .., N$, let M_i be a maximal monotone operator, by setting

$$T_i = J_\lambda^{M_i} := (I_i + \lambda M_i)^{-1} \text{ and } S_i = I_i - \gamma_i \nabla \psi_i,$$

where ψ_i is a convex function such that $\nabla \psi_i$ is η_i-Lipschitzian with

$$\gamma_i \in \left(0, \frac{2}{\max\limits_i \{\eta_i\}}\right],$$

and using the fact that $\mathrm{Fix}(J_\lambda^{M_i}) = M_i^{-1}(0)$, SUHFPP(1.1) is reduced to the following system of unrelated mathematical programming problems with generalized equation constraints:

$$\min_{0 \in M_i(x)} \psi_i(x), \tag{1.4}$$

which is a generalization of the problem studied by Luo et al. [10]. By taking $M_i = \partial \varphi_i$, where $\partial \varphi_i$ is the subdifferential of a lower semicontinuous convex function, then problem (1.4) is reduced to the system of unrelated hierarchical minimization problem considered by Cabot [2] with $N = 1$.

If we set $N = 1$ then SUHFPP(1.1) is reduced to the hierarchical fixed point problem (in short, HFPP) considered and studied by Moudafi and Mainge [12]. For further work on HFPP, see [7, 8, 11, 13].

On the other hand, we consider another new class of problems so called the system of unrelated generalized mixed equilibrium problems (in short, SUGMEP):

Find $x \in \bigcap_{i=1}^{N} K_i$ such that

$$G_i(x, y_i) + \langle B_i x, y_i - x\rangle + \phi_i(x, y_i) - \phi_i(x, x) \geq 0, \ \forall y_i \in K_i, \ i = 1, 2, ..., N, \quad (1.5)$$

where $B_i : K_i \to H$ is a nonlinear mapping and $G_i : K_i \times K_i \to \mathbb{R}$, $\phi_i : K_i \times K_i \to \mathbb{R}$ are bifunctions for each $i = 1, 2, .., N$, where $\mathbb{R}$ is the set of real numbers. The solution set of SUGMEP(1.5) is denoted by $\Theta = \bigcap_{i=1}^{N} \Gamma_i$, where Γ_i is the solution set of generalized mixed equilibrium problem (in short, GMEP$_i$): Find $x \in K_i$ such that (1.5) holds.

The significance of studying the SUGMEP(1.5) lies in the fact that besides its enabling a unified treatment of such well-known problems as the CFP and the CFPP, the variational inequality problem (in short, VIP), the SUGMEP(1.5) also opens a path to a variety of new system of problems that are created from various special cases of the SUGMEP(1.5).

If we set $\phi_i = 0$ and $G_i = 0$ then SUGMEP(1.5) is reduced to the system of unrelated variational inequality problems (in short, SUVIP) considered and studied by Censor et al. [3] for set-valued version of mappings B_i: Find $x \in \bigcap_{i=1}^{N} K_i$ such that

$$\langle B_i x, y_i - x\rangle \geq 0, \ \forall y_i \in K_i, \ i = 1, 2, ..., N. \quad (1.6)$$

We denote the solution set of SUVIP (1.6) by $\Theta_1 = \bigcap_{i=1}^{N} \Psi_i$ where Ψ_i is the solution set of variational inequality problem (in short, VIP(K_i, B_i)): Find $x \in K_i$ such that (1.6) holds.

If we set $N = 1$,

$$G_i(x, y_i) = j_i^0(x; y_i - x) - < f, y_i - x >, \forall x, y_i \in K_i,$$

where $j_i^0(x; y_i)$ is the Clarke's generalized directional derivative of j at x in the direction y_i for a locally Lipschitz continuous function $j : H \to R$ at a given point $x \in H$ and v be any other vector in H and $f \in H^*$ then SUGMEP(1.5) is reduced to the following variational-hemivariational inequality problem of second kind which is a model of contact problem with normal compliance (See Problems 19,44 on pp. 142, 213 [17]): Find $x \in K_1$ such that

$$\langle B_1 x, y_1 - x\rangle + \phi_1(x, y_1) - \phi_1(x, x) + j_1^0(x; y_1 - x) \geq < f, y_1 - x >, \ \forall y_1 \in K_1. \quad (1.7)$$

Further, if we set $j = 0$ then SUGMEP(1.5) is reduced to the elliptic quasivariational inequality problem of second kind which is model of frictional contact problem with normal compliance (see (2.58), Problem 5.36, (5.187) [16]).

In 2006, by combining a hybrid iterative method due to Nakajo and Takahashi [15] with the extra-gradient iterative method due to Korpelevich [9], Nadezhkina and Takahashi [14] introduced the following extra-gradient hybrid iterative method for approximating a common solution of a fixed point problem for a nonexpansive mapping T_1 and VIP(K_1, B_1) for a monotone and Lipschitz continuous mapping and proved a strong convergence theorem: The sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ generated by iterative schemes:

$$\begin{cases} x^0 = x \in K_1, \\ y^n = P_{K_1}(x^n - \lambda^n B_1 x^n), \\ z^n = \alpha^n x^n + (1-\alpha^n) T_1 P_{K_1}(x^n - \lambda^n B_1 y^n), \\ C^n = \{z \in K_1 : \|z^n - z\|^2 \leq \|x^n - z\|^2\}, \\ Q^n = \{z \in K_1 : \langle x^n - z, x - x^n \rangle \geq 0\}, \\ x^{n+1} = P_{C^n \bigcap Q^n} x, \ \forall n \geq 0. \end{cases} \tag{1.8}$$

For the related work, see Djafari-Rouhani et al. [5]

Motivated by the work of Nadezhkina and Takahashi[14], we propose a hybrid extra-gradient iterative method for approximating a common solution to SUGMEP(1.5) for monotone and Lipschitz continuous mappings and SUHFPP(1.1) for nonexpansive mappings in Hilbert space. We prove that the sequences generated by the proposed iterative method converge strongly to the common solution to these problems. Further, we give some applications of our main result. Furthermore, we discuss a theoretical numerical example to demonstrate the applicability of the iterative algorithm of the main result. Our iterative algorithm is new and different from the iterative algorithm due to Nadezhkina and Takahashi[14]. We also give a comparison of a particular case of our iterative algorithm with the iterative algorithm due to [14]. The method and results presented in this paper extend and unify the related known results of this area, see for example [6].

2. Preliminaries

We recall some concepts and results which are needed in the sequel. Let the symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively, and $\omega_w(x^n)$ denote the set of all weak limits of the sequence $\{x^n\}$.

Definition 2.1. A mapping $A : H \to H$ is said to be:

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \forall x, y \in H;$$

(ii) λ-Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$\|Ax - Ay\| \leq \lambda \|x - y\|, \ \forall x, y \in H;$$

(iii) β-inverse strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|Ax - Ay\|^2, \ \forall x, y \in H.$$

We note that β-inverse strongly monotone mapping is monotone and $\frac{1}{\beta}$-Lipschitz continuous but converse need not be true in general.

Lemma 2.1. [1]

(i) *Let M be a maximal monotone operator then $\{(t^n)^{-1}M\}$ graph converges to $N_{M^{-1}(0)}$ as $t^n \to 0$ provided that $M^{-1}(0) \neq \emptyset$;*

(ii) *Let $\{B^n\}$ be a sequence of maximal monotone operators which graph converges to an operator B. If M is a Lipschitz maximal monotone operator then $\{M + B^n\}$ graph converges to $M + B$ and $M + B$ is maximal monotone.*

Assumption 2.1. The bifunctions $G : K \times K \longrightarrow \mathbb{R}$ and $\phi : K \times K \to \mathbb{R}$ satisfy the following assumptions:

(i) $G(x, x) = 0, \ \forall x \in K$;

(ii) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0, \ \forall x, y \in K$;

(iii) For each $y \in K$, $x \to G(x, y)$ is hemi-upper semicontinuous, i.e., for each $x, y, z \in K$, $\limsup_{t\to 0^+} G(tz + (1 - t)x, y) \leq G(x, y)$;

(iv) For each $x \in K$, $y \to G(x, y)$ is convex and lower semicontinuous;

(v) $\phi(\cdot, \cdot)$ is weakly continuous and convex;

(vi) ϕ is skew symmetric, i.e., $\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, x) \geq 0, \ \forall x, y \in K$;

(vii) for each $z \in H$ and for each $x \in K$, there exists a bounded subset $D_x \subseteq K$ and $z_x \in K$ such that for any $y \in K \setminus D_x$,

$$G(y, z_x) + \phi(z_x, y) - \phi(y, y) + \frac{1}{r}\langle z_x - y, y - z\rangle < 0.$$

Assumption 2.2. [5] The bifunction $G : K \times K \longrightarrow \mathbb{R}$ is 2-monotone, i.e.,

$$G(x, y) + G(y, z) + G(z, x) \leq 0, \ \forall x, y, z \in K. \tag{2.1}$$

In particular, if we set $z = x$ or $x = y$ or $y = z$ in (2.1) then 2-monotone bifunction becomes a monotone bifunction. For example, if $G(x, y) = x(y - x)$ then G is a 2-monotone bifunction.

Now, we give the concept of 2-skew-symmetric bifunction.

Definition 2.2. The bifunction $\phi : K \times K \to \mathbb{R}$ is said to be 2-skew-symmetric if

$$\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, z) + \phi(z, z) - \phi(z, x) \geq 0, \ \forall x, y, z \in K. \tag{2.2}$$

We observe that if we set $z = x$ or $x = y$ or $y = z$ in (2.2) then 2-skew-symmetric bifunction becomes a skew-symmetric bifunction.

Theorem 2.1. [4] *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let the bifunctions $G : K \times K \longrightarrow \mathbb{R}$ and $\phi : K \times K \to \mathbb{R}$ satisfying Assumption 2.1. For $r > 0$ and $z \in H$, define a mapping $T_r : H \to K$ as follows:*

$$T_r(z) = \{x \in K : G(x, y) + \phi(y, x) - \phi(x, x) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in K\},$$

for all $z \in H$. Then the following conclusions hold:

(a) $T_r(z)$ *is nonempty for each* $z \in H$;
(b) T_r *is single valued;*
(c) T_r *is firmly nonexpansive mapping, i.e., for all* $z_1, z_2 \in H$,

$$\|T_r z_1 - T_r z_2\|^2 \le \langle T_r z_1 - T_r z_2, z_1 - z_2\rangle;$$

(d) $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{GMEP}(1.7))$;
(e) $\mathrm{Sol}(\mathrm{GMEP}(1.7))$ *is closed and convex.*

Remark 2.1. It follows from Theorem 2.1 (a)-(b) that

$$rG(T_r x, y) + r\phi(y, T_r(x)) - r\phi(T_r(x), T_r(x)) + \langle T_r(x) - x, y - T_r(x)\rangle \ge 0, \ \forall y \in K, \ x \in H. \tag{2.3}$$

Further, Theorem 2.1 (c) implies the nonexpansivity of T_r, i.e.,

$$\|T_r(x) - T_r(y)\| \le \|x - y\|, \ \forall x, y \in H. \tag{2.4}$$

Furthermore, (2.3) implies the following inequality

$$\begin{aligned} \|T_r(x) - y\|^2 \quad \le \quad & \|x - y\|^2 - \|T_r(x) - x\|^2 + 2rG(T_r(x), y) \\ & +2r[\phi(y, T_r(x)) - \phi(T_r(x), T_r(x))], \ \forall y \in K, x \in H. \end{aligned} \tag{2.5}$$

3. Main results

We prove a strong convergence theorem to approximate a common solution to SUGMEP(1.5) for monotone and Lipschitz continuous mappings and SUHFPP(1.1) for nonexpansive mappings in Hilbert space. First, we prove the following Minty type lemma.

Lemma 3.1. *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let the bifunctions $G : K \times K \longrightarrow \mathbb{R}$ and $\phi : K \times K \longrightarrow \mathbb{R}$ satisfy Assumption 2.1(i)-(iv) and Assumption 2.1(v)-(vi), respectively and let $B : K \to H$ be a monotone and Lipschitz continuous mapping. Then the solution set of problem: Find $x \in K$ such that*

$$G(x, y) + \langle Bx, y - x\rangle + \phi(x, y) - \phi(x, x) \ge 0, \ \forall y \in K, \tag{3.1}$$

is closed and convex. Further, it is also the solution set of problem: Find $x \in K$ such that

$$G(y, x) - \langle By, y - x\rangle - \phi(y, y) + \phi(y, x) \le 0, \ \forall y \in K. \tag{3.2}$$

Proof. Under the given assumptions on G, B and ϕ, we can easily prove that the solution set of problem (3.1) is closed and convex. Next, we show that both problems have the same solution set. In order to prove this, we prove that problem (3.1) is equivalent to problem (3.2). By the monotonicity of G, B and skew-symmetry of ϕ, it immediately follows that the inequality (3.1) implies inequality (3.2). Hence the solution of problem (3.1) is the solution of the problem (3.2).

Conversely, let $x \in K$ be a solution of problem (3.2) then we have

$$G(y, x) - \langle By, y - x\rangle - \phi(y, y) + \phi(y, x) \le 0, \ \forall y \in K.$$

For t with $0 < t \leq 1$ and $y \in K$, let $y_t = ty + (1-t)x \in K$, we have

$$G(y_t, x) - \langle By_t, y_t - x\rangle - \phi(y_t, y_t) + \phi(y_t, x) \geq 0.$$

Further, by the convexity of G and ϕ, we have

$$\begin{aligned} 0 &= G(y_t, y_t) \\ &\leq tG(y_t, y) + (1-t)G(y_t, x) \\ &\leq tG(y_t, y) + (1-t)[\langle By_t, y_t - x\rangle + \phi(y_t, y_t) - \phi(y_t, x)] \\ &\leq t[G(y_t, y) + (1-t)\langle By_t, y - x\rangle + \phi(y_t, y) - \phi(y_t, x)] \end{aligned}$$

and therefore, dividing by $t > 0$, we get

$$0 \leq G(y_t, y) + (1-t)\langle By_t, y - x\rangle + \phi(y_t, y) - \phi(y_t, x).$$

Letting $t \to 0^+$, and using the hemi-upper semicontinuity of G in the first variable and continuity of B, we get

$$G(x, y) + \langle Bx, y - x\rangle + \phi(x, y) - \phi(x, x) \geq 0, \ \forall y \in K,$$

and hence we get the desired result. □

Theorem 3.1. *For each $i = 1, 2, ..., N$, let K_i be a nonempty, closed and convex subset of a real Hilbert space H with $\bigcap_{i=1}^{N} K_i \neq \emptyset$. Let $G_i : K_i \times K_i \longrightarrow \mathbb{R}$ be a 2-monotone bifunction, let $\phi_i : K_i \times K_i \longrightarrow \mathbb{R}$ be a 2-skew symmetric bifunction satisfying Assumption 2.1(i),(iii)-(v),(vii) and let $B_i : K_i \to H$ be a monotone and Lipschitz continuous mapping with Lipschitz constant $\rho_i > 0$. For each i, let $S_i : K_i \to K_i$ and $T_i : K_i \to K_i$ be nonexpansive mappings. Assume that $\Omega = \Theta \bigcap \Phi \neq \emptyset$. Let the sequences $\{x^n\}$, $\{y_i^n\}$ and $\{z_i^n\}$ be generated by the following iterative schemes:*

$$\begin{cases} x^0 = x \in \overline{K} = \bigcap\limits_{i=1}^{N} K_i, \\ y_i^n = T_{r_i^n}(x^n - r_i^n B_i x^n), \\ u_i^n = T_{r_i^n}(x^n - r_i^n B_i y_i^n), \\ z_i^n = (1-\alpha_i^n)u_i^n + \alpha_i^n[\sigma_i^n S_i u_i^n + (1-\sigma_i^n)T_i u_i^n], \\ C_i^n = \{z \in K_i : \|z_i^n - z\|^2 \leq (1-\alpha_i^n\sigma_i^n)\|x^n - z\|^2 + \alpha_i^n\sigma_i^n\|S_i u_i^n - z\|^2\}, \\ C^n = \bigcap_{i=1}^{N} C_i^n, \\ Q^n = \{z \in \overline{K} : \langle x - x^n, x^n - z\rangle \geq 0\}, \\ x^{n+1} = P_{C^n \bigcap Q^n} x, \ n \geq 0, \end{cases} \tag{3.3}$$

for each $i = 1, 2, ..., N$, where $\{r_i^n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{2\rho})$, $\rho = \max\limits_{1 \leq i \leq N} \rho_i$, and $\{\alpha_i^n\}$, $\{\sigma_i^n\}$ are real sequences in $(0, 1)$. If the following conditions:

(i) $\lim\limits_{n\to\infty} \sigma_i^n = 0$;

(ii) $\lim\limits_{n\to\infty} \frac{\|u_i^n - z_i^n\|}{\alpha_i^n\sigma_i^n} = 0$, *for each i,* *hold, then the sequences* $\{x^n\}$, $\{y_i^n\}$ *and* $\{z_i^n\}$ *converge strongly to* $\hat{x} \in \Omega$, *where* $\hat{x} = P_\Omega x$, *a metric projection of* H *onto* Ω.

Proof. We divide the proof into several steps.

Step I. Ω and $C^n \cap Q^n$ for all $n \geq 0$ both are closed and convex and $\{x^n\}$ is well defined.

Proof of Step I. In order to prove that Ω is closed and convex, it is enough to show that for each $i = 1, 2, ..., N$ the solution set Γ_i of GMEP$_i$ (3.1), i.e.,

$$\Gamma_i = \{x \in K_i \ : \ G_i(x, y_i) + \langle B_i x, y_i - x\rangle + \phi_i(x, y_i) - \phi_i(x, x) \geq 0, \ \forall y_i \in K_i\}$$

is closed and convex, which is followed by Lemma 3.1. Further, it is evident that Φ is closed and convex, since $\Phi = \mathrm{Fix}(P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} \circ S_i) \neq \emptyset$. Thus Ω is nonempty, closed and convex and $P_\Omega x$ is then well defined. Next, we show that $C^n \cap Q^n$ is closed and convex. From the definition of Q^n, it is clear that Q^n is closed and convex for each $n \geq 0$. Next we show that C^n is closed and convex for all $n \geq 0$. It suffices to show that, for any fixed but arbitrary i, C_i^n is closed and convex for every $n \geq 0$. Indeed, for any $z \in C_i^n$, we see that $z \in K_i$ and

$$\|z_i^n - z\|^2 \leq (1 - \alpha_i^n \sigma_i^n)\|x^n - z\|^2 + \alpha_i^n \sigma_i^n \|S_i u_i^n - z\|^2$$

$$\Leftrightarrow \ \|z_i^n - x^n\|^2 + \|x^n - z\|^2 + 2\langle z_i^n - x^n, x^n - z\rangle \leq (1 - \alpha_i^n \sigma_i^n)\|x^n - z\|^2 + \alpha_i^n \sigma_i^n [\|S_i u_i^n - x^n\|^2 + \|x^n - z\|^2 + 2\langle S_i u_i^n - x^n, x^n - z\rangle]$$

$$\Leftrightarrow \ \|z_i^n - x^n\|^2 + 2\langle z_i^n - x^n, x^n - z\rangle - \alpha_i^n \sigma_i^n \langle S_i u_i^n - x^n, S_i u_i^n + x^n - 2z\rangle \leq 0,$$

which implies that C_i^n is closed and convex for all $n \geq 0$ and $i = 1, 2, .., N$. Consequently, $C^n \bigcap Q^n$ is closed and convex for all $n \geq 0$, and hence $x^{n+1} = P_{C^n \bigcap Q^n} x$ is well defined.

Step II. $\Omega \subset C^n \cap Q^n$ for each $n \geq 0$ and the sequences $\{x_n\}$, $\{u_i^n\}$ and $\{z_i^n\}$ are bounded.

Proof of Step II. Let $\bar{x} \in \Omega$ then $\bar{x} \in \Theta$ which implies that

$$G_i(\bar{x}, y_i^n) + \langle B_i \bar{x}, y_i^n - \bar{x}\rangle + \phi_i(y_i^n, \bar{x}) - \phi_i(\bar{x}, \bar{x}) \geq 0, \ \forall y_i^n \in K_i, \ i = 1, 2, ..., N. \quad (3.4)$$

Applying (2.5) with $x^n - r_i^n B_i y_i^n$ and $\bar{x}$, we get

$$\begin{aligned} \|u_i^n - \bar{x}\|^2 &= \|T_{r_i^n}(x^n - r_i^n B_i y_i^n) - \bar{x}\|^2 \\ &\leq \|x^n - r_i^n B_i y_i^n - \bar{x}\|^2 - \|u_i^n - (x^n - r_i^n B_i y_i^n\|^2 + 2r_i^n G_i(u_i^n, \bar{x}) \\ &\quad + 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n)] \\ &\leq \|x^n - \bar{x}\|^2 - \|u_i^n - x^n\|^2 + 2r_i^n\langle B_i y_i^n, \bar{x} - u_i^n\rangle + 2r_i^n G_i(u_i^n, \bar{x}) \\ &\quad + 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n)] \\ &\leq \|x^n - \bar{x}\|^2 - \|u_i^n - x^n\|^2 + 2r_i^n[\langle B_i y_i^n - B_i \bar{x}, \bar{x} - y_i^n\rangle \\ &\quad + \langle B_i \bar{x}, \bar{x} - y_i^n\rangle + \langle B_i y_i^n, y_i^n - u_i^n\rangle + 2r_i^n G_i(u_i^n, \bar{x}) \\ &\quad + 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n)]. \end{aligned} \quad (3.5)$$

Now, using monotonicity of B_i and (3.4) in above inequality, we obtain

$$\begin{aligned}
\|u_i^n - \bar{x}\|^2 &\leq \|x^n - \bar{x}\|^2 - \|u_i^n - x^n\|^2 + 2r_i^n\langle B_i y_i^n, y_i^n - u_i^n\rangle \\
&+ 2r_i^n[G_i(\bar{x}, y_i^n) + G_i(u_i^n, \bar{x})] + 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n) \\
&+ \phi_i(y_i^n, \bar{x}) - \phi_i(\bar{x}, \bar{x})] \\
&\leq \|x^n - \bar{x}\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - u_i^n\|^2 - 2\langle x^n - y_i^n, y_i^n - u_i^n\rangle \\
&+ 2r_i^n\langle B_i y_i^n, y_i^n - u_i^n\rangle + 2r_i^n[G_i(\bar{x}, y_i^n) + G_i(u_i^n, \bar{x})] \\
&+ 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n) + \phi_i(y_i^n, \bar{x}) - \phi_i(\bar{x}, \bar{x})] \\
&\leq \|x^n - \bar{x}\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - u_i^n\|^2 \\
&- 2\langle y_i^n - (x^n - r_i^n B_i x^n), u_i^n - y_i^n\rangle + 2r_i^n\langle B_i x^n - B_i y_i^n, u_i^n - y_i^n\rangle \\
&+ 2r_i^n[G_i(\bar{x}, y_i^n) + G_i(u_i^n, \bar{x})] + 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n) \\
&+ \phi_i(y_i^n, \bar{x}) - \phi_i(\bar{x}, \bar{x})] \\
&\leq \|x^n - \bar{x}\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - u_i^n\|^2 + 2r_i^n\langle B_i x^n - B_i y_i^n, u_i^n - y_i^n\rangle \\
&+ 2r_i^n[G_i(\bar{x}, y_i^n) + G_i(y_i^n, u_i^n) + G_i(u_i^n, \bar{x})] + 2r_i^n[\phi_i(\bar{x}, u_i^n) - \phi_i(u_i^n, u_i^n) \\
&+ \phi_i(y_i^n, \bar{x}) - \phi_i(\bar{x}, \bar{x}) + \phi_i(u_i^n, y_i^n) - \phi_i(y_i^n, y_i^n)]. \qquad (3.6)
\end{aligned}$$

For each i, since G_i is 2-monotone and ϕ_i is 2-skew symmetric then it follows from (3.6) that

$$\begin{aligned}
\|u_i^n - \bar{x}\|^2 &\leq \|x^n - \bar{x}\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - u_i^n\|^2 + 2r_i^n\langle B_i x^n - B_i y_i^n, u_i^n - y_i^n\rangle \\
&\leq \|x^n - \bar{x}\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - u_i^n\|^2 + 2r_i^n\rho\|x^n - y_i^n\|\|u_i^n - y_i^n\| \\
&\leq \|x^n - \bar{x}\|^2 - (1 - r_i^n\rho)\|x^n - y_i^n\|^2 - (1 - r_i^n\rho)\|y_i^n - u_i^n\|^2, \qquad (3.7)
\end{aligned}$$

where we have used ρ-Lipschitz continuity of B_i with $\rho = \max\limits_{1\leq i\leq N} \rho_i$ in the second inequality.

Further, since $r_i^n \in [a, b]$ and $a, b \in (0, \frac{1}{2\rho})$, we obtain

$$\|u_i^n - \bar{x}\|^2 \leq \|x^n - \bar{x}\|^2. \qquad (3.8)$$

Again, since $\bar{x} \in \Omega$ then $\bar{x} \in K_i$ and $\bar{x} \in \Phi$ which implies that $\bar{x} = T_i\bar{x}$ for each $i = 1, 2, ..., N$. Then using (3.3) and (3.8), we get

$$\begin{aligned}
\|z_i^n - \bar{x}\|^2 &= \|(1 - \alpha_i^n)u_i^n + \alpha_i^n(\sigma_i^n S_i u_i^n + (1 - \sigma_i^n)T_i u_i^n) - \bar{x}\|^2 \\
&= \|(1 - \alpha_i^n)(u_i^n - \bar{x}) + \alpha_i^n(\sigma_i^n S_i u_i^n + (1 - \sigma_i^n)T_i u_i^n - \bar{x})\|^2 \\
&\leq (1 - \alpha_i^n)\|u_i^n - \bar{x}\|^2 + \alpha_i^n(\sigma_i^n\|S_i u_i^n - \bar{x}\|^2 + (1 - \sigma_i^n)\|T_i u_i^n - \bar{x}\|^2) \\
&\leq (1 - \alpha_i^n)\|u_i^n - \bar{x}\|^2 + \alpha_i^n(\sigma_i^n\|S_i u_i^n - \bar{x}\|^2 + (1 - \sigma_i^n)\|u_i^n - \bar{x}\|^2) \\
&\leq (1 - \alpha_i^n\sigma_i^n)\|u_i^n - \bar{x}\|^2 + \alpha_i^n\sigma_i^n\|S_i u_i^n - \bar{x}\|^2 \qquad (3.9) \\
&\leq (1 - \alpha_i^n\sigma_i^n)\|x^n - \bar{x}\|^2 + \alpha_i^n\sigma_i^n\|S_i u_i^n - \bar{x}\|^2. \qquad (3.10)
\end{aligned}$$

This implies that $\bar{x} \in C^n$ and hence $\Omega \subset C^n$, $\forall n \geq 0$. Further, since $\Omega \subset C^0$ and $\Omega \subset Q^0 = H$, we have $\Omega \subset C^0 \cap Q^0$. Now, suppose that $\Omega \subset C^{n-1} \cap Q^{n-1}$ for some $n > 1$. Since Ω is nonempty, $C^{n-1} \cap Q^{n-1}$ is a nonempty, closed and convex set.

So there exists a unique element $x^n \in C^{n-1} \cap Q^{n-1}$ such that

$$x^n = P_{C^{n-1} \cap Q^{n-1}} x.$$

Again, since $\Omega \subseteq C^n$ and for any $\bar{x} \in \Omega$, it follows from (2.4) that

$$\langle x - x^n, x^n - \bar{x} \rangle = \langle x - P_{C^{n-1} \cap Q^{n-1}} x, P_{C^{n-1} \cap Q^{n-1}} x - \bar{x} \rangle \geq 0,$$

and hence $\bar{x} \in Q^n$. Therefore $\Omega \subset C^n \cap Q^n$, $\forall n \geq 0$.

Next, let $d = P_\Omega x$. From $x^{n+1} = P_{C^n \cap Q^n} x$ and $d \in \Omega \subset C^n \cap Q^n$, we have

$$\|x^{n+1} - x\| \leq \|d - x\|, \ \forall n \geq 0. \tag{3.11}$$

Therefore $\{x^n\}$ is bounded. It also follows from (3.8) that the sequence $\{u_i^n\}$ is bounded for each $i = 1, 2, ..., N$. Further the nonexpansivity of S_i and T_i imply that the sequences $\{S_i u_i^n\}$ and $\{T_i u_i^n\}$ are bounded for each $i = 1, 2, ..., N$. Since $\{\alpha_i^n\}$, $\{\sigma_i^n\}$ are bounded, it follows from (3.10) that the sequence $\{z_i^n\}$ is bounded for each $i = 1, 2, ..., N$.

Step III. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$; $\lim_{n\to\infty} \|x_n - z_i^n\| = 0$; $\lim_{n\to\infty} \|x_n - y_i^n\| = 0$;

$$\lim_{n\to\infty} \|x_n - u_i^n\| = 0; \ \lim_{n\to\infty} \|u_i^n - y_i^n\| = 0; \ \lim_{n\to\infty} \|z_i^n - u_i^n\| = 0.$$

Proof of Step III. Since $x^{n+1} \in C^n \cap Q^n$ and $x^n = P_{Q^n} x$, we have

$$\|x^n - x\| \leq \|x^{n+1} - x\|, \ \forall n \geq 0. \tag{3.12}$$

Therefore, it follows from (3.12) that the sequence $\{\|x^n - x\|\}$ is monotonically increasing and bounded, and hence convergent. Therefore, $\lim_{n\to\infty} \|x^n - x\|$ exists and finite. Now, the characterization of $P_{Q^n} x$ with $x^n = P_{Q^n} x$ and $x^{n+1} \in Q^n$ gives

$$\|x^{n+1} - x^n\|^2 \leq \|x^{n+1} - x\|^2 - \|x^n - x\|^2, \ \forall n \geq 0,$$

which implies that

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0. \tag{3.13}$$

Since $x^{n+1} \in C_i^n$, we have

$$\|z_i^n - x^{n+1}\|^2 \leq (1 - \alpha_i^n \sigma_i^n) \|x^n - x^{n+1}\|^2 + \alpha_i^n \sigma_i^n \|S_i u_i^n - x^{n+1}\|^2. \tag{3.14}$$

Since $\{x^n\}$, $\{u_i^n\}$ and $\{S_i u_i^n\}$ are bounded, there exists a number $L > 0$ such that $\|S_i u_i^n - x^{n+1}\| \leq L$, $\forall n$. Hence, it follows from (3.13), (3.14) and $\lim_{n\to\infty} \sigma_i^n = 0$ that

$$\lim_{n\to\infty} \|z_i^n - x^{n+1}\| = 0. \tag{3.15}$$

Further, it follows from the inequality

$$\|x^n - z_i^n\| \leq \|x^n - x^{n+1}\| + \|x^{n+1} - z_i^n\|, \tag{3.16}$$

(3.13) and (3.15) that

$$\lim_{n\to\infty} \|x^n - z_i^n\| = 0. \tag{3.17}$$

Since $(1-\alpha_i^n\sigma_i^n)<1$, it follows from (3.7) and (3.9) that

$$\begin{aligned}\|z_i^n-\bar{x}\|^2 &\leq \|u_i^n-\bar{x}\|^2+\alpha_i^n\sigma_i^n\|S_iu_i^n-\bar{x}\|^2\\ &\leq \|x^n-\bar{x}\|^2-(1-r_i^n\rho)\|x^n-y_i^n\|^2-(1-r_i^n\rho)\|y_i^n-u_i^n\|^2\\ &\quad +\alpha_i^n\sigma_i^n\|S_iu_i^n-\bar{x}\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}(1-r_i^n\rho)\|x^n-y_i^n\|^2 &\leq \|x_n-z_i^n\|(\|x^n-\bar{x}\|+\|z_i^n-\bar{x}\|)+\alpha_i^n\sigma_i^n\|S_iu_i^n-\bar{x}\|^2\\ &\leq \|x_n-z_i^n\|M_1+\alpha_i^n\sigma_i^nM_2 \qquad (3.18)\end{aligned}$$

and

$$(1-r_i^n\rho)\|y_i^n-u_i^n\|^2 \leq \|x_n-z_i^n\|M_1+\alpha_i^n\sigma_i^nM_2 \qquad (3.19)$$

where $M_1:=\max_i\sup_n\{\|x^n-\bar{x}\|+\|z_i^n-\bar{x}\|\}$ and $M_2:=\max_i\sup_n\{\|S_iu_i^n-\bar{x}\|^2\}$. Since $\{x^n\}$, $\{u_i^n\}$ and $\{z_i^n\}$ are bounded, $\{r_i^n\}\subset[a,b]$ for some $a,b\in(0,\frac{1}{2\rho})$, $\rho=\max_{1\leq i\leq N}\rho_i$ then it follows from (3.17), (3.18), (3.19) and $\lim_{n\to\infty}\sigma_i^n=0$ that

$$\lim_{n\to\infty}\|x^n-y_i^n\|=0,\ \text{for each } i=1,2,...,N. \qquad (3.20)$$

$$\lim_{n\to\infty}\|u_i^n-y_i^n\|=0,\ \text{for each } i=1,2,...,N. \qquad (3.21)$$

Since

$$\|x_n-u_i^n\|\leq\|x_n-y_n\|+\|y_n-u_i^n\|, \qquad (3.22)$$

it follows from (3.20), (3.21) and (3.22), we have

$$\lim_{n\to\infty}\|x^n-u_i^n\|=0,\ \text{for each } i=1,2,...,N. \qquad (3.23)$$

Now, it follows from (3.17) and (3.23) that

$$\lim_{n\to\infty}\|z_i^n-u_i^n\|=0,\ \text{for each } i=1,2,...,N. \qquad (3.24)$$

Step IV: $\lim_{n\to\infty}\|u_i^n-T_iu_i^n\|=0$ for each $i=1,2,...,N$.

Proof of Step IV. We have

$$\|u_i^n-T_iu_i^n\|\leq\|u_i^n-z_i^n\|+\|z_i^n-T_iu_i^n\|. \qquad (3.25)$$

Since $\{S_iu_i^n\}$ and $\{T_iu_i^n\}$ are bounded for each $i=1,2,...,N$ then there exists a $L_1>0$ such that $\|S_iu_i^n-T_iu_i^n\|\leq L_1,\ \forall n\geq 0$. Now, by making use of (3.25), we estimate

$$\begin{aligned}\|z_i^n-T_iu_i^n\| &= \|(1-\alpha_i^n)u_i^n+\alpha_i^n(\sigma_i^nS_iu_i^n+(1-\sigma_i^n)T_iu_i^n)-T_iu_i^n\|\\ &= \|(1-\alpha_i^n)(u_i^n-T_iu_i^n)+\alpha_i^n(\sigma_i^nS_iu_i^n-\sigma_i^nT_iu_i^n)\|\\ &\leq (1-\alpha_i^n)\|u_i^n-T_iu_i^n\|+\alpha_i^n\sigma_i^n\|S_iu_i^n-T_iu_i^n\|\\ &\leq (1-\alpha_i^n)\|u_i^n-z_i^n\|+(1-\alpha_i^n)\|z_i^n-T_iu_i^n\|+\alpha_i^n\sigma_i^n\|S_iu_i^n-T_iu_i^n\|,\end{aligned}$$

which implies that

$$\|z_i^n-T_iu_i^n\|\leq\frac{\|u_i^n-z_i^n\|}{\alpha_i^n}+\sigma_i^nL_1. \qquad (3.26)$$

Since $\lim_{n\to\infty} \frac{\|z_i^n - u_i^n\|}{\alpha_i^n \sigma_i^n} = 0$, then $\lim_{n\to\infty} \frac{\|z_i^n - u_i^n\|}{\alpha_i^n} = \lim_{n\to\infty} \sigma_i^n \frac{\|z_i^n - u_i^n\|}{\alpha_i^n \sigma_i^n} = 0$ and hence, using (3.24) and $\lim_{n\to\infty} \sigma_i^n = 0$ in (3.26), we have

$$\lim_{n\to\infty} \|z_i^n - T_i u_i^n\| = 0, \text{ for each } i = 1, 2, ..., N. \tag{3.27}$$

Thus, it follows from (3.24), (3.25) and (3.27) that

$$\lim_{n\to\infty} \|u_i^n - T_i u_i^n\| = 0, \text{ for each } i = 1, 2, ..., N. \tag{3.28}$$

Step V: $\hat{x} \in \Omega$.

Proof of Step V. Since $\{x^n\}$ is bounded, there exists a $\hat{x} \in \omega_w(x_n)$. Further, since every Hilbert space satisfies Opial's condition, Opial's condition guarantees that $\omega_w(x_n)$ is singleton. Thus, $\{x_n\}$ converges weakly to $\hat{x}$. Further, it follows from (3.17), (3.20) and (3.23) that the sequences $\{x^n\}$, $\{y_i^n\}$, $\{u_i^n\}$ and $\{z_i^n\}$ all have same asymptotic behavior and hence $\{y_i^n\}$, $\{u_i^n\}$ and $\{z_i^n\}$ converge weakly to $\hat{x}$.

Now, it follows from demiclosed principle and (3.28) that $\hat{x} \in \text{Fix}(T_i)$ for each $i = 1, 2, ..., N$. Next, we show that $\hat{x} \in \Phi$. It follows from algorithm (3.3) that

$$z_i^n - u_i^n = \alpha_i^n(\sigma_i^n(S_i u_i^n - u_i^n) + (1 - \sigma_i^n)(T_i u_i^n - u_i^n)),$$

and hence

$$\frac{1}{\alpha_i^n \sigma_i^n}(u_i^n - z_i^n) = \left((I - S_i) + \frac{1 - \sigma_i^n}{\sigma_i^n} \left(I - T_i \right) \right) u_i^n. \tag{3.29}$$

Since, for each $i = 1, 2, ..., N$, S_i, T_i are nonexpansive, we have that $(I - S_i)$, $(I - T_i)$ are maximal monotone operators [1] and hence Lemma 2.1(i) assures that the operator sequence $\left\{ \left(\frac{1 - \sigma_i^n}{\sigma_i^n} \left(I - T_i \right) \right) \right\}$ graph converges to $N_{\text{Fix}(T_i)}$ and hence it follows from Lemma 2.1(ii) that the operator sequence $\left\{ (I - S_i) + \frac{1 - \sigma_i^n}{\sigma_i^n} \left(I - T_i \right) \right\}$ graph converges to $(I - S_i) + N_{\text{Fix}(T_i)}$.

Now, passing to the limit in (3.29) as $n \to \infty$ and by taking into account the fact that $\frac{\|u_i^n - z_i^n\|}{\alpha_i^n \sigma_i^n} \to 0$ and that the graph of $(I - S_i) + N_{\text{Fix}(T_i)}$ is weakly-strongly closed, we obtain $0 \in (I - S_i)\hat{x} + N_{\text{Fix}(T_i)}\hat{x}$ and thus $\hat{x} \in \Phi$.

Next, we show that $\hat{x} \in \Theta$. Since K_i is closed and convex, $y_i^n \in K_i$ and $y_i^n \rightharpoonup \hat{x}$, it follows that $\hat{x} \in K_i$ and hence $\hat{x} \in \bigcap_{i=1}^{N} K_i$. Now, the relation $y_i^n = T_{r_i^n}(x^n - r_i^n B_i x^n)$ is equivalent to

$$G_i(y_i^n, y_i) + \langle B_i x^n, y_i - y_i^n \rangle + \phi_i(y_i, y_i^n) - \phi_i(y_i^n, y_i^n) + \frac{1}{r_i^n}\langle y_i - y_i^n, y_i^n - x^n \rangle \geq 0, \ \forall y_i \in K_i.$$

Since G_i is 2-monotone and hence monotone, the above inequality implies

$$\langle B_i x^n, y_i - y_i^n \rangle + \phi_i(y_i, y_i^n) - \phi_i(y_i^n, y_i^n) + \frac{1}{r_i^n}\langle y_i - y_i^n, y_i^n - x^n \rangle \geq G_i(y_i, y_i^n), \ \forall y_i \in K_i.$$

For t with $0 < t \leq 1$ and $y_i \in K_i$, let $y_{i,t} := ty_i + (1-t)\hat{x} \in K_i$, we have

$$\begin{aligned}\langle y_{i,t} - y_i^n, B_i y_{i,t}\rangle &\geq \langle y_{i,t} - y_i^n, B_i y_{i,t}\rangle - \phi_i(y_{i,t}, y_i^n) + \phi_i(y_i^n, y_i^n) - \langle y_{i,t} - y_i^n, B_i x^n\rangle \\ &\quad - \left\langle y_{i,t} - y_i^n, \frac{y_i^n - x^n}{r_i^n}\right\rangle + G_i(y_{i,t}, y_i^n) \\ &= \langle y_{i,t} - y_i^n, B_i y_{i,t} - B_i y_i^n\rangle + \langle y_{i,t} - y_i^n, B_i y_i^n - B_i x^n\rangle \\ &\quad - \phi_i(y_{i,t}, y_i^n) + \phi_i(y_i^n, y_i^n) - \left\langle y_{i,t} - y_i^n, \frac{y_i^n - x^n}{r_i^n}\right\rangle + G_i(y_{i,t}, y_i^n).\end{aligned}$$

Since $\lim_{n\to\infty} \|y_i^n - x^n\| = 0$ and B_i is Lipschitz continuous, we have

$$\lim_{n\to\infty} \|B_i y_i^n - B_i x^n\| = 0, \quad \text{for each } i = 1, 2, ..., N.$$

Further, from the monotonicity of B_i, the convexity and lower semicontinuity of G_i in the second variable and the weak lower semi-continuity of ϕ_i and the fact that $\frac{\|y_i^n - x^n\|}{r_i^n} \to 0$ and $y_i^n \rightharpoonup \hat{x}$, by letting $n \to \infty$, we deduce that

$$\langle y_{i,t} - \hat{x}, B_i y_{i,t}\rangle \geq -\phi_i(y_{i,t}, \hat{x}) + \phi_i(\hat{x}, \hat{x}) + G_i(y_{i,t}, \hat{x}). \tag{3.30}$$

Further, by the convexity of G_i, we have

$$\begin{aligned} 0 &= G_i(y_{i,t}, y_{i,t}) \\ &\leq tG_i(y_{i,t}, y_i) + (1-t)G_i(y_{i,t}, \hat{x}) \\ &\leq tG_i(y_{i,t}, y_i) + (1-t)[\phi_i(y_{i,t}, \hat{x}) - \phi_i(\hat{x}, \hat{x}) + \langle y_{i,t} - \hat{x}, B_i y_{i,t}\rangle] \\ &\leq tG_i(y_{i,t}, y_i) + (1-t)t[\phi_i(y_i, \hat{x}) - \phi_i(\hat{x}, \hat{x})] + (1-t)t\langle y_i - \hat{x}, B_i y_{i,t}\rangle \end{aligned}$$

and therefore, dividing by $t > 0$, we get

$$0 \leq G_i(y_{i,t}, y_i) + (1-t)[\phi_i(y_i, \hat{x}) - \phi_i(\hat{x}, \hat{x})] + (1-t)\langle y_i - \hat{x}, B_i y_{i,t}\rangle, \text{ for each } i = 1, 2, ..., N.$$

Letting $t \to 0^+$ and using the hemi-upper semicontinuity of G_i in the first variable, we get

$$G_i(\hat{x}, y_i) + \langle y_i - \hat{x}, B_i\hat{x}\rangle + \phi_i(y_i, \hat{x}) - \phi_i(\hat{x}, \hat{x}) \geq 0, \ \forall y_i \in K_i.$$

This implies that $\hat{x} \in \Theta$.

Step VI: Finally, we show that $x^n \to \hat{x}$, where $\hat{x} = P_\Omega x$.

Proof of Step VI. Since $x^n = P_{Q^n} x$ and $\hat{x} \in \Omega \subset Q^n$, we have

$$\|x^n - x\| \leq \|\hat{x} - x\|.$$

It follows from $d = P_\Omega x$, (3.11) and lower semicontinuity of the norm that

$$\|d - x\| \leq \|\hat{x} - x\| \leq \liminf_{n\to\infty} \|x^n - x\| \leq \limsup_{n\to\infty} \|x^n - x\| \leq \|d - x\|.$$

Thus, we have

$$\lim_{n\to\infty} \|x^n - x\| = \|d - x\| = \|\hat{x} - x\|.$$

Since $x^n - x \rightharpoonup \hat{x} - x$ and $\|x^n - x\| \to \|\hat{x} - x\|$ then from the Kadec-Klee property of Hilbert space, we have $\lim_{n\to\infty} x^n = \hat{x} = d$. Thus, we conclude that $\{x^n\}$ converges strongly to $\hat{x}$, where $\hat{x} = P_\Omega x$. □

4. Application

We have the following strong convergence theorem for an iterative method to approximate a common solution of SUVIP(1.6) and a common fixed point problem (CFPP) for a finite family of nonexpansive mappings T_i.

Theorem 4.1. *For each $i = 1, 2, ..., N$, let K_i be a nonempty, closed and convex subset of a real Hilbert space H with $\bigcap_{i=1}^{N} K_i \neq \emptyset$. Let $B_i : K_i \to H$ be a monotone and Lipschitz continuous mapping with Lipschitz constant $\rho_i > 0$. For each i, let $T_i : K_i \to K_i$ be a nonexpansive mapping. Assume that*

$$\Omega_2 = \Theta_1 \cap \left(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \right) \neq \emptyset.$$

Let the sequences $\{x^n\}$, $\{y_i^n\}$ and $\{z_i^n\}$ be generated by the following iterative schemes:

$$\begin{cases} x^0 = x \in \overline{K} = \bigcap\limits_{i=1}^{N} K_i, \\ y_i^n = P_{K_i}(x^n - r_i^n B_i x^n), \\ u_i^n = P_{K_i}(x^n - r_i^n B_i y_i^n), \\ z_i^n = (1 - \alpha_i^n) u_i^n + \alpha_i^n [\sigma_i^n u_i^n + (1 - \sigma_i^n) T_i u_i^n], \\ C_i^n = \{ z \in K_i : \| z_i^n - z \|^2 \leq (1 - \alpha_i^n \sigma_i^n) \| x^n - z \|^2 + \alpha_i^n \sigma_i^n \| u_i^n - z \|^2 \}, \\ C^n = \bigcap_{i=1}^{N} C_i^n, \\ Q^n = \{ z \in \overline{K} : \langle x - x^n, x^n - z \rangle \geq 0 \}, \\ x^{n+1} = P_{C^n \bigcap Q^n} x, \ n \geq 0, \end{cases} \tag{4.1}$$

for each $i = 1, 2, ..., N$, where $\{r_i^n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{2\rho})$, $\rho = \max\limits_{1 \leq i \leq N} \rho_i$, and $\{\alpha_i^n\}$, $\{\sigma_i^n\}$ are real sequences in $(0, 1)$. If the following conditions:

(i) $\lim\limits_{n \to \infty} \sigma_i^n = 0$;

(ii) $\lim\limits_{n \to \infty} \frac{\| u_i^n - z_i^n \|}{\alpha_i^n \sigma_i^n} = 0$, *for each i,*

hold, then the sequences $\{x^n\}$, $\{y_i^n\}$ and $\{z_i^n\}$ converge strongly to $\hat{x} \in \Omega_2$, where $\hat{x} = P_{\Omega_2} x$.

Proof. For each $i = 1, 2, ..., N$, set $G_i = 0$, $\phi_i = 0$ and $S_i = I_i$ then $T_{r_i^n} = P_{K_i}$ and hence by Theorem 3.1 we obtain the desired result. □

Remark 4.1. The iterative algorithm (4.1) with $N = 1$ approximates a common element of the solution set of VIP(K_1, B_1) and the fixed point set of T_1. This is new and different from the iterative algorithm (1.8) due to Nadezhkina and Takahashi [14]. Further, we observe through an example (see Remark 5.1) that it is more rapidly convergent than the iterative algorithm (1.8) [14].

Finally, if we set $N = 1$, $G_i = 0$, $\phi_i = 0$ and $S_i = T_i = I_i$ then $T_{r_i^n} = P_{K_i}$ in Theorem 3.1 we obtain the result due to Iiduka et al. [6] for the case when the mapping B_i is β_i-inverse strongly monotone.

5. Numerical example

Now, we give a theoretical numerical example which justifies Theorem 3.1.

Example 5.1. Let $H = \mathbb{R}$ with the usual inner product $< \cdot, \cdot >$ and induced norm $|\cdot|$. Let $i = 1, 2, 3, ..., 10$ and let $K_i = (-\infty, \frac{1}{3}]$ so that

$$\overline{K} = \bigcap_{1\le i\le 10} K_i = (-\infty, \frac{1}{3}].$$

For each i, let the mappings $G_i : K_i \times K_i \to \mathbb{R}$ and $\phi_i : K_i \times K_i \to \mathbb{R}$ be defined by

$$G_i(x, y) = i(x + 1)(y - x) \text{ and } \phi_i(x, y) = i(x - y), \forall x, y \in K_i,$$

respectively; let the mapping $B_i : K_i \to \mathbb{R}$ be defined by

$$B_i(x) = i(2x - 3), \forall x \in K_i$$

and let the mappings $S_i, T_i : K_i \to K_i$ be defined by

$$S_i x = \frac{x + 2i}{1 + 6i},\ T_i x = \frac{x + i}{1 + 3i}\ \forall x \in K_i,$$

respectively. Setting $\alpha_i^n = \frac{0.9}{i\sqrt{n}}$ and $\sigma_n = \frac{0.9}{in}$, $\forall n \ge 1$. Then the sequence $\{x_n\}$ generated by iterative scheme (3.3) converges to $\hat{x} = \frac{1}{3} \in \Omega$.

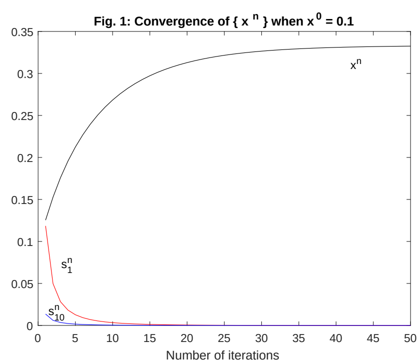
Proof. We observe that, for each i, G_i and ϕ_i both satisfy all the conditions given in Theorem 3.1. Further, it is easy to prove that B_i is monotone and $2i$-Lipschitz continuous and hence $\rho = \max_{1\le i\le 10} \rho_i = 20$. Now, we can choose $a = 0.01$, $b = 2.4$ and therefore, choose $r_i^n = \frac{1}{5i} \in [a, b]$ for $a, b \in (0, \frac{1}{2\rho})$. We observe that S_i, T_i are nonexpansive mappings with $\text{Fix}(S_i) = \text{Fix}(T_i) = \{\frac{1}{3}\}$. Furthermore, it is easy to prove that $\Gamma = \Phi = \{\frac{1}{3}\}$ and hence $\Omega = \{\frac{1}{3}\}$. Now, the iterative scheme (3.3) is reduced to the following scheme:

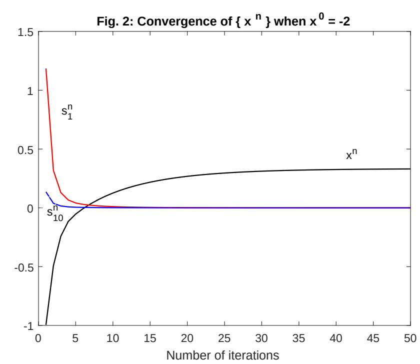
$$\begin{cases} x^0 = x \in \overline{K}, \\ y_i^n = \frac{3x^n+1}{6}, \\ u_i^n = \frac{5x^n-2y_i^n+1}{6}, \\ z_i^n = (1-\alpha_i^n)u_i^n + \alpha_i^n[\sigma_i^n S_i u_i^n + (1-\sigma_i^n)T_i u_i^n], \\ C_i^n = [e_i^n, +\infty) \text{ where } e_i^n = \dfrac{{z_i^n}^2 - {x^n}^2 + \alpha_i^n\sigma_i^n\left({x^n}^2 - \left(\frac{u_i^n+2i}{1+6i}\right)^2\right)}{2(z_i^n - x^n) + 2\alpha_i^n\sigma_i^n\left(x^n - \left(\frac{u_i^n+2i}{1+6i}\right)\right)}, \\ C^n = \bigcap_{i=1}^{10} C_i^n, \\ Q^n = [x^n, +\infty), \\ x^{n+1} = P_{C^n \bigcap Q^n} x,\ n \ge 0. \end{cases} \tag{5.1}$$

Next, using the software Matlab 7.8, we have following table and figures which show that the sequence $\{x_n\}$ converges to $\hat{x} = \frac{1}{3} \in \Omega$. Denote

$$s_1^n := \frac{\|u_1^n - z_1^n\|}{\alpha_1^n \sigma_1^n} \text{ and } s_{10}^n := \frac{\|u_{10}^n - z_{10}^n\|}{\alpha_{10}^n \sigma_{10}^n}.$$

n	x_n $x_0 = -2$	s_1^n	s_{10}^n	n	x_n $x_0 = 0.1$	s_1^n	s_{10}^n
1	-0.995000	1.185000	0.135684	**1**	0.125351	0.118500	0.013568
5	-0.052812	0.041445	0.005215	**5**	0.212394	0.012720	0.001601
10	0.125809	0.010675	0.001360	**10**	0.268338	0.003343	0.000426
15	0.218102	0.003915	0.000501	**15**	0.297243	0.001226	0.000157
20	0.268076	0.001654	0.000212	**20**	0.312895	0.000518	0.000066
25	0.295871	0.000757	0.000097	**25**	0.321600	0.000237	0.000030
30	0.311610	0.000365	0.000047	**30**	0.326530	0.000114	0.000015
35	0.320636	0.000182	0.000023	**35**	0.329357	0.000057	0.000007
40	0.325864	0.000094	0.000012	**40**	0.330994	0.000029	0.000004
45	0.328916	0.000049	0.000006	**45**	0.331950	0.000015	0.000002
46	0.329355	0.000043	0.000006	**46**	0.332087	0.000014	0.000002
47	0.329749	0.000038	0.000005	**47**	0.332211	0.000012	0.000002
48	0.330103	0.000034	0.000004	**48**	0.332322	0.000011	0.000001
49	0.330422	0.000030	0.000004	**49**	0.332422	0.000009	0.000001
50	0.330709	0.000026	0.000003	**50**	0.332512	0.000008	0.000001





This completes the proof. □

Remark 5.1. For the $N = 1$, we demonstrate that the iterative algorithm (4.1) with conditions given in Theorem4.1, approximates a common element of the solution set of VIP(K_1, B_1) and the fixed point set of T_1. Further, we observe that it is faster than the iterative algorithm (1.8) due to Nadezhkina and Takahashi[14].

Set $N = 1$ and $S_i = I_i$, the identity operators, in Example 5.1, we have $K_1 = (-\infty, \frac{1}{3}]$, $B_i x = B_1 x = 3x - 1$, $T_i x = T_1 x = \frac{x+1}{4}$, $\alpha_1^n = \frac{0.9}{\sqrt{n}}$ and $\sigma_n = \frac{0.9}{n}$, $\forall n \geq 1$, $r_1^n = \frac{1}{8} \in [a,b]$ for $a, b \in (0, \frac{1}{2\rho})$, $\forall n \geq 1$.. We easily observe that the $B_1, T_1, \alpha_1^n, \sigma_1^n$ satisfy all the conditions given in Theorem 4.1 and in Theorem 3.1[14]. It is clear that $\mathrm{Sol(VIP(K_1, B_1))} = \{\frac{1}{3}\}$, $\mathrm{Fix(T_1)} = \{\frac{1}{3}\}$ and hence $\Omega = \mathrm{Sol(VIP(K_1, B_1)))} \bigcap \mathrm{Fix(T_1)} = \{\frac{1}{3}\}$. In this case, the iterative algorithm (4.1) and the iterative algorithm (1.8) reduce to the following iterative algorithms:

Iterative Algorithm 5.2:

$$\begin{cases} x^0 = x \in K_1, \\ y_1^n = P_{K_1}(x^n - r_1^n B_1 x^n) = \begin{cases} 0, \ \text{if} x < 0, \\ 1, \ \text{if} x > 1, \\ x^n - \frac{1}{8}(3x^n - 1), \ \text{otherwise}, \end{cases} \\ u_1^n = P_{K_1}(x^n - r_1^n B_1 y_1^n)) = \begin{cases} x^n + \frac{1}{8}, \ \text{if} x < 0, \\ x^n - \frac{1}{4}, \ \text{if} x > 1, \\ x^n - \frac{1}{8}(3y_1^n - 1), \ \text{otherwise}, \end{cases} \\ z_1^n = (1 - \alpha_1^n) u_1^n + \alpha_1^n [\sigma_1^n u_1^n + (1 - \sigma_1^n) T_1 u_1^n], \\ C_1^n = [e_1^n, +\infty) \text{ where } e_1^n = \frac{{z_1^n}^2 - {x^n}^2 + \alpha_1^n \sigma_1^n ({x^n}^2 - {u_1^n}^2)}{2(z_1^n - x^n) + 2\alpha_1^n \sigma_1^n (x^n - z_1^n)}, \\ C^n = \bigcap C_1^n, \\ Q^n = [x^n, +\infty), \\ x^{n+1} = P_{C^n \bigcap Q^n} x, \ n \geq 0. \end{cases} \tag{5.2}$$

and

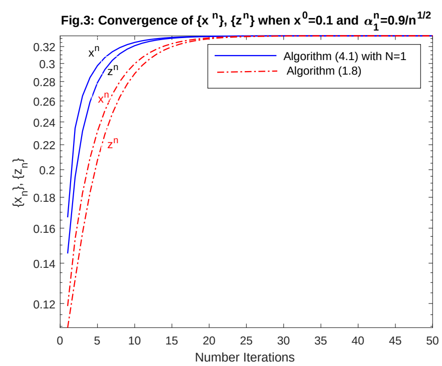
Iterative Algorithm 5.3:

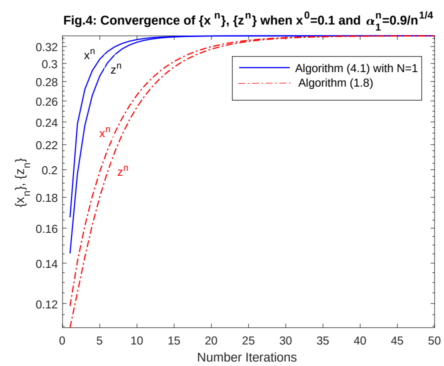
$$\begin{cases} x^0 = x \in K_1, \\ y_1^n = P_{K_1}(x^n - r_1^n B_1 x^n) = \begin{cases} 0, \ \text{if} x < 0, \\ 1, \ \text{if} x > 1, \\ x^n - \frac{1}{8}(3x^n - 1), \ \text{otherwise}, \end{cases} \\ u_1^n = P_{K_1}(x^n - r_1^n B_1 y_1^n)) = \begin{cases} x^n + \frac{1}{8}, \ \text{if} x < 0, \\ x^n - \frac{1}{4}, \ \text{if} x > 1, \\ x^n - \frac{1}{8}(3y_1^n - 1), \ \text{otherwise}, \end{cases} \\ z_1^n = \alpha_1^n x^n + (1 - \alpha_1^n) T_1 u_1^n, \\ C_1^n = [e_1^n, +\infty) \text{ where } e_1^n = \frac{z_1^n + x^n}{2}, \\ C^n = \bigcap C_1^n, \\ Q^n = [x^n, +\infty), \\ x^{n+1} = P_{C^n \bigcap Q^n} x, \ n \geq 0, \end{cases} \tag{5.3}$$

respectively.

Hence, the sequence $\{x_n\}$ defined by Iterative Algorithm 5.2 as well as defined by Iterative Algorithm 5.3 converges strongly to $\hat{x} = \frac{1}{3}$.

Finally, using the software Matlab 7.8, we have following figures which show that the sequences $\{x_n\}, \{z_n\}$ converges to $\hat{x} = \frac{1}{3} \in \Omega$. Figure 3 shows the convergence of $\{x^n\}$ and $\{z^n\}$ when $x^0 = 0.1$ and $\alpha_1^n = \frac{0.9}{\sqrt{n}}$, while Figure 4 shows the convergence of $\{x^n\}$ and $\{z^n\}$ when $x^0 = 0.1$ and $\alpha_1^n = \frac{0.9}{n^{\frac{1}{4}}}$. It is evident from figures that the sequence $\{x_n\}$ obtained by Iterative Algorithm 5.2 converges faster than the sequence $\{x_n\}$ obtained by Iterative Algorithm 5.3.





6. Conclusion

Future directions to be pursued in the context of this research include the investigation of the problem when the mappings B_i are set-valued mappings as in [3] as well as the investigation of the problem to extend the hybrid extra-gradient iterative method to find the common solution of HFPP (1.2) and monotone variational inclusion problem ([8], inclusion (1.8)) for monotone and Lipschitz continuous mapping.

Acknowledgments. The authors are very thankful to the anonymous referees for their critical comments and helpful suggestions which led to substantial improvements in the original version of the manuscript.

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Received: June 7, 2018 ; Accepted: October 31, 2019.