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APPLICATIVE APPROACH OF FIXED POINT THEOREMS TOWARDS VARIOUS ENGINEERING PROBLEMS

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Abstract. Utilization of fixed point theory, especially to Engineering problems, is of prime concern in recent times. In this article, we aim to firstly establish some original fixed point results in the metric like spaces and then utilize the same to solve those problems which emphasize primarily the applications for the existence of the solution of equations arising in Rocket science, Electrical engineering and, Mechanical engineering. In this article, offered contractive conditions are of general type, having index $l \in \mathbb{N}$ on underlying mapping which refine and expand various results in the existing theory and thereby giving a pathway to applications-based problems. Moreover, to address conceptual depth within this approach, we supply illustrative examples where necessary, which is, of course, of interest of Engineers and Mathematicians. Computer simulation is adopted to verify the contractive conditions giving more-in-depth insight. We suggest some open problems for future work on the application of fixed point theory.
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1. Introduction and basic facts

These days fixed point theory plays an integral part in applied mathematics and utilized in various disciplines of science and engineering *e.g.*, optimization theory, physics, chemistry, computer science, civil engineering, Mechanical Engineering, Economics, Medical Science, etc. Recently many researchers focused on the applicative approach of fixed point theorems. For noteworthy contributions we refer some of them mentioned below:
• Halabi [4], utilizing the fixed point technique in the area of pattern recognition.
• Abdon and Baleanu [1] modified the nonlinear Schrodinger equation by using the new Caputo-Liouville derivative with fractional order and established the stability of the iteration scheme using the fixed point theorem.

• Ozgur and Karaca [9] introduced the digital version of the Banach fixed point theorem and validated a vital application to digital images.
• M. K. Moghadam [7] established the existence of a non-trivial solution for fourth-order elastic beam equations occurring in structure engineering problems.
• Jung et al. [5] developed fixed point results to ascertain the stability of a functional equation of the spiral of Theodorus,

$$f(x+1) = \left(1 + \frac{1}{\sqrt{x+1}}\right) f(x).$$

• Singh et al. [15] utilized their results to the equations arising in the Oscillation of spring for Mechanical Engineering problem.
• Singh et al. in [14] invoked their results to establish the existence of solution of first-order periodic boundary value problem

$$u'(t) = k(t, u(t)), \quad t \in I = [0, T], \quad \text{and} \quad u(0) = u(t).$$

Where $T > 0$ and $k : I \times R \to R$ is a continuous function.

One of the most important generalizations of metric space is metric like space (in short mls) introduced by Amini Harandi [3], where the authors replaced the triangular inequality with the weaker one gives as follows.

Definition 1.1. [3] *A function $\sigma : X \times X \to [0, \infty)$ is called a metric-like if for all $x, y, z \in X$, the following conditions are satisfied:*
$(\sigma 1)$ $\sigma(x, y) = 0$ *implies* $x = y$;
$(\sigma 2)$ $\sigma(x, y) = \sigma(y, x)$;
$(\sigma 3)$ $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.
The pair (X, σ) is called a metric-like space. Note that, a metric-like satisfies all the conditions of metric except that $\sigma(x, x)$ may be positive for $x \in X$.

For more synthesis on the space along with the concept like Cauchy sequence, convergence and example etc., we refer the reader to [3]. In 2012, Wardowski [16] introduced a new contraction called F-contraction by defining certain properties (F_1, F_2 and F_3) of mapping $F : R^+ \to R$ and proved a fixed point result as a generalization of the Banach contraction principle in different way. On the same line Secelean [13] changed the condition F_2 of [16] by F_2' and later on Piri and Kumam [10] replaced condition F_3 of [16] by F_3'.
In this present work, we take F_1 of [16]) and F_3' of [10] and denote the class of functions satisfying F_1 and F_3' by Δ_F.

We note some features of the present work in the following.
• We consider Boyd-Wong type $l - F$ Suzuki contraction and its variants.
• We utilize Boyd-Wong type $l - F$ Suzuki contraction and its variants for the existence of fixed point.
• We present some revolutionary examples which validate the hypothesis of proved results.
• Theorems proved in this article generalize several findings in the existing literature.
• Utilizing obtained results, we establish the existence of solution of boundary value problems, which are mathematical models of Electrical circuits, buckling of a tapered column, the motion of rocket, and deformation of an elastica.

• Finally, we discuss an open problem related to our results.

In the process, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}^*$ will represent the set of all real numbers, natural numbers and $\{0\} \cup \mathbb{N}$, respectively.

Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that

(1) ϕ is upper semi-continuous i.e. for any sequence $\{t_n\}$ in $[0, \infty)$ such that $t_n \to t$ as $n \to \infty$, we have $\limsup_{n\to\infty} \phi(t_n) \le \phi(t)$;

(2) $\phi(t) < t$ for each $t > 0$.

Let Ψ is the collection of all continuous functions $\psi : (0, \infty) \to (0, \infty)$.

2. Boyd-Wong type $l - F$ Suzuki contractive mappings and fixed point results

We start this section by introducing our very first and important subsequent definition.

Definition 2.1. *Let (X, σ) be metric like space. A self mapping $T : X \to X$ is said to be Boyd-Wong type l-F Suzuki contraction of type I, if there exists $F \in \Delta_F, \psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $\sigma(T^{l+1}x, Ty) > 0$,*

$$\frac{1}{2}\sigma(T^l x, T^{l+1}x) < \sigma(T^l x, y) \Rightarrow F(\sigma(T^{l+1}x, Ty)) \le F\Big(\phi(M_l(x, y))\Big) - \psi\Big(M_l(x, y)\Big), \tag{2.1}$$

in which

$$M_l(x, y) = \max\Big\{\sigma(T^l x, y), \sigma(T^l x, T^{l+1}x), \sigma(y, Ty), \frac{\sigma(T^l x, Ty) + \sigma(T^{l+1}x, y)}{4}\Big\}, \tag{2.2}$$

and $l \in \mathbb{N}^*(= \{0\} \cup \mathbb{N})$, such that $T^0x = Ix$, where I is identity mapping.

Theorem 2.1. *Let (X, σ) be a complete metric-like space. If T is Boyd-Wong type l-F Suzuki contraction of type I . Then T has a unique fixed point $x^* \in X$.*

Proof. Let $x_0 \in X$ be arbitrary point in X. We construct a sequence $\{x_n\}$ by

$$Tx_n = x_{n+1}. \tag{2.3}$$

If there exists $n_0 \in \mathbb{N}$ such that $\sigma(x_{n_0}, x_{n_0+1}) = 0$, then x_{n_0} is the desired fixed point and proof is completed. Then for the subsequent discussion, suppose $\sigma(x_n, x_{n+1}) > 0$ for every $n \in \mathbb{N}^*$ Thus for $x = x_{n-1}, y = x_{n+l}$, for all $n \in \mathbb{N}^*$, we have

$$\sigma(T^{l+1}x, Ty) = \sigma(T^{l+1}x_{n-1}, Tx_{n+l}) = \sigma(x_{n+l}, x_{n+l+1}) > 0, \text{ as } n + l \in \mathbb{N}^*. \tag{2.4}$$

Hence from (2.4), we have

$$\begin{aligned}\frac{1}{2}\sigma(T^l x_{n-1}, T^{l+1}x_{n-1}) = \frac{1}{2}\sigma(x_{n+l-1}, x_{n+l}) &\le \sigma(x_{n+l-1}, x_{n+l}) \\ &= \sigma(T^l x_{n-1}, x_{n+l}) \text{ for all } n,\ l \in \mathbb{N}^*.\end{aligned}$$

Then by the assumption of the theorem, we have

$$\begin{aligned} F(\sigma(x_{n+l}, x_{n+l+1})) &= F(\sigma(T^{l+1}x_{n-1}, Tx_{n+l})) \\ &\le \big(F(\phi(M_l(x_{n-1}, x_{n+l})))\big) - \psi\big(M_l(x_{n-1}, x_{n+l})\big). \end{aligned} \tag{2.5}$$

Here

$$M_l(x_{n-1}, x_{n+l}) = \max\Big\{\sigma(T^l x_{n-1}, x_{n+l}), \sigma(T^l x_{n-1}, T^{l+1} x_{n-l}), \sigma(x_{n+l}, Tx_{n+l}), \frac{\sigma(T^l x_{n-1}, Tx_{n+l}) + \sigma(T^{l+1} x_{n-1}, x_{n+l})}{4}\Big\}.$$

With routine calculation, we obtain that

$$M_l(x_{n-1}, x_{n+l}) = \max\Big\{\sigma(x_{n+l-1}, x_{n+l}), \sigma(x_{n+l}, x_{n+l+1})\Big\}.$$

Now, if $\max\Big\{\sigma(x_{n+l-1}, x_{n+l}), \sigma(x_{n+l}, x_{n+l+1})\Big\} = \sigma(x_{n+l}, x_{n+l+1})$. Then from (2.5), one can get

$$\begin{aligned} F(\sigma(x_{n+l}, x_{n+l+1})) &\leq F(\phi(\sigma(x_{n+l}, x_{n+l+1}))) - \psi(\sigma(x_{n+l}, x_{n+l+1})) \\ &< F(\sigma(x_{n+l}, x_{n+l+1})) - \psi(\sigma(x_{n+l}, x_{n+l+1})). \end{aligned}$$

Which is a contradiction in view of (2.4) and the hypothesis of ψ.
Thus we deduce that

$$\max\Big\{\sigma(x_{n+l-1}, x_{n+l}), \sigma(x_{n+l}, x_{n+l+1})\Big\} = \sigma(x_{n+l-1}, x_{n+l}). \tag{2.6}$$

Then from (2.5) and by the concept of ϕ and ψ,

$$\begin{aligned} F(\sigma(x_{n+l}, x_{n+l+1})) &\leq F(\phi(\sigma(x_{n+l-1}, x_{n+l}))) - \psi(\sigma(x_{n+l-1}, x_{n+l})) \\ &< F(\sigma(x_{n+l-1}, x_{n+l})). \end{aligned} \tag{2.7}$$

In view of F_1, it immediately follows that

$$\sigma(x_{n+l}, x_{n+l+1}) < \sigma(x_{n+l-1}, x_{n+l}). \tag{2.8}$$

Now we claim that $u = 0$. Suppose to the contrary that $u > 0$. On making $n \to \infty$ in (2.7), we have $F(u) \leq F(\phi(u)) - \psi(u) < F(u) - \psi(u)$. This leads to a contradiction. Then we must have $u = 0$. i.e.

$$\lim_{n\to\infty} \sigma(x_{n+l}, x_{n+l+1}) = 0. \tag{2.9}$$

Further, we maintain that $\{x_n\}$ is a σ-Cauchy sequence.
Suppose on the contrary that $\{x_n\}$ is not a σ-Cauchy sequence, then there exists $\epsilon > 0$ and two sub-sequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$, such that for every $n(k) \geq m(k) > k$,

$$\sigma(x_{m(k)+l}, x_{n(k)+l}) \geq \epsilon \tag{2.10}$$

or

$$\sigma(T^{l+1} x_{n(k)-1}, Tx_{m(k)+l-1}) \geq \epsilon$$

Now, corresponding to $m(k)$, we can select $n(k)$ in such a manner that it is the smallest integer with $n(k) > m(k)$ satisfying (2.10).
Thus, we have

$$\sigma(x_{m(k)+l}, x_{n(k)+l-1}) < \epsilon.$$

Utilizing (2.10), one can obtain

$$\begin{aligned}\epsilon &\leq \sigma(x_{n(k)+l}, x_{m(k)+l})\\ &\leq \sigma(x_{n(k)+l}, x_{n(k)+l-1}) + \sigma(x_{n(k)+l-1}, x_{m(k)+l})\\ &< \sigma(x_{n(k)+l}, x_{n(k)+l-1}) + \epsilon.\end{aligned}$$

Thus, we obtain

$$\epsilon \leq \sigma(x_{n(k)+l}, x_{m(k)+l}) < \sigma(x_{n(k)+l}, x_{n(k)+l-1}) + \epsilon. \tag{2.11}$$

Passing limit $n \to \infty$ on (2.11) with (2.9), we have

$$\lim_{n\to\infty} \sigma(x_{n(k)+l}, x_{m(k)+l}) = \epsilon. \tag{2.12}$$

Again for all $n \in \mathbb{N}^*$, we have

$$\begin{aligned}\sigma(x_{n(k)+l+1}, x_{m(k)+l+1}) &\leq \sigma(x_{n(k)+l+1}, x_{m(k)+l}) + \sigma(x_{m(k)+l}, x_{m(k)+l+1})\\ &\leq \sigma(x_{n(k)+l+1}, x_{n(k)+l}) + \sigma(x_{n(k)+l}, x_{m(k)+l})\\ &+ \sigma(x_{m(k)+l}, x_{m(k)+l+1}).\end{aligned} \tag{2.13}$$

Letting $n \to \infty$ to (2.13), one can get

$$\lim_{n\to\infty} \sigma(x_{n(k)+l+1}, x_{m(k)+l+1}) = \epsilon. \tag{2.14}$$

From (2.10) and (2.14), we have

$$\sigma(T^{l+1}x_{n(k)}, Tx_{m(k)+l}) > \epsilon > 0$$

and from (2.9) and (2.11), we must have

$$\frac{1}{2}\sigma(T^l x_{n(k)}, T^{l+1}x_{n(k)}) = \frac{1}{2}\sigma(x_{n(k)+1}, x_{n(k)+l+1}) < \frac{\epsilon}{2} < \sigma(T^l x_{n(k)}, x_{m(k)+l}).$$

Then by assumption of theorem, we have

$$\begin{aligned}F(\sigma(x_{n(k)+l+1}, x_{m(k)+l+1})) &= F(\sigma(T^{l+1}x_{n(k)}, Tx_{m(k)+l}))\\ &\leq F\Big(\phi(M_l(x_n(k), x_{m(k)+l}))\Big) - \psi\Big(M_l(x_n(k), x_{m(k)+l})\Big).\end{aligned} \tag{2.15}$$

Here, with elementary calculation, we arrive at

$$\lim_{n\to\infty} M_l(x_{n(k)}, x_{m(k)+l}) = \max\{\epsilon, 0, 0\} = \epsilon. \tag{2.16}$$

Then passing limit sup. as $n \to \infty$ to (2.15), one can get from (2.14) and (2.16) that

$$\begin{aligned}F(\epsilon) &\leq F(\phi(\epsilon)) - \psi(\epsilon)\\ &< F(\epsilon) - \psi(\epsilon),\end{aligned}$$

a contradiction since $\psi(\epsilon) > 0$ as $\epsilon > 0$.

Hence $\{x_n\}$ is a σ-Cauchy sequence in X. Since (X, σ) is a complete metric-like space, then there exists x^* in X such that

$$\lim_{n\to\infty} x_n = x^* \text{ or } \lim_{n\to\infty} x_{n+l} = x^*, \quad l \in \mathbb{N}^*.$$

Then

$$\sigma(x^*, x^*) = \lim_{n\to\infty} \sigma(x_{n+l}, x^*) = \lim_{n,m\to\infty} \sigma(x_{n+l}, x_{m+l}) = 0. \tag{2.17}$$

Now we assert that, for every $n \in \mathbb{N}$,

$$\frac{1}{2}\sigma(x_{n+l}, Tx_{n+l}) < \sigma(x_{n+l}, x^*) \text{ or } \frac{1}{2}\sigma(Tx_{n+l}, T^2x_{n+l}) < \sigma(Tx_{n+l}, x^*). \tag{2.18}$$

Arguing by contradiction, we assume that there exists $p \in \mathbb{N}$ such that

$$\frac{1}{2}\sigma(x_{p+l}, Tx_{p+l}) \geq \sigma(x_{p+l}, x^*) \text{ and } \frac{1}{2}\sigma(Tx_{p+l}, T^2x_{p+l}) \geq \sigma(Tx_{p+l}, x^*). \tag{2.19}$$

Therefore

$$2\sigma(x_{p+l}, x^*) \leq \sigma(x_{p+l}, Tx_{p+l}) \leq \sigma(x_{p+l}, x^*) + \sigma(x^*, Tx_{p+l}),$$

this implies that

$$\sigma(x_{p+l}, x^*) \leq \sigma(x^*, Tx_{p+l}). \tag{2.20}$$

Utilizing (2.8) and (2.18), we have

$$\begin{aligned} \sigma(Tx_{p+l}, T^2x_{p+l}) &< \sigma(x_{p+l}, Tx_{p+l}) \\ &\leq \sigma(x_{p+l}, x^*) + \sigma(x^*, Tx_{p+l}) \\ &= 2\sigma(Tx_{p+l}, x^*) \end{aligned} \tag{2.21}$$

Which is a contradiction in view of (2.19). Thus (2.18) holds.
Consider, if part I of (2.18) is true and $\sigma(x^*, Tx^*) > 0$, then one has

$$\begin{aligned} F(\sigma(x_{n+l+1}, Tx^*)) &= F(\sigma(T^{l+1}x_n, Tx^*)) \\ &\leq F(\phi(M_l(x_n, x^*))) - \psi(M_l(x_n, x^*)). \end{aligned} \tag{2.22}$$

Letting $n \to \infty$ and employing (2.9) and (2.17), we have

$$\lim_{n\to\infty}(M_l(x_n, x^*) = \sigma(x^*, Tx^*).$$

Then from (2.22), on passing limit sup. as $n \to \infty$, we get

$$\begin{aligned} F(\sigma(x^*, Tx^*)) &\leq F(\phi(\sigma(x^*, Tx^*))) - \psi(\sigma(x^*, Tx^*)) \\ &< F(\sigma(x^*, Tx^*)) - \psi(\sigma(x^*, Tx^*)), \end{aligned}$$

which is a contradiction in light of F_1 and ψ, thus we must have

$$\sigma(x^*, Tx^*) = 0 \implies Tx^* = x^*$$

i.e. x^* is a fixed point of T.
If part II of (2.18) is true and $\sigma(x^*, Tx^*) > 0$ then employing a similar approach as above, we conclude that $Tx^* = x^*$. Hence x^* is a fixed point of T.
With elementary and regular calculation one can easily establish the uniqueness of obtained fixed point. □

2.1. **Consequences of theorems.** Subsequent result is an easy consequence of Theorem 2.1

Corollary 2.1. *Let (X, σ) be a complete metric-like space. A mapping $T : X \to X$ be such that, for all $x, y \in X$ with $\sigma(T^{l+1}x, Ty) > 0$,*

$$\frac{1}{2}\sigma(T^l x, T^{l+1}x) < \sigma(T^l x, y) \Rightarrow F(\sigma(T^{l+1}x, Ty)) \leq F\Big(\alpha M_l(x, y)\Big) - \psi\Big(M_l(x, y)\Big), \tag{2.23}$$

where $M_l(x, y)$ is defined by (2.2), $F \in \Delta_F, \psi \in \Psi, l \in \mathbb{N}^*$ and $0 \leq \alpha < 1$. Then T has a unique fixed point $x^* \in X$.

Proof. Proof is immediate by putting $\phi(t) = \alpha t$ for $0 \leq \alpha < 1$ in Theorem 2.1. □

If we put $\psi(t) = \tau$, where $\tau > 0$ is constant in Theorem 2.1, then following corollary is obtained which is considered as $\phi - l$ Wardowski type result in metric-like spaces.

Corollary 2.2. *Let (X, σ) be a complete metric-like space. A mapping $T : X \to X$ be such that, for all $x, y \in X$ with $\sigma(T^{l+1}x, Ty) > 0$,*

$$\frac{1}{2}\sigma(T^l x, T^{l+1}x) < \sigma(T^l x, y) \Rightarrow F(\sigma(T^{l+1}x, Ty)) \leq F\Big(\alpha\phi(M_l(x, y)\Big) - \tau, \tag{2.24}$$

where $M_l(x, y)$ is defined by (2.2), $F \in \Delta_F, \phi \in \Phi, l \in \mathbb{N}^*, \tau > 0$ and $0 \leq \alpha < 1$. Then T has a unique fixed point $x^* \in X$.

Remark 2.1. *By choosing function ϕ, ψ and the value of l suitably in Theorem 2.1 and Corollaries 2.1, 2.2, one can deduce a multitude of results from the existing literature which includes many celebrated results i.e., if we put $\psi(t) = \tau > 0$ and $l = 0$ in Corollary 2.1, we get Theorem(2.2) of* [6] *and accordingly other results. Due to analogy, we skip the mentioning of all the effects here.*

2.2. **Validation of results.** To show the substantiation of our findings, we expound an example which demonstrates the superiority of our results.

Example 2.1. Let $X = [0, 1]$ and let the function $\sigma : X^2 \to [0, \infty)$ be defined by $\sigma(x, y) = x^2 + y^2$, for all $x, y \in X$. It is clear that $\sigma(x, y)$ is a complete metric-like space.
Let the mapping $T : X \to X$ be defined by $Tx = \frac{x^2}{1+x}$. In order to check the validity of condition (2.1) with $F(p) = \log p + p$, clearly $F \in \Delta_F$. Let $\phi : (0, \infty) \to (0, \infty)$ be given by $\phi(t) = \frac{50t}{51}$ and function $\psi : [0, \infty) \to [0, \infty)$ be defined by $\psi(t) = \frac{t}{200}$. Here we must notice that

$$\sigma(T^{l+1}x, Ty) > 0 \text{ if } (x = 0 \wedge y > 0) \vee (x > 0 \wedge y = 0) \vee (x > 0, y > 0).$$

Also for all $l \in \mathbb{N}^*$ and $x \in X$, it is evident that $T^{l+1}x \leq T^l x$, then we have

$$\frac{1}{2}\sigma(T^l x, T^{l+1}x) < \sigma(T^l x, y).$$

Then subsequent cases are discussed.

Case I. When $x = 0$ and $y > 0$, then for sure $T^l x = 0$ for $l \in \mathbb{N}^*$. In this case we calculate the terms involved in the inequality (2.1), we obtain that the left hand side of (2.1) becomes

$$F(\sigma(T^{l+1}x, Ty)) = \log\left(\frac{y^2}{1+y}\right)^2 + \left(\frac{y^2}{1+y}\right)^2$$

and right hand side of (2.1) comes out

$$F\Big(\phi(M_l(x,y))\Big) - \psi\Big(M_l(x,y)\Big) = \frac{50}{51}\left(y^2 + \left(\frac{y^2}{1+y}\right)^2\right) + \log\left(\frac{50}{51}\left(y^2 + \left(\frac{y^2}{1+y}\right)^2\right)\right) - \frac{y^2 + \left(\frac{y^2}{1+y}\right)^2}{200}.$$

From the following figure, it is verified that right hand side with red surface surpassing the purple surface representing left hand side, thus left hand side is less than right hand side and Condition (2.1) is verified in this case.

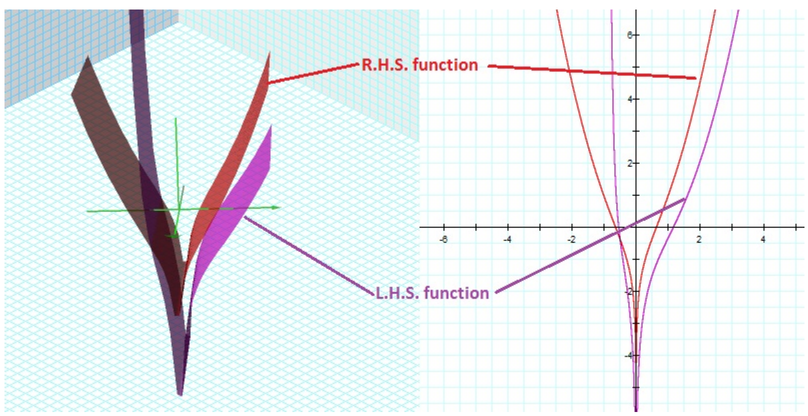

FIGURE 1. Domination of right hand side over left hand side for Case I

Case II. For $x > 0$ and $y = 0$. Calculating as above for $l = 0$, with routine calculation one can easily observe that left hand side is dominated by right hand side for $l = 0$. Similar result obtain when we substitute different values of $l \in \mathbb{N}$ in this case.

Case III. When $x > 0, y > 0$, then without loss of generality we assume that $x > y$, then following sub-cases arise.

Sub-Case I. If $x > y > T^l x > Ty$, then **for** $l = 0$,, then it is easily concluded that for all $x, y \in X$, equation (2.1) is satisfied. Furthermore for all $l \in \mathbb{N}$ in this sub-case, the same result is obtained.

Sub-Case II. If $x > T^l x > y > T^{l+1}x$, then the similar conclusion as in sub-case I is obtained.

Sub-Case III. If $x > y > Ty > T^l x$, then Figure 2 is showing the validity of (2.1) in this sub-case.

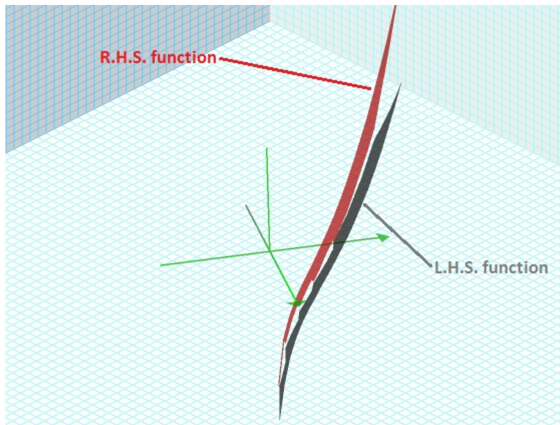

FIGURE 2. plot of left hand side and right hand side of contractive condition for sub-case III when l=0

Similarly for other values of $l \in \mathbb{N}^*$ and for all possible sub-cases, one can easily deduce that condition (2.1) is satisfied.
Thus T is a Boyd-Wong type $l-F$- Suzuki contraction of type I.Thus all the conditions of Theorem 2.1 are fulfilled and hence T has a unique fixed point $x = 0$ which is demonstrated in the Figure 3.

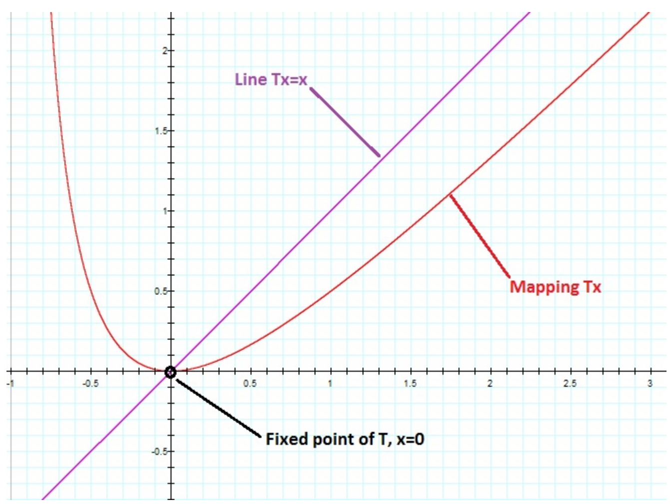

FIGURE 3. Plot showing that x=0 is the unique fixed point in [0,1]

3. Applications of fixed point results

3.1. **Application to electrical engineering through a fractional differential equation.** We observe that many standard Engineering and physics models are today employing fractional differential equations. In [2], the authors maintained the connection between basic equations of electric circuits involving resistors, capacitors, and inductors and fractional differential equations. They analysed the following equations for three different types of circuits

$$I''(t) + \frac{I(t)}{LC} = 0 \tag{3.1}$$

$$CV'(t) + \frac{V(t)}{R} = 0 \tag{3.2}$$

$$LI'(t) + RI(t) = V \tag{3.3}$$

where $I(t)$ is the current in the circuit at time t, L is the inductance, C, the capacitance, R is the resistance and V is the voltage drop across the circuit. Equation (3.1) represents the LC (inductor-capacitor) circuit, equation (3.2) represents RC(resistor-capacitor) circuit and equation (3.3) represents LR (inductor-resistor) circuit.
Using suitable replacements, authors [2] convert the above equations as fractional differential equation of the type (3.4). Thus , in this section, we establish the existence of solution of equation of various electrical circuits through the presence of solution of fractional differential equation.
Firstly, some definitions from the theory of fractional calculus are provided.
The Reiman-Liouville fractional derivative of order $\beta > 0$ for a function $u \in C[0, 1]$ is defined by

$$D^{\beta}u(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)\, ds}{(t-s)^{\beta-n+1}}$$

provided that the right hand side is point-wise defined on [0,1]. Where $n = [\beta] + 1$ and $[\beta]$ means the integral part of the number β and Γ is the Euler gamma function.
Next, consider the following fractional boundary value problem

$$\begin{aligned} &{}^{c}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 \geq t \geq 1, \quad 1 < \alpha \geq 2; \\ &u(0) = u(1) = 0, \end{aligned} \tag{3.4}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous function and ${}^{c}D^{\alpha}$ represents the Caputo fractional derivative of order α and it is defined by

$${}^{c}D^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)\, ds}{(t-s)^{\alpha-n+1}}.$$

Consider the space $X = C([0, 1], \mathbb{R})$ of continuous functions defined on [0,1] such that X is endowed with the metric-like mapping

$$\sigma(u, v) = \sup_{t\in[0,1]} (|u(t)| + |v(t)|), \quad u, v \in C([0, 1], \mathbb{R}).$$

Clearly $(X, \sigma)$is a complete metric-like space.

Theorem 3.1. *Consider, the nonlinear fractional differential equation* (3.4)*. Assume that the following assertions hold:*

(i) there exists $\tau > 0$, $\epsilon \in [0,1)$ and mapping $T : X \to X$ such that for all $x, y \in C([0,1],\mathbb{R})$ and for $l \in \mathbb{N}$,

$$\begin{aligned} |f^{l+1}(t,x)| + |f(t,y)| \leq e^{-\tau}\epsilon \max\{&|T^l(x)| + |y|, |T^l(x)| + |T^{l+1}(x)|, \\ &|T^l(y)| + |y|, \frac{|T^l(x)| + |T^{l+1}(x)| + |T^l(y)| + |y|}{4}\}; \end{aligned} \tag{3.5}$$

(ii) $\sup_{t\in[0,1]} \int_0^1 (G(t,s))^{l+1}\, ds \leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\, ds \leq 1.$

Then the problem (3.4) *has a unique solution in X.*

Proof. The problem (3.4) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(s,x(s))\, ds,$$

for all $x \in X$ and $t \in [0,1]$. In which

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq T, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Consider the mapping $T : X \to X$ defined by

$$Tx(t) = \int_0^1 G(t,s) f(s,x(s))\, ds.$$

It is easy to notice that if $u^* \in X$ is a fixed point of T then x^* is a solution of the problem (3.4).
Now, for $x, y \in X$, we obtain

$$\begin{aligned} |T^{l+1}x(t)| + |Ty(t)| &= \left[\left|\int_0^1 G(t,s) f(s,x(s))\, ds\right|\right]^{l+1} + \left|\int_0^1 G(t,s) f(s,y(s))\, ds\right| \\ &\leq \int_0^1 G(t,s) |f^{l+1}(s,x(s))| + |f(s,y(s))| ds \\ &\leq \int_0^1 G(t,s)\, e^{-\tau}\epsilon \end{aligned}$$

$$\max\left\{|T^l(x)| + |y|, |T^l(x)| + |T^{l+1}(x)|, |T^l(y)| + |y|, \tfrac{|T^l(x)|+|T^{l+1}(x)|+|T^l(y)|+|y|}{4}\right\} ds$$

$$\leq \left(\epsilon e^{-\tau} \int_0^1 G(t,s) ds\right) \max\left\{\sigma(T^l x, y), \sigma(T^l x, T^{l+1}x), \sigma(y, Ty), \tfrac{\sigma(T^l x, Ty) + \sigma(T^{l+1}x, y)}{4}\right\}.$$

Now, on taking supremum over $[0,1]$, finally we get

$$\sigma(T^{l+1}x, Ty) \leq \epsilon e^{-\tau} M_l(x,y),$$

where $M_l(x,y)$ is defined in (2.2). By passing through a logarithms, we have

$$\log \sigma(T^{l+1}x, Ty) \leq \log\left(\epsilon e^{-\tau} M_l(x,y)\right),$$

which implies

$$\log \sigma(T^{l+1}x, Ty) \leq \log\left(\epsilon M_l(x,y)\right) - \tau.$$

This conclude that the contractive Condition (2.1) of Theorem 2.1 is satisfied with $F(t) = \log(t), \phi(u) = \epsilon$ and $\psi(p) = \tau > 0$. Hence T has a unique fixed point in X, and therefore the fractional equation (3.4) has a unique solution. $\square$

3.2. **Application to buckling of a tapered column.** Consider the stability of a tapered column of length L fixed at the base $x = 0$ at the top $x = L$. Apply an axial compressive load P at the top of the column. The cross-section of the column is of circular, with radii r_0 at the base and $r_1 < r_0$ at the top, respectively, varying linearly along the length x. The modulus of elasticity for the column material is E. Then by the routine calculation(please see [17]), the buckling load P_{cr} when the column loses its stability is given by the following Bessel's type differential equation

$$\begin{cases} \xi^2 \dfrac{d^2\eta}{d\xi^2} + \xi \dfrac{d\eta}{d\xi} = K(\xi, \eta(\xi)); \\ \eta(0) = \eta(1) = 0, \end{cases} \tag{3.6}$$

where $k : [0, 1] \times R^+ \to R$ is a continuous function.
Utilizing the fixed point result obtained in this article, we can find the existence of solution of (3.6).
Above problem (3.6) is equivalent to the integral equation

$$\eta(t) = \int_0^1 G(\xi, p)K(p, \eta(p))dp, \quad \xi \in [0, 1], \tag{3.7}$$

where $G(\xi, p)$ is the Green's function

$$G(\xi, p) = \begin{cases} \dfrac{p}{2\xi}(1 - \xi^2), & 0 \le p < \xi \le 1; \\ \dfrac{\xi}{2p}(1 - p^2), & 0 \le \xi < p \le 1. \end{cases} \tag{3.8}$$

Let $X = C(I), (I = [0, 1])$ be the space of all continuous functions defined on I. For an arbitrary $\eta \in X$, we define

$$\|\eta_1 - \eta_2\|_\tau = \sup_{\xi \in [0,1]} \{|\eta_1(\xi) - \eta_2(\xi)|\} \text{ where } \tau > 0. \tag{3.9}$$

Define metric-like $\sigma : X \times X \to R^+$ by

$$\sigma(x, y) = \|x - y\|_\tau + \|x\|_\tau + \|y\|_\tau, \quad for \;\; all \;\; x, y \in X, \tag{3.10}$$

where $\|x - y\|_\tau$ is defined by (3.9).
It is clear that equivalent metric to metric-like space is given by

$$d_\sigma(x, y) = 2\sigma(x, y) - \sigma(x, x) - \sigma(y, y) = 2\|x - y\|_\tau \quad \text{(by using (3.9))}.$$

Clearly $d_\sigma(x, y)$ is complete and hence (X, σ) is also complete.
Consider the self map $T : X \to X$ defined by

$$T(\eta(\xi)) = \int_0^1 G(\xi, p)f(p, \eta(p))dp, \;\; \xi \in I \text{ and for all } \eta \in X. \tag{3.11}$$

It is evident that η is a solution of (3.7) if and only if η is fixed point of T. Now succeeding theorem is established for the guarantee of the existence of fixed point of T.

Theorem 3.2. *Suppose the following conditions hold:*

(1) *there exists $\tau > 0$ and $u : I \to R^+$, such that*

$$|f(p,\eta_1) - f(p,\eta_2)| \leq 10e^{-\tau}u(p)|\eta_1 - \eta_2| \text{ for every } p \in I \text{ and } \eta_1, \eta_2 \in R^+;$$

(2) *there exists a continuous function $v : I \to R^+$ such that*

$$|f(p,\eta_1)| \leq 10e^{-\tau}v(p)|\eta_1| \text{ for every } p \in I \text{ and } \eta_1 \in R^+;$$

(3) $\sup_{p\in[0,1]} u(p) = \alpha_1 < \frac{1}{3}$ *and* $\sup_{p\in[0,1]} v(p) = \alpha_2 < \frac{1}{3}$.

Then the integral equation has a solution in X.

Proof. With the routine calculation as in Theorem 3.1 . Subsequently T has a fixed point which is the solution of Integral equation (3.7) and hence the equation represent the buckling load when the column loses its stability has a solution. □

3.3. **Application to ascending motion of Rocket.** Consider the ascending motion of a rocket of initial motion m_0. The fuel in the rocket consumed at a constant rate $q = \frac{-dm}{dt}$ and is expelled at a constant speed u relative to rocket. At any instance t, the mass of rocket is $m(t) = m_0 - qt$. Then the equation of motion of a rocket moving upward at high speed during the propelled phase is given by (see [17]).

$$m(t)\frac{dv(t)}{dt} + \beta v^2(t) + m(t)g - qu = 0.$$

Which is further by utilizing the substitution $v(t) = \frac{m(t)\dot{V}(t)}{\beta V(t)}$, reduced to equation of the form (3.6) (Please refer [17]). Thus on the similar line as in Application (3.2), one can easily establish the existence of solution of the equation representing ascending motion of rocket.

3.4. **Application of fixed point theorem to deformation of an elastica.** The transverse deformation of a thin elastic in-extensional rod subjected to an axial loading and clamped at its end is governed by the following boundary value problem.

$$\begin{aligned} -\frac{d^2\eta}{ds^2} &= f(s,\eta(s)),\ 0 < s < 1; \\ \eta(0) &= \eta(1) = 0, \end{aligned} \tag{3.12}$$

here θ is the angle that the deformed rod makes with the initial undeformed axis and $f : [0,1] \times R \to R$ is a continuous function.
Aforementioned problem (3.12) is equivalent to the integral equation

$$\theta(t) = \int_0^1 G(s,p)f(p,\theta(p))dp, \qquad \forall\, t \in [0,1], \tag{3.13}$$

where $G(s,p)$ is Green's function, given by

$$G(s,p)=\begin{cases} s(1-p) & \text{if } 0\le s\le p\le 1,\\ p(1-s) & \text{if } 0\le p\le s\le 1\end{cases} \tag{3.14}$$

Let $X=C(I),(I=[0,1])$ be the space of all continuous functions defined on I. For an arbitrary $\theta\in X$, we define

$$\|\theta_1-\theta_2\|_\tau=\sup_{s\in[0,1]}\{|\theta_1(s)-\theta_2(s)|\} \text{ where } \tau>0. \tag{3.15}$$

Now onwards, following the process applied in Application (3.2), we conclude that the function $F:R^+\to R$ defined by $F(\beta)=log(\beta)$ for every $\beta\in C(I)$ and for $\tau>0$ is in Δ_F. Thus all the conditions of Corollary 2.2 are satisfied with $\theta(t)=\alpha\ \ t,\ \ 0\le\alpha<1$ and for $l=0$. Consequently T has a fixed point which is the solution of Integral equation (3.13) and hence the equation represent the deformation of an elastica has a solution.

Open problem: For further applications of results obtained in this article, an open problem is suggested as follows:

Consider the functional differential equation of the form

$$x'(t)=-\int_{t-r(t)}^{t}p(t,v)G(v,x(v))dv,$$

where $r(t):[0,\infty)\to[0,\infty)$ and $p(t,v):[0,\infty)\times[r_0,\infty)\to R$ are continuous functions. Moreover there exists an $l>0$ such that G satisfies a Lipschitz condition with respect to x on $[r_0,\infty)\times[0,l]$, that is, there exists a constant $L>0$, such that $|G(v,x)-G(v,y)|\le L|x-y|$ for $v\ge r_0$ and $x,y\in[0,l]$.

Whether from our results, the existence of the solution of the above functional differential equation can achieve.

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