Fixed Point Theory, 21(2020), No. 2, 549-560
DOI: 10.24193/fpt-ro.2020.2.39
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

HALPERN TYPE ITERATION WITH TWO MAPPINGS IN A COMPLETE GEODESIC SPACE

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Abstract. In this paper, we show a strong convergence theorem for the Halpern iteration procedure in a complete CAT(1) space with two quasinonexpansive Δ-demiclosed mappings. We consider a sequence of coefficients for convex combination in the iterative scheme and find a certain discontinuity of the limit.
Key Words and Phrases: CAT(1) space, quasinonexpansive mapping, Δ-demiclosed mapping, Halpern itaration.
2010 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

Halpern type method is a technique to approximate a fixed point of a nonlinear mapping and has been studied by many mathematicians in various spaces. In 1992, strong convergence of a Halpern type iteration with a nonexpansive mapping was obtained by Wittmann [8] in a Hilbert space. In 2010, Saejung [6] proved a convergence theorem in a complete CAT(0) space. In 2013, Kimura and Satô [4] proved the same result in the setting of a complete CAT(1) space.

On the other hand, in 2015 Nakagawa [5] proved the following theorem with two strongly quasinonexpansive and Δ-demiclosed mappings in complete CAT(1) space:

Theorem 1.1. *Let X be a complete* CAT(1) *space with $d(v, v') < \pi/2$ for all $v, v' \in X$ and S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F = F(S) \cap F(T) \neq \emptyset$. Let P_F be a metric projection from X onto F. Let $\{\alpha_n\}, \{\beta_n\} \subset]0, 1[$ be real sequences satisfying $\alpha_n \to \alpha \in]0, 1[, \beta_n \to 0$ and*

$$\sum_{n=1}^{\infty} \beta_n = \infty.$$

Define $\{x_n\} \subset X$ *by* $x_1 = u \in X$ *and*

$$x_{n+1} = \alpha_n(\beta_n u \oplus (1-\beta_n)Sx_n) \oplus (1-\alpha_n)(\beta_n u \oplus (1-\beta_n)Tx_n)$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges to* $P_F u$.

In the studies of this kind of iterative schemes, we often consider the conditions of the coefficient sequences generating the approximating sequence. We focus on the limit $\alpha \in]0,1[$ of a sequence $\{\alpha_n\}$, and we attempt to deal with the extremal cases $\alpha = 0$ or 1, and other cases, simultaneously.

In this paper, we obtain that $\{x_n\}$ converges to a fixed point of T or S and its limit point depend on the limit of the coefficient sequence $\{\alpha_n\}$. From this result, we find a certain discontinuity of the limit of the iterative sequence concerning the coefficient sequence at the endpoints of $[0,1]$.

2. Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $\gamma : [0,l] \to X$ is called a geodesic with endpoints x, y if γ satisfies $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(s), \gamma(t)) = |s-t|$ for all $s, t \in [0,l]$. If a geodesic with endpoints x, y exists for any $x, y \in X$, then we call X a geodesic metric space. Moreover, if a geodesic exists uniquely for each $x, y \in X$, then we call X a uniquely geodesic space. In this case, the image $[x,y]$ of γ is uniquely detarmined for every $x, y \in X$ and it is called a geodesic segment joining x and y.

Let X be a uniquely geodesic metric space such that $d(v, v') < \pi/2$ for all $v, v' \in X$. A geodesic triangle is defined by $\triangle(x,y,z) = [x,y] \cup [y,z] \cup [z,x]$. Let $\mathbb{S}^2$ be the two-dimentional unit sphere in $\mathbb{R}^3$. For $\bar{x}, \bar{y}, \bar{z} \in \mathbb{S}^2$, a triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{S}^2$ is called a camparison triangle for $\triangle(x,y,z)$ if

$$d_{\mathbb{S}^2}(\bar{x}, \bar{y}) = d(x,y), \ d_{\mathbb{S}^2}(\bar{y}, \bar{z}) = d(y,z), \ d_{\mathbb{S}^2}(\bar{z}, \bar{x}) = d(z,x).$$

A point $\bar{p} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ is called a comparison point for $p \in [y,z]$ on the edge of $\triangle(x,y,z)$ if $\bar{p} \in [\bar{y}, \bar{z}]$ and $d(y,p) = d(\bar{y}, \bar{p})$. If, for any $p, q \in \triangle(x,y,z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality $d(p,q) \le d_{\mathbb{S}^2}(\bar{p}, \bar{q})$ is satisfied for all triangles in X, then X is called a CAT(1) space.

Let X be a geodesic metric space and $\{x_n\} \subset X$ a bounded sequence. For $x \in X$, we put

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n) \text{ and } r(\{x_n\}) = \inf_{x \in X} r(x, \{x_n\}).$$

If there exists $x \in X$ such that $r(x, \{x_n\}) = r(\{x_n\})$, we call x an asymptotic center of $\{x_n\}$. Let $\{x_n\}$ be a bounded sequence of X and $x_0 \in X$. If x_0 is a unique asymptotic center of all subsequences of $\{x_n\}$, then we say that $\{x_n\}$ is Δ-converges to x_0. We denote it by $x_n \overset{\Delta}{\rightharpoonup} x_0$. Let X be a CAT(1) space and T a mapping from X into itself. If $x_n \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ imply $x_0 \in F(T)$, we say T is Δ-demiclosed.

Let X be a metric space and $T : X \to X$. The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{z \in X : Tz = z\}$. A mapping T with $F(T) \ne \emptyset$ is said to be quasinonexpansive if $d(Tx, z) \le d(x, z)$ for any $x \in X$ and $F(T)$. Further, T is said to be strongly quasinonexpansive if it is quasinonexpansive and, for every

$p \in F(T)$ and every sequence $\{x_n\}$ in X satisfying that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, p)/\cos d(Tx_n, p)) = 1$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let X be a CAT(1) space such that $d(v, v') \le \pi/2$ for every $v, v' \in X$. Let F be a nonempty closed convex subset of X. Then, for any $x \in X$, there exists unique $p_x \in F$ such that $d(x, p_x) = \inf_{y\in F} d(x, y)$. Therefore we can define a mapping $P_F : X \to F$ by $P_F x = p_x$ for $x \in X$, and it is called a metric projection onto F.

3. Tools for the main results

In this section, we introduce some tools for the main theorems.

Lemma 3.1 ([1], [7]). *Let $\{\alpha_n\} \subset [0, \infty[$, $\{d_n\} \subset \mathbb{R}$ and $\{\gamma_n\} \subset]0, 1[$ such that*

$$\sum_{n=1}^{\infty} \gamma_n = \infty.$$

Define a set $\Phi = \{\varphi : \mathbb{N} \to \mathbb{N},$ nondecreasing and $\lim_{i\to\infty} \varphi(i) = \infty\}$. Suppose that

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n d_n$$

for any $n \in \mathbb{N}$. If $\overline{\lim}_{i\to\infty} d_{\varphi(i)} \le 0$ for any $\varphi \in \Phi$ satisfying

$$\varliminf_{i\to\infty} (a_{\varphi(i+1)} - a_{\varphi(i)}) \ge 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 3.2 (Kimura and Satô [4]). *Let X be a complete CAT(1) space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $\alpha \in [0, 1]$ and $u, y, z \in X$. Then*

$$\begin{aligned} &1 - \cos d(\alpha u \oplus (1 - \alpha)y, z) \\ &\quad \le (1 - \beta)(1 - \cos d(y, z)) + \beta \left(1 - \frac{\cos d(u, z)}{\sin d(u, y) \tan((\frac{\alpha}{2})d(u, y)) + \cos d(u, y)}\right), \end{aligned}$$

where

$$\beta = \begin{cases} 1 - \dfrac{\sin((1 - \alpha)d(u, y))}{\sin d(u, y)} & (u \ne y), \\ \alpha & (u = y). \end{cases}$$

Lemma 3.3 (Nakagawa [5]). *Let θ be a real number in $]0, \pi/2[$ and $\{\beta_n\}$ a real sequence in $]0, 1[$ such that $\lim_{n\to\infty} \beta_n = 0$. Then the following holds:*

$$\lim_{n\to\infty} \frac{1 - \cos(\beta_n\theta)}{\beta_n} = 0.$$

Lemma 3.4 (Nakagawa [5]). *Suppose $\{s_n\}$ and $\{t_n\} \subset]-\infty, 0]$ satisfy*

$$\lim_{n\to\infty} (s_n + t_n) = 0.$$

Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0$.

Lemma 3.5 (He, Fang, Lopez, and Li [3]). *Let X be a complete* CAT(1) *space and $u \in X$. If a sequence $\{x_n\}$ in X satisfies that $\overline{\lim}_{n\to\infty} d(u,x_n) < \pi/2$ and $x_n \overset{\Delta}{\rightharpoonup} x \in X$, then*

$$\underline{\lim}_{n\to\infty} d(u,x_n) \geq d(u,x).$$

Lemma 3.6 (Kimura and Satô [4]). *Let $\triangle(x,y,z)$ be a geodecic triangle in a* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x) < 2\pi$. Let $u = tx \oplus (1-t)y$ for some $t \in [0,1]$. Then*

$$\cos d(u,z) \geq t\cos d(x,z) + (1-t)\cos d(y,z).$$

Lemma 3.7 (Nakagawa [5]). *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v,v' \in X$ and $u \in X$. Let T be a Δ-demiclosed mapping from X into itself such that $F(T) \neq \emptyset$ is closed and convex. Let $\{x_n\} \subset X$ such that $\overline{\lim}_{n\to\infty} d(u,x_n) < \pi/2$. If $d(x_n,Tx_n) \to 0$, then*

$$\underline{\lim}_{n\to\infty} d(u,x_n) \geq d(u,P_{F(T)}u),$$

where $P_{F(T)}$ is a metric projection from X onto $F(T)$.

4. Main results

In this section, we prove our main results. We begin with the following lemma, which is essentially obtained by Nakagawa [5].

Lemma 4.1. *Let $\{\alpha_n\},\{\beta_n\} \in]0,1[$ such that $\sum_{n=1}^{\infty}\beta_n = \infty$, and let $\{d_n\} \in [0,\pi/2[$ such that $M = \sup_{n\in\mathbb{N}} d_n < \pi/2$. Then*

$$\sum_{n=1}^{\infty}(\alpha_n\sigma_n + (1-\alpha_n)\tau_n) = \infty,$$

where

$$\sigma_n = \begin{cases} 1 - \dfrac{\sin(1-\beta_n)d_n}{\sin d_n} & (d_n \neq 0), \\ \beta_n & (d_n = 0), \end{cases}$$
$$\tau_n = \begin{cases} 1 - \dfrac{\sin(1-\beta_n)d'_n}{\sin d'_n} & (d'_n \neq 0), \\ \beta_n & (d'_n = 0). \end{cases}$$

The following Theorem generalizes Theorem 1.1. We do not assume $\{\alpha_n\}$ to be convergent.

Theorem 4.2. *Let X be a complete* CAT(1) *space such that $M = \sup_{p,q\in X} d(p,q) < \pi/2$. Let S,T be strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F = F(S)\cap F(T) \neq \emptyset$. Let P_F be a metric projection from X onto F. Let*

$\{\alpha_n\} \subset [a,b] \subset]0,1[$ and $\{\beta_n\} \subset]0,1[$ satisfying $\beta_n \to 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Define $\{x_n\} \subset X$ by $x_1 = u \in X$ and

$$\begin{aligned} s_n &= \beta_n u \oplus (1-\beta_n) S x_n, \\ t_n &= \beta_n u \oplus (1-\beta_n) T x_n, \\ x_{n+1} &= \alpha_n s_n \oplus (1-\alpha_n) t_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u$.

Proof. Put $p = P_F u$,

$$\begin{aligned} a_n &= 1 - \cos d(x_n, p), \\ b_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Sx_n) \tan(\frac{\beta_n}{2} d(u, Sx_n)) + \cos d(u, Sx_n)}, \\ c_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Tx_n) \tan(\frac{\beta_n}{2} d(u, Tx_n)) + \cos d(u, Tx_n)}, \\ \sigma_n &= \begin{cases} 1 - \dfrac{\sin(1-\beta_n) d(u, Sx_n)}{\sin d(u, Sx_n)} & (u \neq Sx_n), \\ \beta_n & (u = Sx_n), \end{cases} \\ \tau_n &= \begin{cases} 1 - \dfrac{\sin(1-\beta_n) d(u, Tx_n)}{\sin d(u, Tx_n)} & (u \neq Tx_n), \\ \beta_n & (u = Tx_n) \end{cases} \end{aligned}$$

for $n \in \mathbb{N}$. Since $\alpha_n \sigma_n + (1-\alpha_n)\tau_n > 0$ for any $n \in \mathbb{N}$, by Lemmas 3.2 and 3.6, we have

$$\begin{aligned} a_{n+1} &= 1 - \cos d(\alpha_n s_n \oplus (1-\alpha_n) t_n, p) \\ &\leq 1 - (\alpha_n \cos d(s_n, p) + (1-\alpha_n) \cos d(t_n, p)) \\ &= \alpha_n (1 - \cos d(s_n, p)) + (1-\alpha_n)(1 - \cos d(t_n, p)) \\ &\leq \alpha_n ((1-\sigma_n) a_n + \sigma_n b_n) + (1-\alpha_n)((1-\tau_n) a_n + \tau_n c_n) \\ &= (1 - (\alpha_n \sigma_n + (1-\alpha_n)\tau_n)) a_n \\ &\quad + (\alpha_n \sigma_n + (1-\alpha_n)\tau_n) \left(\frac{\alpha_n \sigma_n b_n + (1-\alpha_n)\tau_n c_n}{\alpha_n \sigma_n + (1-\alpha_n)\tau_n} \right) \end{aligned}$$

for any $n \in \mathbb{N}$. To apply Lemma 3.1, we will show the following:

(i) $\sum_{n=1}^{\infty} (\alpha_n \sigma_n + (1-\alpha_n)\tau_n) = \infty$,

(ii) $\overline{\lim}_{i\to\infty} \left(\dfrac{\alpha_{\varphi(i)} \sigma_{\varphi(i)} b_{\varphi(i)} + (1-\alpha_{\varphi(i)}) \tau_{\varphi(i)} c_{\varphi(i)}}{\alpha_{\varphi(i)} \sigma_{\varphi(i)} + (1-\alpha_{\varphi(i)}) \tau_{\varphi(i)}} \right) \leq 0$ for any nondecreasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_{i\to\infty} \varphi(i) = \infty$ and

$$\underline{\lim}_{i\to\infty} (a_{\varphi(i)+1} - a_{\varphi(i)}) \geq 0.$$

The condition (i) is a direct result from Lemma 4.1. We consider (ii). For $\varphi : \mathbb{N} \to \mathbb{N}$ satisfying the conditions in (ii), we write $n_i = \varphi(i)$ for any $i \in \mathbb{N}$. Then it follows that $\varliminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \geq 0$, and we get

$$\begin{aligned}
0 &\leq \varliminf_{i\to\infty} (a_{n_i+1} - a_{n_i}) \\
&= \varliminf_{i\to\infty} (\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \\
&\leq \varliminf_{i\to\infty} (\cos d(x_{n_i}, p) - (\alpha_{n_i} \cos d(s_{n_i}, p) + (1-\alpha_{n_i}) \cos d(t_{n_i}, p))) \\
&= \varliminf_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(s_{n_i}, p)) \\
&\quad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(t_{n_i}, p))) \\
&\leq \varliminf_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\quad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))) \\
&= \varliminf_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\quad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))) \\
&\leq \varlimsup_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\quad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))) \\
&\leq 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\qquad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))) = 0.
\end{aligned}$$

Further, by Lemma 3.4, we get

$$\begin{aligned}
&\lim_{i\to\infty} \alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&= \lim_{i\to\infty} (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p)) = 0.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
\left|1 - \frac{\alpha_{n_i} \cos d(x_{n_i}, p)}{\alpha_{n_i} \cos d(Sx_{n_i}, p)}\right| &= \left|\frac{\alpha_{n_i}(\cos d(Sx_{n_i}, p) - \cos d(x_{n_i}, p))}{\alpha_{n_i} \cos d(Sx_{n_i}, p)}\right| \\
&= \frac{\cos d(Sx_{n_i}, p) - \cos d(x_{n_i}, p)}{\cos d(Sx_{n_i}, p)} \\
&\leq \frac{\cos d(Sx_{n_i}, p) - \cos d(x_{n_i}, p)}{\cos M} \\
&\to 0
\end{aligned}$$

as $i \to \infty$. In the same way, we have

$$\left|1 - \frac{\alpha_{n_i} \cos d(x_{n_i}, p)}{\alpha_{n_i} \cos d(Tx_{n_i}, p)}\right| \to 0$$

as $i \to \infty$. Therefore, we get

$$\lim_{i\to\infty} \frac{\cos d(x_{n_i}, p)}{\cos d(Sx_{n_i}, p)} = \lim_{i\to\infty} \frac{\cos d(x_{n_i}, p)}{\cos d(Tx_{n_i}, p)} = 1.$$

Since S and T are strongly quasinonexpansive, we get

$$\lim_{i\to\infty} d(x_{n_i}, Sx_{n_i}) = \lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0. \tag{4.1}$$

Let $\{n_{i_j}\}$ be a subsequence of $\{n_i\}$ such that

$$\overline{\lim_{i\to\infty}} \frac{\alpha_{n_i}\sigma_{n_i}b_{n_i} + (1-\alpha_{n_i})\tau_{n_i}c_{n_i}}{\alpha_{n_i}\sigma_{n_i} + (1-\alpha_{n_i})\tau_{n_i}} = \lim_{j\to\infty} \frac{\alpha_{n_{i_j}}\sigma_{n_{i_j}}b_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\alpha_{n_{i_j}}\sigma_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}}.$$

Further, We may find a subsequence $\{z_k\}$ of $\{x_{n_{i_j}}\}$ satisfying that $z_k \overset{\Delta}{\rightharpoonup} x_0 \in X$. Then by (4.1) and Δ-demiclosedness of S and T, we get $x_0 \in F(S) \cap F(T)$. Moreover, by Lemma 3.7, there exist δ and a subsequence $\{z_{k_l}\}$ of $\{z_k\}$ such that

$$\delta = \lim_{l\to\infty} d(u, z_{k_l}) = \underline{\lim}_{k\to\infty} d(u, z_k) \geq d(u, x_0) \geq d(u, p).$$

Also, we obtain

$$\begin{aligned}
\lim_{l\to\infty} d(u, z_{k_l}) &\leq \lim_{l\to\infty} (d(u, Sz_{k_l}) + d(Sz_{k_l}, z_{k_l})) = \lim_{l\to\infty} d(u, Sz_{k_l}) \\
&\leq \lim_{l\to\infty} (d(u, z_{k_l}) + d(z_{k_l}, Sz_{k_l})) = \lim_{l\to\infty} d(u, z_{k_l}) \\
&\leq \lim_{l\to\infty} (d(u, Tz_{k_l}) + d(Tz_{k_l}, z_{k_l})) = \lim_{l\to\infty} d(u, Tz_{k_l}) \\
&\leq \lim_{l\to\infty} (d(u, z_{k_l}) + d(z_{k_l}, Tz_{k_l})) = \lim_{l\to\infty} d(u, z_{k_l}).
\end{aligned}$$

Therefore, we obtain $\lim_{l\to\infty} d(u, z_{k_l}) = \lim_{l\to\infty} d(u, Sz_{k_l}) = \lim_{l\to\infty} d(u, Tz_{k_l})$. We can also obtain $\{\sigma_{k_l}/\tau_{k_l}\}$ converges to 1. Indeed, we have

$$\begin{aligned}
\frac{\sigma_{k_l}}{\tau_{k_l}} &= \frac{1 - \dfrac{\sin(1-\beta_{k_l})d(u, Sz_{k_l})}{\sin d(u, Sz_{k_l})}}{1 - \dfrac{\sin(1-\beta_{k_l})d(u, Tz_{k_l})}{\sin d(u, Tz_{k_l})}} \\
&= \frac{1 - \dfrac{\sin d(u, Sz_{k_l}) \cos \beta_{k_l} d(u, Sz_{k_l}) - \cos d(u, Sz_{k_l}) \sin \beta_{k_l} d(u, Sz_{k_l})}{\sin d(u, Sz_{k_l})}}{1 - \dfrac{\sin d(u, Tz_{k_l}) \cos \beta_{k_l} d(u, Tz_{k_l}) - \cos d(u, Tz_{k_l}) \sin \beta_{k_l} d(u, Tz_{k_l})}{\sin d(u, Tz_{k_l})}} \\
&= \frac{1 - \cos \beta_{k_l} d(u, Sz_{k_l}) - \dfrac{\cos d(u, Sz_{k_l}) \sin \beta_{k_l} d(u, Sz_{k_l})}{\sin d(u, Sz_{k_l})}}{1 - \cos \beta_{k_l} d(u, Tz_{k_l}) - \dfrac{\cos d(u, Tz_{k_l}) \sin \beta_{k_l} d(u, Tz_{k_l})}{\sin d(u, Tz_{k_l})}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\dfrac{1-\cos\beta_{k_l} d(u,Sz_{k_l})}{\beta_{k_l}} - \dfrac{d(u,Sz_{k_l})}{\tan d(u,Sz_{k_l})} \cdot \dfrac{\sin\beta_{k_l} d(u,Sz_{k_l})}{d(u,Sz_{k_l})}}{\dfrac{1-\cos\beta_{k_l} d(u,Tz_{k_l})}{\beta_{k_l}} - \dfrac{d(u,Tz_{k_l})}{\tan d(u,Tz_{k_l})} \cdot \dfrac{\sin\beta_{k_l} d(u,Tz_{k_l})}{d(u,Tz_{k_l})}} \\
&\to \frac{0 - \dfrac{\delta}{\tan\delta}\cdot 1}{0 - \dfrac{\delta}{\tan\delta}\cdot 1} \\
&= 1.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\lim_{l\to\infty}\left(\frac{\alpha_{k_l}\frac{\sigma_{k_l}}{\tau_{k_l}} b_{k_l} + (1-\alpha_{k_l})c_{k_l}}{\alpha_{k_l}\frac{\sigma_{k_l}}{\tau_{k_l}} + (1-\alpha_{k_l})}\right) \\
&= \lim_{l\to\infty} \frac{\alpha_{k_l}\cdot 1\cdot b_{k_l} + (1-\alpha_{k_l})c_{k_l}}{\alpha_{k_l}\cdot 1 \cdot + (1-\alpha_{k_l})} \\
&= \lim_{l\to\infty} (\alpha_{k_l} b_{k_l} + (1-\alpha_{k_l})c_{k_l}) \\
&= \lim_{k\to\infty} \left(\alpha_{k_l}\left(1 - \frac{\cos d(u,p)}{0+\cos\delta}\right) + (1-\alpha_{k_l})\left(1 - \frac{\cos d(u,p)}{0+\cos\delta}\right)\right) \\
&= 1 - \frac{\cos d(u,p)}{\cos\delta} \\
&\le 0.
\end{aligned}
$$

Thus, we get

$$
\overline{\lim_{i\to\infty}}\left(\frac{\alpha_{\varphi(i)}\sigma_{\varphi(i)} b_{\varphi(i)} + (1-\alpha_{\varphi(i)})\tau_{\varphi(i)} c_{\varphi(i)}}{\alpha_{\varphi(i)}\sigma_{\varphi(i)} + (1-\alpha_{\varphi(i)})\tau_{\varphi(i)}}\right) \le 0,
$$

and hence (ii) holds. By Lemma 3.1, we get $\lim_{n\to\infty} a_n = 0$. It implies that $\{x_n\}$ converges to $P_F u$. □

Next, we consider the case where the coefficient sequence $\{\alpha_n\}$ is convergent to an endpoint of $[0,1]$.

Theorem 4.3. *Let X be a complete $\mathrm{CAT}(1)$ space such that $M = \sup_{p,q\in X} d(p,q) < \pi/2$ and S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Let $P_{F(S)}$ and $P_{F(T)}$ be metric projections from X onto $F(S)$ and $F(T)$, respectively. Let $\{\alpha_n\}, \{\beta_n\} \subset]0,1[$ be real sequences satisfying $\beta_n \to 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Define $\{x_n\} \subset X$ by $x_1 = u \in X$ and*

$$
\begin{aligned}
s_n &= \beta_n u \oplus (1-\beta_n) S x_n, \\
t_n &= \beta_n u \oplus (1-\beta_n) T x_n, \\
x_{n+1} &= \alpha_n s_n \oplus (1-\alpha_n) t_n
\end{aligned}
$$

for all $n \in \mathbb{N}$. *Then*

- *if* $\alpha_n \to 0$, *then* $x_n \to P_{F(T)}u$;
- *if* $\alpha_n \to 1$, *then* $x_n \to P_{F(S)}u$.

Proof. We consider the case that $\alpha_n \to 0$. Put $p = P_{F(T)}u$,

$$\begin{aligned}
a_n &= 1 - \cos d(x_n, p),\\
b_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Sx_n)\tan(\frac{\beta_n}{2} d(u, Sx_n)) + \cos d(u, Sx_n)},\\
c_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Tx_n)\tan(\frac{\beta_n}{2} d(u, Tx_n)) + \cos d(u, Tx_n)},\\
\sigma_n &= \begin{cases} 1 - \dfrac{\sin(1-\beta_n)d(u, Sx_n)}{\sin d(u, Sx_n)} & (u \neq Sx_n),\\ \beta_n & (u = Sx_n),\end{cases}\\
\tau_n &= \begin{cases} 1 - \dfrac{\sin(1-\beta_n)d(u, Tx_n)}{\sin d(u, Tx_n)} & (u \neq Tx_n),\\ \beta_n & (u = Tx_n).\end{cases}
\end{aligned}$$

for $n \in \mathbb{N}$. Then, by the same calculation as in Theorem 4.2, we have

$$a_{n+1} \leq (1-(\alpha_n\sigma_n+(1-\alpha_n)\tau_n))a_n + (\alpha_n\sigma_n+(1-\alpha_n)\tau_n)\left(\frac{\alpha_n\sigma_n b_n + (1-\alpha_n)\tau_n c_n}{\alpha_n\sigma_n + (1-\alpha_n)\tau_n}\right).$$

To apply Lemma 3.1, we will show the following:

(i) $\displaystyle\sum_{n=1}^{\infty}(\alpha_n\sigma_n + (1-\alpha_n)\tau_n) = \infty$,

(ii) $\overline{\lim}_{i\to\infty}\left(\dfrac{\alpha_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)} + (1-\alpha_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\alpha_{\varphi(i)}\sigma_{\varphi(i)} + (1-\alpha_{\varphi(i)})\tau_{\varphi(i)}}\right) \leq 0$ for any nondecreasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_{i\to\infty} \varphi(i) = \infty$ and

$$\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1} - a_{\varphi(i)}) \geq 0.$$

The condition (i) is obtained from Lemma 4.1. We consider (ii). For $\varphi : \mathbb{N} \to \mathbb{N}$ satisfying the conditions in (ii), we write $n_i = \varphi(i)$ for any $i \in \mathbb{N}$. Then it follows that $\underline{\lim}_{i\to\infty}(a_{n_i+1} - a_{n_i}) \geq 0$, and we get

$$
\begin{aligned}
0 &\le \varliminf_{i\to\infty} (a_{n_i+1} - a_{n_i}) \\
&= \varliminf_{i\to\infty} (\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \\
&\le \varliminf_{i\to\infty} (\cos d(x_{n_i}, p) - (\alpha_{n_i} \cos d(s_{n_i}, p) + (1-\alpha_{n_i}) \cos d(t_{n_i}, p))) \\
&= \varliminf_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(s_{n_i}, p)) \\
&\quad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(t_{n_i}, p))) \\
&\le \varliminf_{i\to\infty} (\alpha_{n_i}(\cos d(x_{n_i}, p) - \cos d(Sx_{n_i}, p)) \\
&\quad + (1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p))) \\
&= \varliminf_{i\to\infty} (\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p)) \\
&\le \varlimsup_{i\to\infty} (\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p)) \\
&\le 0.
\end{aligned}
$$

Therefore, we have $\lim_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p)) = 0$. On the other hand, we obtain

$$
\begin{aligned}
\left|1 - \frac{\cos d(x_{n_i}, p)}{\cos d(Tx_{n_i}, p)}\right| &= \left|\frac{\cos d(Tx_{n_i}, p) - \cos d(x_{n_i}, p)}{\cos d(Tx_{n_i}, p)}\right| \\
&= \frac{1}{\cos d(Tx_{n_i}, p)} |\cos d(Tx_{n_i}, p) - \cos d(x_{n_i}, p)| \\
&\le \frac{1}{\cos M} |\cos d(Tx_{n_i}, p) - \cos d(x_{n_i}, p)| \\
&\to 0
\end{aligned}
$$

as $i \to \infty$. Therefore, we get

$$\lim_{i\to\infty} \frac{\cos d(x_{n_i}, p)}{\cos d(Tx_{n_i}, p)} = 1.$$

Since T is strongly quasinonexpansive, we get

$$\lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0. \tag{4.2}$$

There exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$\varlimsup_{i\to\infty} \frac{\alpha_{n_i}\sigma_{n_i}b_{n_i} + (1-\alpha_{n_i})\tau_{n_i}c_{n_i}}{\alpha_{n_i}\sigma_{n_i} + (1-\alpha_{n_i})\tau_{n_i}} = \lim_{j\to\infty} \frac{\alpha_{n_{i_j}}\sigma_{n_{i_j}}b_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\alpha_{n_{i_j}}\sigma_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}}.$$

Further, We may find a subsequence $\{z_k\}$ of $\{x_{n_{i_j}}\}$ satisfying that $z_k \overset{\Delta}{\rightharpoonup} x_0 \in X$. Then by (4.1) and Δ-demiclosedness of T, we get $x_0 \in F(T)$. Moreover, by Lemma 3.7, there exist δ and a subsequence $\{z_{k_l}\}$ of $\{z_k\}$ such that

$$\delta = \lim_{l\to\infty} d(u, z_{k_l}) = \varliminf_{k\to\infty} d(u, z_k) \ge d(u, x_0) \ge d(u, p).$$

Also, we obtain

$$\begin{aligned}\lim_{l\to\infty} d(u, z_{k_l}) &\le \lim_{l\to\infty} (d(u, Tz_{k_l}) + d(Tz_{k_l}, z_{k_l})) = \lim_{l\to\infty} d(u, Tz_{k_l})\\ &\le \lim_{l\to\infty} (d(u, z_{k_l}) + d(z_{k_l}, Tz_{k_l})) = \lim_{l\to\infty} d(u, z_{k_l}).\end{aligned}$$

Therefore, we obtain $\lim_{l\to\infty} d(u, z_{k_l}) = \lim_{l\to\infty} d(u, Tz_{k_l})$. Then, we obtain that

$$\begin{aligned}\overline{\lim_{i\to\infty}} \left(\frac{\alpha_{n_i}\sigma_{n_i}b_{n_i} + (1-\alpha_{n_i})\tau_{n_i}c_{n_i}}{\alpha_{n_i}\sigma_{n_i} + (1-\alpha_{n_i})\tau_{n_i}} \right) &= \lim_{j\to\infty} \left(\frac{\alpha_{n_{i_j}}\sigma_{n_{i_j}}b_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\alpha_{n_{i_j}}\sigma_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}} \right)\\ &= \overline{\lim_{j\to\infty}} \left(\frac{\alpha_{n_{i_j}}\sigma_{n_i}b_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\alpha_{n_{i_j}}\sigma_{n_{i_j}} + (1-\alpha_{n_{i_j}})\tau_{n_{i_j}}} \right)\\ &= \lim_{k\to\infty} \left(\frac{\alpha_k\sigma_k b_k + (1-\alpha_k)\tau_k c_k}{\alpha_k\sigma_k + (1-\alpha_k)\tau_k} \right)\\ &= \underline{\lim}_{k\to\infty} \left(\frac{\alpha_k\sigma_k b_k + (1-\alpha_k)\tau_k c_k}{\alpha_k\sigma_k + (1-\alpha_k)\tau_k} \right)\\ &= \lim_{l\to\infty} \left(\frac{\alpha_{k_l}\sigma_{k_l}b_{k_l} + (1-\alpha_{k_l})\tau_{k_l}c_{k_l}}{\alpha_{k_l}\sigma_{k_l} + (1-\alpha_{k_l})\tau_{k_l}} \right)\\ &= \lim_{l\to\infty} c_{k_l} = 1 - \frac{\cos d(u,p)}{\cos\delta} \le 0.\end{aligned}$$

Thus, we get

$$\overline{\lim_{i\to\infty}} \left(\frac{\alpha_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)} + (1-\alpha_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\alpha_{\varphi(i)}\sigma_{\varphi(i)} + (1-\alpha_{\varphi(i)})\tau_{\varphi(i)}} \right) \le 0.$$

By Lemma 3.1, we get $\lim_{n\to\infty} a_n = 0$. It implies that $\{x_n\}$ converges to $P_{F(T)}u$. In a similar fashion, we have $\{x_n\}$ converges to $P_{F(S)}u$ if $\alpha_n \to 1$. Hence we obtain the desired result. □

From the results above, we observe that the limit point of the iterative scheme behaves discontinuously at the endpoints of $[0,1]$ for α, which is a limit of the coefficient sequence $\{\alpha_n\}$. It seems to be curious, however, we notice that the limit of an iterative sequence can be represented by a single mapping $U_\alpha = \alpha S \oplus (1-\alpha)T$. We know that it is pointwise continuous for α; for a sequence $\alpha_n \subset [0,1]$ with a limit $\alpha \in [0,1]$, $\{U_{\alpha_n}x\}$ converges to $U_\alpha x$ for each $x \in X$. On the other hand, the set $F(U_\alpha)$ of fixed points of U_α does not behave continuously at $\alpha = 0$ or $\alpha = 1$; we have

$$F(U_\alpha) = \begin{cases} F(T) & (\alpha = 0),\\ F(S)\cap F(T) & \alpha \in\,]0,1[,\\ F(S) & (\alpha = 1).\end{cases}$$

Consequently, we obtain the following result generalizing Theorems 1.1 and 4.3.

Theorem 4.4. *Let X be a complete CAT(1) space such that $M = \sup_{p,q\in X} d(p,q) < \pi/2$ and S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself* *with* $F = F(S) \cap F(T) \neq \emptyset$. *Let* P_F *be a metric projection from* X *onto* F. *Let* $\{\alpha_n\}, \{\beta_n\} \subset]0,1[$ *satisfying* $\alpha_n \to \alpha \in [0,1], \beta_n \to 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Define* $\{x_n\} \subset X$ *by* $x_1 = u \in X$ *and*

$$\begin{aligned} s_n &= \beta_n u \oplus (1-\beta_n) S x_n, \\ t_n &= \beta_n u \oplus (1-\beta_n) T x_n, \\ x_{n+1} &= \alpha_n s_n \oplus (1-\alpha_n) t_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges to* $P_{F(\alpha S \oplus (1-\alpha)T)} u$.

References

[1] K. Aoyama, Y. Kimura, F. Kohsaka, *Strong convergence theorems for strongly relatively nonexpansive sequences and applications*, J. Nonlinear Anal. Optim., Theory and Applications, **3**(2012), 67-77.

[2] M.R. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, vol. 319 of "Grundlehren der. Mathematischen Wissenschaften", Springer, Verlag, Berlin, Germany, 1999.

[3] J.S. He, D.H. Fang, G. Lopez, C. Li, *Mann's algorithm for nonexpansive mappings in* CAT(κ)-*spaces*, Nonlinear Anal., **75**(2012), 445-452.

[4] Y. Kimura, K. Satô, *Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one*, Fixed Point Theory Appl., **2013**(2013), 14 pages.

[5] K. Nakagawa, *Convergence Theorems to a Common Fixed Point of Two Mappings in* CAT(1) *spaces*, Master Thesis, Toho University, 2015.

[6] S. Saejung, *Halpern's iteration in* CAT(0) *spaces,* Fixed Point Theory Appl., **2010** (2010), Art. ID 471781.

[7] S. Saejung, P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal., **75**(2012), 742-750.

[8] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. (Basel), **58**(1992), 486-491.

Received: November 4, 2019; Accepted: January 11, 2020.